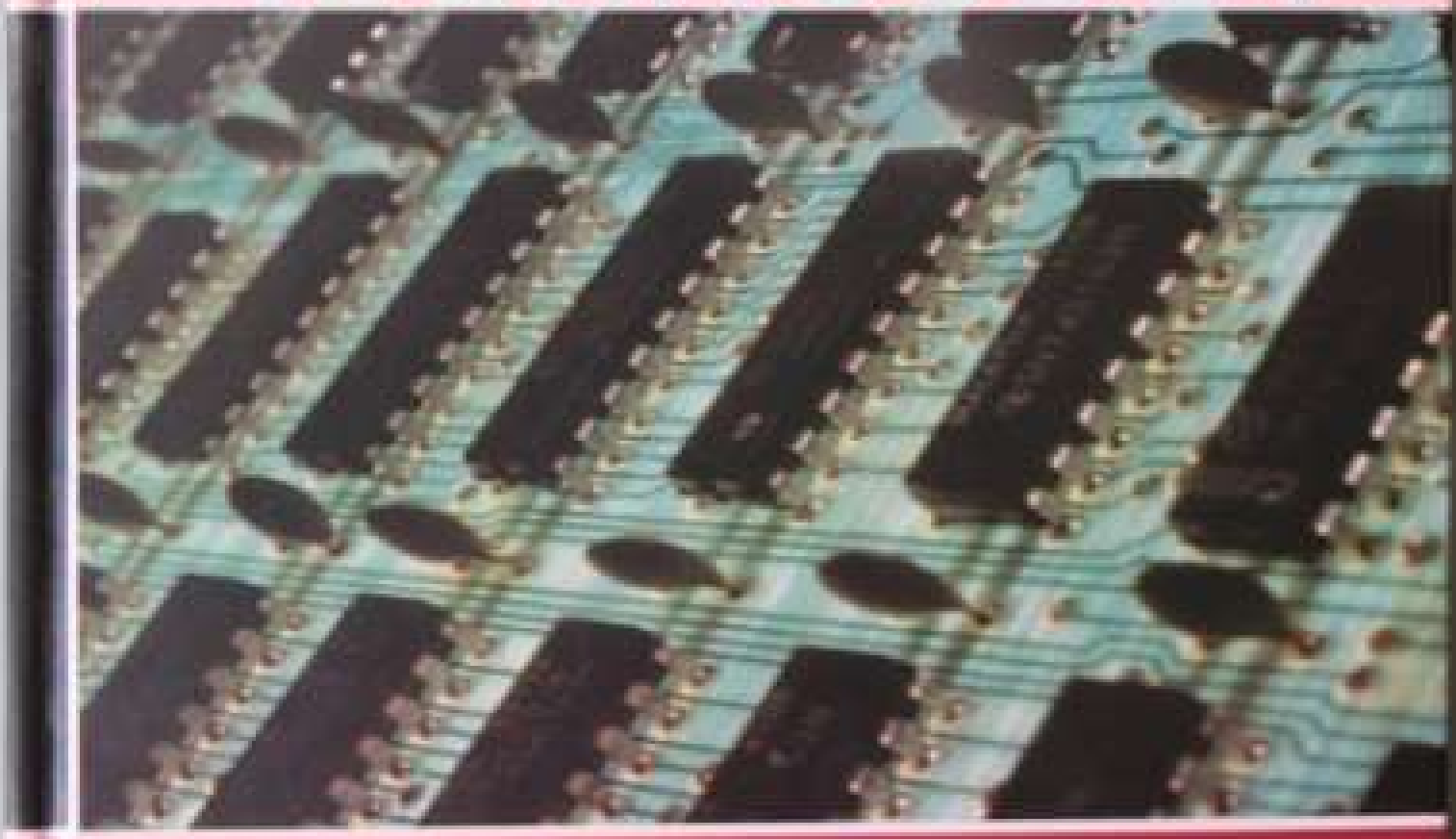
Mathematics Applied to TECHNOLOGY



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Chapter 0

Natural Numbers, Fractions, and Decimals

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See Problem Set 0.4, Exercise 114

Natural Numbers, Fractions, and Decimals

CONNECTIONS

Arithmetic books dating back to the Middle Ages have attempted to teach students how to calculate with counting numbers and fractions. For thousands of years, people have sought ways to calculate more rapidly and accurately. Various devices have been invented to carry out computations automatically. The oldest such device is the abacus which often consisted of a series of beads strung on rows of wire each representing a power of 10.

Today calculators and computers are used to make calculations in fields ranging from engineering to coaching basketball. Computers make thousands of arithmetic calculations for meteorologists in preparing a weather forecast for the National Weather Service.

So, why should we study arithmetic? Understanding how to add, subtract, multiply, and divide with natural numbers, fractions, and decimals is the foundation upon which algebra is developed. Algebra can be described as arithmetic with letters used to represent numbers. To understand the connection between arithmetic and algebra, we must first understand the concepts of arithmetic. This chapter contains a review of arithmetic.

0.1 ORDER OF OPERATIONS

Arithmetic is a study of numbers. The symbols that we use to name numbers are called **numerals.** For example, 2 and II are names for the number two. The symbol 2 is an Arabic numeral, whereas II is a Roman numeral. We use Arabic numerals almost exclusively in mathematics.

There are many kinds of numbers. In this chapter we will use the counting numbers, or natural numbers. They are 1, 2, 3, ... The three dots (called an ellipsis) indicate that the numbers continue.

In arithmetic, we perform four operations (addition, subtraction, multiplication, and division) with numbers. We use symbols to indicate these operations. The following chart indicates the symbols that we use for each of the operations with the numbers 7 and 5.

OPERATION	SYMBOL(S)	READ
Addition	$7 + 5$	7 plus 5
Subtraction	$7 - 5$	7 minus 5
Multiplication	$7 \cdot 5$; $7(5)$; $(7)5$; $(7)(5)$	7 times 5
Division	$7 \div 5$; $\frac{7}{5}$; $7/5$; $5\overline{)7}$	7 divided by 5

The **sum** of two numbers is the result of adding them. The **difference** in two numbers is the result of subtracting one from the other. The **product** of two numbers is the result of multiplying them. The **quotient** is the result of dividing one number (dividend) by another (divisor).

EXAMPLE 1. Express each phrase in symbols.

(a) The sum of 8 and 3.
(b) The difference when 6 is subtracted from 9.
(c) The difference when 9 is subtracted from 6.
(d) The product of 2 and 7.
(e) The quotient when 12 is divided by 19.
(f) The quotient when 19 is divided by 12.

Solutions:

(a) $8 + 3$ (b) $9 - 6$ (c) $6 - 9$
(d) $2 \cdot 7$ (e) $12 \div 19$ (f) $19 \div 12$ □

Mathematicians have developed a shorthand notation for writing products with the same number. Instead of writing $2 \cdot 2 \cdot 2$, we write 2^3. When we write 3^5, we mean $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.

In an expression such as 3^5, we call 3 the **base,** and 5 the **exponent.** The exponent tells us how many times to use the base in the product. We read 3^5 as 3 to the fifth **power.** We use the second and third powers so often that we read 7^2 as 7 *squared* and 4^3 as 4 *cubed.*

EXAMPLE 2. Find the value of each expression.

(a) 3^2 (b) 2^3

Solutions:

(a) $3^2 = 3 \cdot 3$
$= 9$

(b) $2^3 = 2 \cdot 2 \cdot 2$
$= 8$ □

Many times more than one operation is involved in simplifying an expression. For example, $2 + 3 \cdot 4$ has multiplication and addition. We must decide which operation to perform first. **Grouping symbols** such as parentheses ( ), brackets [ ], or braces { } are used to make the order of operations clear. We perform operations inside grouping symbols first.

If we write $(2 + 3) \cdot 4$, then we perform the addition first and then the multiplication. So,

$$(2 + 3) \cdot 4 = 5 \cdot 4$$
$$= 20$$

However, if we write $2 + (3 \cdot 4)$, then we perform the multiplication first followed by the addition. So,

$$2 + (3 \cdot 4) = 2 + 12$$
$$= 14$$

If there are no grouping symbols, then we do multiplications and divisions from left to right before we do additions and subtractions. So,

$$2 + 3 \cdot 4 = 2 + 12$$
$$= 14$$

Fraction bars are also grouping symbols. For example,

$$\frac{2+10}{2+4} = \frac{12}{6}$$
$$= 2$$

This expression is the same as $(2 + 10) \div (2 + 4)$. The fraction allows us to omit the parentheses.

The following rules determine which operation to perform first.

Order of Operations

If grouping symbols are present, perform operations inside them, starting with the innermost grouping symbol, in the following order.

1. Perform any exponentiations.
2. Perform all multiplications and divisions in order from left to right.
3. Perform all additions and subtractions in order from left to right.

If grouping symbols are *not* present, perform operations in the above order.

A commonly used memory aid is the sentence, "Please excuse my dear Aunt Sally." The first letters of each word give the order of operations.

Parentheses
Exponents
Multiplication and Division from left to right
Addition and Subtraction from left to right

Be Careful!

Note that multiplication does not have to be done before division, nor addition before subtraction. Multiplications and divisions are done in order as they occur working from left to right. Additions and subtractions are done in order as they occur working from left to right.

EXAMPLE 3. Perform the operations indicated.

(a) $6 \cdot 3 - \frac{4}{2}$ (b) $10 - 3 + 2^2$

(c) $14/2 \cdot 3$ (d) $12 - 2(6 - 3)$

(e) $(3^2 - 4)^2$ (f) $\frac{2 \cdot 3 + 8}{10 - 3}$

Solutions:

(a) We perform the multiplication and division first.

$$6 \cdot 3 - \frac{4}{2} = 18 - 2$$
$$= 16$$

(b) We work with the exponent first.

$$10 - 3 + 2^2 = 10 - 3 + 4$$
$$= 11$$

(c) We perform multiplications and divisions from left to right in the order they occur. So, we divide first and then multiply.

$$14/2 \cdot 3 = 7 \cdot 3$$
$$= 21$$

(d) We work inside the parentheses first.

$$12 - 2(6 - 3) = 12 - 2 \cdot 3$$
$$= 12 - 6$$
$$= 6$$

(e) $(3^2 - 4)^2 = (9 - 4)^2$
$= 5^2$
$= 25$

(f) We simplify the numerator and denominator first.

$$\frac{2 \cdot 3 + 8}{10 - 3} = \frac{6 + 8}{10 - 3}$$
$$= \frac{14}{7}$$
$$= 2$$ □

PROBLEM SET 0.1

Warm-ups

In Problems 1 through 15, express each phrase in symbols. See Example 1.

1. The sum of 3 and 4.
2. 7 more than 5.
3. 3 times 9.
4. 6 subtracted from 11.
5. The quotient when 21 is divided by 8.
6. Twice 14.
7. One-half of 17.
8. 8 squared.
9. 5 more than 1.
10. The quotient when 5 is divided by 9.
11. The product of 2 and 5.
12. 4 less than 6.
13. 2 cubed.
14. 11 squared plus 6.
15. One less than twice 13.

Find the value of each. See Example 2.

16. 2^3 17. 3^2 18. 4^2 19. 5^3 20. 10^2

In Problems 21 through 36, perform the operations indicated. See Example 3.

21. $6 + 3 \cdot 2$
22. $12 - 2^2$
23. $\frac{16 - 2}{7}$
24. $(3 + 4)(6 - 4)$
25. $(3 + 4)6 - 4$
26. $3 + 4(6 - 4)$
27. $12 - 2(8 - 4)$
28. $\frac{8^2 - 6 \cdot 7}{2 \cdot 3 + 5}$
29. $9 + 42/7 - 2^2$

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30. $2 \cdot 3^2 - 2^2$

31. $\dfrac{2\left(\frac{12}{4}\right) - 2}{2^2}$

32. $(6 - 3)^2$

33. $45 - (2 \cdot 3)^2$

34. $18/6 \cdot 2$

35. $28 \div 4 - 2 \cdot 3$

36. $2^2 \cdot (2 \cdot 3)^2$

Practice Exercises

In Problems 37 through 50, find each value.

37. The product of 3 and 4.

38. 5 less than 17.

39. 3 added to 53.

40. 1 subtracted from 6.

41. The quotient when 77 is divided by 11.

42. One-third of 12.

43. 12 squared.

44. 11 less than 15.

45. The quotient when 6 is divided by 2.

46. The square of the sum of 5 and 3.

47. Twice the sum of 8 and 7.

48. 4 cubed.

49. 13 squared minus 6.

50. One more than twice 7.

In Problems 51 through 76, perform the operations indicated.

51. $6 \cdot 3 - 2$

52. $2^2 - 3$

53. $\dfrac{16 - 9}{7}$

54. $(6 - 4)(6 + 4)$

55. $(7 - 4)6 - 4$

56. $12 - 4(6 - 4)$

57. $12 + 2(8 - 7)$

58. $\dfrac{8 \cdot 7 - 6^2}{2 \cdot 5 - 6}$

59. $22 - (20 - 14)$

60. $6^2 - \dfrac{16 - 4}{4}$

61. $23 - 63/7 - 3^2$

62. $2^2 \cdot 3^2 - 2^2$

63. $\dfrac{2\left(\frac{48}{16}\right) - 2}{2^2}$

64. $(6 + 3)^2$

65. $25 - 2 \cdot 3^2$

66. $3 \cdot 8/6$

67. $36 \div 9 \cdot 2 - 3$

68. $2 \cdot (4 \cdot 3)^2$

69. $2(3^3 - 3^2)$

70. $4^2 + 5 \div 5 - 2 \cdot 3$

71. $2 + (3 \cdot 2^3 - 2 \cdot 3^2)$

72. $2 + 3[2 + 3(7 - 6)]$

73. $19 - 3[2 + 2(11 - 3^2)]$

74. $[13 - 2(3 \cdot 5 - 12)] - 2^2$

75. $3(4^2 - 2^3)$

76. $3^2 - 3 \div 3 + 3 \cdot 3$

In Problems 77 through 96, answer each question.

77. Mae has 9 books and Sue has 5 books. How many books do they have together?

78. Jo has 4 times 3 skirts and Geisla has 2 times 8 skirts. How many skirts do they have together?

79. Tony has 24 quarters and Ellen has 10 fewer quarters than Tony. How many quarters does Ellen have?

80. Jorge has 9 sodas and Manuel has 3 fewer sodas than Jorge. How many sodas does Manuel have?

81. John has 6 records and Mary has 10 records. How many records do they have together?

82. Susan has 3 times 9 dollars and Judy has 2 times 8 dollars. How many dollars do they have?

83. Wei has 26 nickels and Choi has 10 more nickels than Wei. How many nickels does Choi have?

84. Juan has 12 marbles and Maria has 7 fewer marbles than Juan. How many marbles does Maria have?

IN YOUR OWN WORDS . . .

85. State the order of operations.

Write in words what each expression means.

86. $8 + 7$

87. $11 - 14$

88. $5 \cdot 23$

89. $6 \div 7$

90. $9/22$

91. $\dfrac{7}{6}$

92. Explain two ways to interpret the statement "The sum of 7 and 2 squared."

0.2 FACTORING

In much of our work in algebra, it is necessary to write a number as a product of numbers. This process is called **factoring.** The numbers in the product are called **factors.** Most often we deal with natural numbers.

A **prime** number is a natural number larger than 1 whose only natural number factors are itself and 1. For example, 5 is prime because $5 = 5 \cdot 1$. This is the only way to write 5 as a product of natural numbers. Likewise, 2, 3, 7, and 11 are prime. We don't call 1 a prime number. The smallest prime number is 2.

Sometimes we need to factor a natural number that is not prime. Often there are several ways to do this. For example 12 can be factored as follows:

$$12 = 2 \cdot 2 \cdot 3 \qquad 12 = 2 \cdot 6 \qquad 12 = 3 \cdot 4 \qquad 12 = 12 \cdot 1$$

Frequently, the most useful of these is $2 \cdot 2 \cdot 3$ because it is a product of prime numbers. There is only one way to factor a natural number using only prime factors. When we write a number as a product of prime numbers, we say that we have **factored it completely,** or that we have the **prime factorization.**

In factoring large numbers, it is helpful to know when a number is divisible by a prime.

Some Tests for Divisibility

1. A natural number is divisible by 2 if it is even — that is, if its last digit is 0, 2, 4, 6, or 8.
2. A natural number is divisible by 3 if the sum of the digits of the number is divisible by 3.
3. A natural number is divisible by 5 if its last digit is 0 or 5.

To factor a number into prime factors, we start with the smallest prime that will divide into the number and continue until we have all prime factors.

EXAMPLE 1. Write each number as a product of prime factors.

(a) 45 (b) 108 (c) 392

Solutions:

(a) We notice that 45 is not even, so it is not divisible by 2. However, the sum of its digits is 9, so it is divisible by 3.

$$\begin{aligned} 45 &= 3 \cdot 15 \\ &= 3 \cdot 3 \cdot 5 \\ &= 3^2 \cdot 5 \end{aligned}$$

(b) Since 108 is even, it is divisible by 2.

$$108 = 2 \cdot 54$$

Now 54 is also divisible by 2. So,

$$108 = 2 \cdot 2 \cdot 27$$

(continued)

$$= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$$
$$= 2^2 \cdot 3^3$$

(c) Since 392 is even, we divide by 2:

$$392 = 2 \cdot 196$$
$$= 2 \cdot 2 \cdot 98$$
$$= 2 \cdot 2 \cdot 2 \cdot 49$$
$$= 2^3 \cdot 7^2$$ □

Short division is long division without all of the arithmetic. It is a convenient way to factor numbers. Let's factor 588 using short division. We begin with the smallest prime that will divide into 588. It is 2:

$$2\overline{)588}\quad 294$$

We do the same thing with the number 294. The smallest prime that will divide into 294 is 2:

$$\begin{array}{r} 147 \\ 2\overline{)294} \\ 2\overline{)\ 588} \end{array}$$

Now 147 is not divisible by 2. However, since the sum of its digits is 12, it is divisible by 3:

$$\begin{array}{r} 49 \\ 3\overline{)147} \\ 2\overline{)\ 294} \\ 2\overline{)\ 588} \end{array}$$

The smallest prime that will divide into 49 is 7:

$$\begin{array}{r} 7 \\ 7\overline{)49} \\ 3\overline{)147} \\ 2\overline{)\ 294} \\ 2\overline{)\ 588} \end{array}$$

Since 7 is prime, we stop the process. Thus, $588 = 2^2 \cdot 3 \cdot 7^2$

PROBLEM SET 0.2

In Problems 1 through 10, write each number as a product of prime factors.

1. 24
2. 36
3. 25
4. 16
5. 8
6. 20
7. 28
8. 18
9. 45
10. 30

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Practice Exercises

In Problems 11 through 40, write each number as a product of prime factors.

11. 12
12. 48
13. 50
14. 32
15. 81
16. 40
17. 70
18. 63
19. 90
20. 125
21. 200
22. 243
23. 56
24. 196
25. 150
26. 100
27. 96
28. 72
29. 108
30. 180
31. 441
32. 144
33. 3969
34. 576
35. 972
36. 625
37. 169
38. 900
39. 960
40. 1008

Answer each question.

41. Martin bought 6 pictures and Rowland bought three times as many as Martin. How many pictures did Rowland buy?
42. Vincent and Kim each ate 2 candy bars every month for one year. How many candy bars did they eat in the year?
43. Ken has 3 rare coins in his collection and Branden has double that many in his collection. How many rare coins does Branden have?
44. Susie has 41 stamps in her collection. If she triples the number of stamps that she has, how many stamps will she have?
45. Tim bought 7 pencils and Irene bought twice as many as Tim. How many pencils did Irene buy?
46. Ky has 6 sandwiches and Nyugen has seven times as many sandwiches as Ky. How many sandwiches does Nyugen have?
47. Pierre cuts a board into three pieces each of length 5 m. How long was the board originally?
48. A basketball team had 43 points at halftime. At the end of the game they had doubled the halftime score. How many points did the team have at the end of the game?

IN YOUR OWN WORDS...

49. What does it mean to factor a natural number?
50. Describe the steps in finding the prime factors of 294.

0.3 FRACTIONS

Fractions are very common in algebra. This section will review operations with fractions in arithmetic.

Throughout this section, we will use the letters p, q, r, and s to represent natural numbers. We do this in order to state the basic rules and principles of operations with fractions. If we divide 5 by 9, we can write the quotient as the fraction $\frac{5}{9}$. In general we say if we divide a *number* p by a *number* q, we can write the quotient as the fraction, $\frac{p}{q}$. We call p the **numerator** and q the **denominator.**

When working problems with fractions, we sometimes see fractions in which the numerator is larger than or equal to the denominator. Such fractions are called **improper fractions.** In *arithmetic* it is common to change improper fractions into mixed numbers. For example,

$$\frac{22}{9} = 2\frac{4}{9}$$

Notice that the mixed number $2\frac{4}{9}$ means $2 + \frac{4}{9}$.

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In algebra we prefer to do arithmetic with a fraction rather than a mixed number (because we indicate *multiplication* when we write numbers next to each other, not addition.) So in algebra, when improper fractions arise, we leave them. When we find mixed numbers, we write them as fractions.

If a problem is stated using mixed numbers, we convert the mixed numbers to fractions and solve the problem. We answer the question using mixed numbers, since we usually answer questions in the form in which they are stated.

The fractions $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$ are different ways of writing the same number. This is the idea behind the Fundamental Principle of Fractions.

Fundamental Principle of Fractions

$$\frac{p}{q} = \frac{p \cdot r}{q \cdot r}$$

If we say $\frac{3}{6} = \frac{1}{2}$, then we are using the Fundamental Principle to reduce the fraction. It tells us that we can remove **factors** that are common to both the numerator and the denominator. If there are no factors common to both the numerator and the denominator, then we say the fraction is **reduced to lowest terms.**

EXAMPLE 1. Reduce $\frac{48}{60}$ to lowest terms.

Solution:

First we factor the numerator and denominator.

$$\frac{48}{60} = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 3 \cdot 5}$$

The common factors in numerator and denominator are 3 and $2 \cdot 2$.

$$\frac{48}{60} = \frac{2 \cdot 2}{5}$$
$$= \frac{4}{5}$$

If we see that 12 is a factor of both 48 and 60, then we can do Example 1 without factoring completely.

$$\frac{48}{60} = \frac{12 \cdot 4}{12 \cdot 5} = \frac{4}{5}$$

If we say $\frac{1}{2} = \frac{3}{6}$, then we are using the Fundamental Principle to build up a fraction. Often we must do this when we add or subtract fractions. The Fundamental Principle tells us that we can multiply both numerator and denominator by the same number and not change the value of a fraction.

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EXAMPLE 2. Write $\frac{2}{15}$ with a denominator of 45.

Solution:

Since $45 = 15 \cdot 3$, we must multiply both numerator and denominator by 3.

$$\frac{2}{15} = \frac{2 \cdot 3}{15 \cdot 3}$$

$$= \frac{6}{45}$$ □

To multiply two fractions, we multiply numerators and multiply denominators.

Multiplication of Fractions

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{p \cdot r}{q \cdot s}$$

EXAMPLE 3. Find the products indicated and reduce answers to lowest terms.

(a) $\left(\frac{2}{3}\right)^3$ (b) $\frac{12}{35} \cdot 1\frac{1}{14}$

Solutions:

(a) $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$

$$= \frac{2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3}$$

$$= \frac{8}{27}$$

(b) $\frac{12}{35} \cdot 1\frac{1}{14} = \frac{12}{35} \cdot \frac{15}{14}$ Express as an improper fraction.

$$= \frac{12 \cdot 15}{35 \cdot 14}$$

Instead of multiplying the numbers in the numerator and in the denominator, we factor so that we can look for common factors to reduce.

$$\frac{12}{35} \cdot 1\frac{1}{14} = \frac{2^2 \cdot 3 \cdot 3 \cdot 5}{7 \cdot 5 \cdot 7 \cdot 2}$$

$$= \frac{2 \cdot 3 \cdot 3}{7 \cdot 7}$$ Common factors are 2 and 5.

$$= \frac{18}{49}$$ □

Notice that when we build up a fraction using the Fundamental Principle, we are actually multiplying the fraction by 1. Suppose we wish to write the fraction $\frac{5}{8}$ with a denominator of 56. Since $56 = 8 \cdot 7$, we need to multiply the denominator by 7.

$$\frac{5}{8} = \frac{5}{8} \cdot 1$$

$$= \frac{5}{8} \cdot \frac{7}{7} \qquad 1 = \frac{7}{7}$$

$$= \frac{35}{56} \qquad \text{Multiplication of fractions}$$

To divide two fractions, multiply the first fraction by the reciprocal of the divisor.

Division of Fractions

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r} = \frac{p \cdot s}{q \cdot r}$$

EXAMPLE 4. Perform the division indicated $\frac{27}{16} \div \frac{9}{8}$, and reduce the answer to lowest terms.

Solution:

$$\frac{27}{16} \div \frac{9}{8} = \frac{27}{16} \cdot \frac{8}{9}$$

$$= \frac{27 \cdot 8}{16 \cdot 9}$$

$$= \frac{3}{2}$$

□

We add fractions that have the same denominator by adding the numerators and using the common denominator as the new denominator. Subtraction of fractions is done in a similar manner.

Addition and Subtraction of Fractions

$$\frac{p}{q} + \frac{r}{q} = \frac{p + r}{q}$$

$$\frac{p}{q} - \frac{r}{q} = \frac{p - r}{q}$$

EXAMPLE 5. Perform the operations indicated.

(a) $\frac{7}{13} + \frac{5}{13}$ (b) $\frac{16}{23} - \frac{11}{23}$

Solutions:

(a) $\frac{7}{13} + \frac{5}{13} = \frac{7+5}{13}$

$= \frac{12}{13}$

(b) $\frac{16}{23} - \frac{11}{23} = \frac{16-11}{23}$

$= \frac{5}{23}$ □

When the denominators are not the same, we must find an appropriate denominator and write each fraction with this denominator. The denominator should be the smallest number divisible by each denominator. We call this denominator the **least common denominator.**

Procedure for Finding the Least Common Denominator (LCD)

1. Factor each denominator completely, using exponents.
2. List all the different prime factors from all the denominators.
3. The LCD is the product of the factors in step 2 each raised to the largest power of that factor in any single denominator.

EXAMPLE 6. Find the least common denominator for each pair of denominators.

(a) 12 and 15 (b) 8 and 24 (c) 72 and 108

Solutions:

(a) First we factor each number.

$12 = 2^2 \cdot 3$

$15 = 3 \cdot 5$

The different prime factors are 2, 3, and 5.

2 occurs twice in 12 and no times in 15.
So, the LCD must contain 2 as a factor twice.

3 occurs one time in each number.
The LCD must contain one factor of 3.

5 occurs no times in 12 and 1 time in 15.
The LCD must contain one factor of 5.

The LCD is $2^2 \cdot 3 \cdot 5$, or 60.

(continued)

$8 = 2^3$

The different prime factors are 2 and 3.

2 occurs three times in both numbers.
The LCD must contain three factors of 2.

3 occurs one time in 24 and no times in 8.
The LCD must contain one factor of 3.

The LCD is $2^3 \cdot 3$, or 24.

(c) $72 = 2^3 \cdot 3^2$

$108 = 2^2 \cdot 3^3$

The different prime factors are 2 and 3.

2 occurs three times in 72 and two times in 108.
The LCD must contain 2^3.

3 occurs two times in 72 and three times in 108.
The LCD must contain 3^3.

The LCD is $2^3 \cdot 3^3$, or 216. □

EXAMPLE 7. Perform the operations indicated and reduce the answers to lowest terms.

(a) $\frac{5}{24} + 1\frac{5}{8}$ (b) $\frac{7}{12} - \frac{8}{15}$

Solutions:

(a) We must first find the LCD. From Example 6(b), it is 24.

$$\frac{5}{24} + 1\frac{5}{8} = \frac{5}{24} + \frac{13}{8}$$ Express as an improper fraction.

$$= \frac{5}{24} + \frac{13 \cdot 3}{8 \cdot 3}$$

$$= \frac{5}{24} + \frac{39}{24}$$

$$= \frac{44}{24}$$

$$= \frac{11}{6}$$ Reduce.

$$= 1\frac{5}{6}$$ Express as a mixed number.

(b) From Example 6(a), the LCD is 60.

$$\frac{7}{12} - \frac{8}{15} = \frac{7 \cdot 5}{12 \cdot 5} - \frac{8 \cdot 4}{15 \cdot 4}$$

$$= \frac{35}{60} - \frac{32}{60}$$

$$= \frac{3}{60}$$

$$= \frac{1}{20}$$

The order of operations that we learned in Section 0.1 applies to fractions.

EXAMPLE 8. Perform the operations indicated and reduce to lowest terms.

(a) $\frac{3}{4} + \frac{2}{3} \cdot \frac{1}{2}$ (b) $\left(\frac{2}{3} - \frac{1}{5}\right) \div \frac{7}{3}$

Solutions:

(a) We multiply before we add.

$$\frac{3}{4} + \frac{2}{3} \cdot \frac{1}{2} = \frac{3}{4} + \frac{1}{3}$$

$$= \frac{9}{12} + \frac{4}{12}$$

$$= \frac{13}{12}$$

(b) Subtract inside the parentheses first.

$$\left(\frac{2}{3} - \frac{1}{5}\right) \div \frac{7}{3} = \left(\frac{10}{15} - \frac{3}{15}\right) \div \frac{7}{3}$$

$$= \frac{7}{15} \div \frac{7}{3}$$

$$= \frac{7}{15} \cdot \frac{3}{7}$$

$$= \frac{7 \cdot 3}{15 \cdot 7}$$

$$= \frac{1}{5}$$

EXAMPLE 9. Jean has $5\frac{1}{2}$ bushels of peaches, which she wants to divide evenly among her three children. How many bushels will each child receive?

Solution:

We must divide $5\frac{1}{2}$ by 3.

$5\frac{1}{2} \div 3 = \frac{11}{2} \div 3$ Express as a fraction.

$= \frac{11}{2} \cdot \frac{1}{3}$ Multiply by the reciprocal.

(continued)

$= \frac{11}{6}$

$= 1\frac{5}{6}$ Express as a mixed number.

Each child will receive $1\frac{5}{6}$ bushels of peaches.

PROBLEM SET 0.3

Warm-ups

In Problems 1 and 2, use the Fundamental Principle to reduce each fraction to lowest terms. See Example 1.

1. $\frac{12}{18}$

2. $\frac{36}{72}$

In Problems 3 through 6, use the Fundamental Principle to write each fraction with the given denominator. See Example 2.

3. $\frac{1}{2}$; denominator of 10

4. $\frac{3}{4}$; denominator of 12

5. $\frac{4}{3}$; denominator of 18

6. $\frac{11}{36}$; denominator of 144

In Problems 7 through 41, perform the operations indicated. Reduce answers to lowest terms.

In Problems 7 through 12, see Example 3.

7. $\frac{3}{4} \cdot \frac{1}{3}$

8. $\left(\frac{1}{3}\right)^4$

9. $\frac{15}{33} \cdot \frac{44}{35}$

10. $\left(1\frac{5}{6}\right) \cdot \left(1\frac{7}{11}\right)$

11. $\left(1\frac{3}{4}\right)^2$

12. $\frac{2}{3} \cdot 6$

In Problems 13 through 16, see Example 4.

13. $\frac{3}{4} \div \frac{1}{3}$

14. $\frac{4}{7} \div 2$

15. $3\left(\frac{7}{9}\right) \div \left(3\frac{2}{3}\right)$

16. $\frac{25}{24} \div \frac{45}{42}$

In Problems 17 through 22, find the LCD for each pair of numbers. See Example 6.

17. 3 and 7

18. 8 and 16

19. 6 and 8

20. 20 and 24

21. 36 and 27

22. 135 and 54

In Problems 23 through 34, see Examples 5 and 7.

23. $\frac{1}{3} + \frac{4}{15}$

24. $\frac{1}{4} + \frac{1}{3}$

25. $\frac{3}{4} - \frac{1}{3}$

26. $\frac{3}{4} - \frac{1}{8}$

27. $\frac{2}{3} - \frac{3}{8}$

28. $\frac{5}{6} - \frac{5}{8}$

29. $\frac{5}{12} + \frac{7}{18}$

30. $1\frac{3}{5} + 2\frac{1}{5}$

31. $5\frac{2}{7} - 3\frac{3}{7}$

32. $2\frac{11}{12} - 1\frac{13}{18}$

33. $\frac{3}{7} + 1$

34. $2 - \frac{4}{5}$

In Problems 35 through 41, see Example 8.

35. $\frac{1}{18} + \frac{7}{12} - \frac{5}{9}$

36. $\frac{3}{8} \cdot \frac{2}{3} \div 4$

37. $\frac{3}{2} \cdot \frac{1}{3} \cdot \frac{4}{5}$

38. $\frac{3}{2} \cdot \frac{1}{3} \div \frac{4}{5}$

39. $\frac{3}{2} \div \frac{1}{3} + \frac{4}{5}$

40. $\left(\frac{3}{2} + \frac{1}{3}\right) \cdot \frac{4}{5}$

41. $5\left(\frac{3}{2}\right)^2 \div \frac{5}{12}$

In Problems 42 through 45, answer each question with a statement. See Example 9.

42. Subtract $3\frac{1}{3}$ from the product of $8\frac{1}{3}$ and $\frac{3}{5}$.

43. Add $\left(\frac{2}{3}\right)^2$ to $6\frac{1}{3}$.

44. Joan cuts a wire that is $2\frac{1}{3}$ yards (yd) long in half. How long will each half be?

45. Marge is in charge of making pancakes at her club's breakfast. She must have 15 pounds (lb) of flour. If Virginia donated $5\frac{1}{2}$ lb, Jane gave $8\frac{1}{3}$ lb, and Madelyn gave $5\frac{1}{6}$ lb of flour, does Marge have enough flour?

Practice Exercises

In Problems 46 through 84, perform the operations indicated. Reduce answers to lowest terms.

46. $\frac{3}{5} + \frac{1}{3}$

47. $\frac{3}{5} - \frac{1}{3}$

48. $\frac{3}{5} \cdot \frac{1}{3}$

49. $\frac{3}{5} \div \frac{1}{3}$

50. $\left(\frac{1}{3}\right)^2$

51. $\left(\frac{1}{2}\right)^4$

52. $\left(\frac{2}{5}\right)^3$

53. $\left(\frac{1}{10}\right)^2$

54. $\frac{1}{3} + \frac{5}{6}$

55. $\frac{3}{7} - \frac{1}{14}$

56. $\frac{2}{3} - \frac{3}{7}$

57. $\frac{5}{6} - \frac{5}{9}$

58. $\frac{7}{12} + \frac{5}{18}$

59. $\frac{24}{50} \cdot \frac{40}{27}$

60. $\frac{36}{35} \div \frac{42}{5}$

61. $2\frac{5}{6} + 1\frac{1}{6}$

62. $4\frac{1}{8} - 3\frac{3}{8}$

63. $\left(1\frac{3}{7}\right) \cdot \left(2\frac{4}{5}\right)$

64. $\left(3\frac{3}{8}\right) \div \left(2\frac{1}{4}\right)$

65. $5\frac{3}{4} - 3\frac{1}{2}$

66. $\left(1\frac{4}{5}\right)^2$

67. $\frac{3}{4} + 1$

68. $3 - \frac{3}{2}$

69. $\frac{4}{5} \cdot 10$

70. $\frac{6}{7} \div 3$

71. $\frac{2}{27} + \frac{35}{54} - \frac{5}{9}$

72. $\frac{5}{8} \cdot \frac{2}{5} \div 6$

73. $\frac{1}{4} \cdot \frac{4}{3} \cdot \frac{3}{4}$

74. $\frac{1}{4} \cdot \frac{4}{3} \div \frac{3}{4}$

75. $\frac{1}{4} \div \frac{4}{3} + \frac{3}{4}$

76. $\frac{1}{4} + \frac{4}{3} \cdot \frac{3}{4}$

77. $\frac{1}{4} \div \left(\frac{4}{3} + \frac{3}{4}\right)$

78. $\left(\frac{1}{4} + \frac{4}{3}\right) \cdot \frac{3}{4}$

79. $\left(\frac{3}{4} - \frac{1}{5}\right) \div \frac{11}{5}$

80. $5\left(\frac{2}{3}\right)^2 \div \frac{10}{3}$

81. $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2$

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82. $\left(\frac{2}{3}-\frac{1}{6}\right)^2$ 83. $\left(\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{3}-\frac{1}{4}\right)$ 84. $\left(\frac{3}{2}-1\right)^2$

In Problems 85 through 88, answer each question with a statement.

85. Spencer bought a 25-lb bag of fertilizer. If Linda gave $12\frac{1}{4}$ lb of it to her friend, how much fertilizer does Spencer have left?
86. What is $7\frac{2}{3}$ squared?
87. Last month Bonnie's phone bill showed that she talked $2\frac{1}{2}$ minutes (min) to her mother, $6\frac{1}{2}$ min to her sister, 30 seconds (s) to her boss, and 12 min to her son. For how many minutes did she talk last month?
88. Shirley wishes to cut a cord that is $2\frac{1}{4}$ yd long into three equal parts. How long will each part be?

IN YOUR OWN WORDS...

Write out in words what each expression means.

89. $\frac{1}{5}\cdot 6+7$ 90. $\frac{1}{5}\cdot 6+7$ 91. $\frac{1}{5}(6+7)$

92. $8+9\div 10$ 93. $(8+9)\div 10$ 94. $\frac{8+9}{10}$

0.4 DECIMALS AND PERCENTS

Fractions are quotients. The fraction $\frac{2}{5}$ is another way of writing $2\div 5$. If we perform long division, we obtain a decimal.

$$\begin{array}{r} 0.4 \\ 5\overline{)2.0} \\ \underline{2\,0} \end{array}$$

Every fraction can be represented as a decimal by dividing the denominator into the numerator.

EXAMPLE 1: Express each fraction as a decimal.

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$

Solutions:

(a) $\begin{array}{r} 0.5 \\ 2\overline{)1.0} \\ \underline{1\,0} \end{array}$

$\frac{1}{2}=0.5$

Notice that we write 0 before the decimal point. This is a good practice. We prefer to write a digit on each side of the decimal point.

(b) $\begin{array}{r} 0.3333 \\ 3\overline{)1.0000} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$

To indicate that the 3 continues without end, we write $\frac{1}{3} = 0.\overline{3}$, or $\frac{1}{3} = 0.333\ldots$.

(c)
$$\begin{array}{r} 0.1666 \\ 6\overline{)1.0000} \\ \underline{6} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array}$$

$\frac{1}{6} = 0.1\overline{6}$, or $\frac{1}{6} = 0.1666\ldots$ □

In Example 1, notice that each problem involved the quotient of two natural numbers. In each case, we wrote each fraction as a decimal, which either terminated or repeated. Any time that we divide a natural number by a natural number, the decimal will either repeat or terminate.

We perform operations with decimals as we do with other numbers. The next example illustrates the operations of addition, subtraction, multiplication, and division with decimals.

EXAMPLE 2. Perform the operations indicated.

(a) $127.32 + 24.1 + 7.003$

(b) $128.73 - 38.658$

(c) $(22.3)(12.25)$

(d) $32.4 \div 8.1$

Solutions:

(a) We line up the decimal points and add.

$$\begin{array}{r} 127.320 \\ 24.100 \\ +\ \ 7.003 \\ \hline 158.423 \end{array}$$

(b) Again, we line up the decimal points.

$$\begin{array}{r} 128.730 \\ -\ \ 38.658 \\ \hline 90.072 \end{array}$$

(c)
$$\begin{array}{rl} 12.25 & \text{(2 decimal places)} \\ \times\ 22.3 & \text{(1 decimal place)} \\ \hline 3675 & \\ 2450 & \\ \underline{2450} & \\ 273.175 & \text{(3 decimal places)} \end{array}$$

Notice that the answer has three decimal places because there are two decimal places in 12.25 and one in 22.3.

$$(22.3)(12.25) = 273.175$$

(continued)

(d) First we move the decimal place to the right in both the dividend and the divisor until the divisor is a natural number.

$$8.1\overline{)32.4}\quad \begin{array}{r}4\\ \\ 32\,4\end{array}$$

$$32.4 \div 8.1 = 4$$ □

EXAMPLE 3. Subtract 8.73 from the sum of 5.2 and 6.9.

Solution:

Since we are to subtract a number from the *sum* of 5.2 and 6.9, we add them first; then we subtract.

$$\begin{aligned}(5.2 + 6.9) - 8.73 &= 12.1 - 8.73\\ &= 3.37\end{aligned}$$ □

Fractions and decimals are often written as *percents*. Writing 15% is another way of writing $\frac{15}{100}$, or 0.15. Although percents are common in writing and talking, they are usually written as decimals or fractions in computation.

Percent

$$1\% = 0.01 = \frac{1}{100}$$

First, let's look at how to write a decimal as a percent. To write 0.25 as a percent, we notice that

$$\begin{aligned}0.25 &= 25(0.01)\\ &= 25\%\end{aligned}$$

Notice that we wrote 0.25 with a factor of 0.01 and that we replaced 0.01 with the % symbol. Another observation is that when we wrote 0.25 as 25%, we *moved the decimal point* ***two*** *places to the right.*

EXAMPLE 4. Write each decimal as a percent.

(a) 0.35 (b) 0.5 (c) 0.125
(d) $0.\overline{3}$ (e) 2.0

Solutions:

(a) We move the decimal point two places to the right and write the % symbol.

$$\begin{aligned}0.35 &= 0.35\%\\ &= 35\%\end{aligned}$$

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(b) $0.5 = 0.50\%$

$= 50\%$

(c) $0.125 = 0.12.5\%$

$= 12.5\%$

(d) First, we write three digits of the repeating decimal so that we can move the decimal two places to the right.

$$0.\overline{3} = 0.333\overline{3}$$

Now we move the decimal point and write the %.

$$0.333\overline{3} = 0.33.3\overline{3}\%$$

$$= 33.\overline{3}\%$$

(e) $2.0 = 2.00\%$

$= 200\%$ □

To write a percent as a decimal we reverse the procedure. That is, we replace the % symbol with the factor 0.01 and multiply.

$$41\% = 41(0.01)$$

$$= 0.41$$

Notice that we *moved the decimal point* ***two*** *places to the left.*

EXAMPLE 5. Write the following percents as decimals.

(a) 17% (b) 88%

(c) 7% (d) 41.6%

(e) $99\frac{1}{2}\%$ (f) $33\frac{1}{3}\%$

Solutions:

(a) We move the decimal point two places to the left and remove the % symbol.

$$17\% = .17$$

$$= 0.17$$

(b) $88\% = .88$

$= 0.88$

(c) $7\% = .07$

$= 0.07$

Notice it was necessary to place a leading zero in front of the 7 in order to move the decimal point *two* places.

(continued)

(d) $41.6\% = 41.6\%$

$= 0.416$

(e) If the percent includes a fraction, we convert it to a decimal.

$99\frac{1}{2}\% = 99.5\%$

$= 99.5$

$= 0.995$

(f) $33\frac{1}{3}\% = 33.\overline{3}\%$

$= 33.\overline{3}$

$= 0.33\overline{3}$

□

Percent-Decimal Conversion

Decimal to Percent:

1. Move the decimal point *two* places to the right.
2. Write the % symbol.

Percent to Decimal:

1. Replace any fraction with an equivalent decimal.
2. Move the decimal point *two* places to the left.
3. Remove the % symbol.

To write a fraction as a percent, we write the fraction as a decimal and then follow the procedure given.

EXAMPLE 6. Write each fraction as a percent.

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{1}{3}$

Solutions:

(a) $\frac{1}{4} = 0.25$

$= 25\%$

(b) $\frac{1}{8} = 0.125$

$= 12.5\%$

(c) $\frac{1}{3} = 0.33\overline{3}$

$= 33.\overline{3}\%$

□

Another useful way to write percents is as fractions. Since

$$0.01 = \frac{1}{100}$$

we can replace the % symbol with the fraction $\frac{1}{100}$ and multiply.

EXAMPLE 7. Write the following percents as fractions.

(a) 17% (b) 88%
(c) 7% (d) 41.6%
(e) $99\frac{1}{2}\%$ (f) $33\frac{1}{3}\%$

Solutions:

(a) $17\% = 17\left(\frac{1}{100}\right) = \frac{17}{100}$

(b) $88\% = 88\left(\frac{1}{100}\right) = \frac{88}{100}$

However, the fraction $\frac{88}{100}$ can be reduced to $\frac{22}{25}$.

$$88\% = \frac{22}{25}$$

(c) $7\% = 7\left(\frac{1}{100}\right) = \frac{7}{100}$

(d) $41.6\% = 41.6\left(\frac{1}{100}\right) = \frac{41.6}{100}$

We should not leave a decimal point in a fraction. So, we multiply numerator and denominator by 10.

$$\frac{41.6}{100} = \frac{416}{1000}$$

Since 8 is a common factor, this fraction can be reduced.

$$\frac{416}{1000} = \frac{52}{125}$$

(e) We can work with $99\frac{1}{2}\%$ using fractions or we can work with 99.5% using decimals.

Using fractions,

$$99\frac{1}{2}\% = \left(99\frac{1}{2}\right)\left(\frac{1}{100}\right) = \frac{199}{2} \cdot \frac{1}{100} = \frac{199}{200}$$

Using decimals,

$$99\frac{1}{2}\% = (99.5)\left(\frac{1}{100}\right) = \frac{99.5}{100} = \frac{995}{1000} = \frac{199}{200}$$

(continued)

(f) Again we may use decimals or fractions. Using fractions,

$$33\tfrac{1}{3}\% = \left(33\tfrac{1}{3}\right)\left(\frac{1}{100}\right) = \frac{100}{3}\cdot\frac{1}{100} = \frac{1}{3}$$ ▭

To say a solution is 32% alcohol is to say $\frac{32}{100}$ of the solution is alcohol. The amount of alcohol in 500 liters (L) of a 32% alcohol solution is given by

$$\frac{32}{100}\cdot 500 = 32\cdot 5 = 160$$

There are 160 L of alcohol in the solution.

To find a specified percent of a number, we change the percent to a fraction or a decimal and then multiply.

EXAMPLE 8. Find each of the following.

(a) 23% of 3300 (b) $16\frac{1}{2}\%$ of 98

(c) $66\frac{2}{3}\%$ of 291 (d) $9\frac{1}{4}\%$ of \$24,000

Solutions:

(a) Since $23\% = \frac{23}{100}$, we have

$$\frac{23}{100}\cdot 3300 = 23\cdot 33 = 759$$

23% of 3300 is 759.

(b) Because $16\frac{1}{2}\% = 0.165$, we have

$$(0.165)(98) = 16.17$$

$16\frac{1}{2}\%$ of 98 is 16.17.

(c) Since $66\frac{2}{3}\% = \frac{200}{3}\cdot\frac{1}{100} = \frac{2}{3}$, we have

$$\frac{2}{3}\cdot 291 = \frac{582}{3} = 194$$

$66\frac{2}{3}\%$ of 291 is 194.

(d) Since $9\frac{1}{4}\% = 9.25\% = 0.0925$, we have

$$(0.0925)(24{,}000) = 2220$$

$9\frac{1}{4}\%$ of \$24,000 is \$2220. ▭

PROBLEM SET 0.4

Warm-ups

In Problems 1 through 5, write each fraction as a decimal. See Example 1.

1. $\frac{3}{5}$ 2. $\frac{3}{8}$ 3. $\frac{7}{6}$ 4. $3\frac{4}{5}$ 5. $\frac{8}{3}$

In Problems 6 through 9, perform the operations indicated. See Example 2.

6. $12.9 + 23.67$
7. $234.9 - 123.45$
8. $(23.45)(2.03)$
9. $100.4 \div 50.2$

In Problems 10 through 12, see Example 3.

10. Subtract 13.8752 from the sum of 24.6 and 8.2.
11. Subtract the sum of 22.09 and 5.345 from 50.
12. One-half of 48.6 added to one third of 66.9.

In Problems 13 through 21, write the given numbers as percents. See Examples 4 and 6.

13. $\frac{91}{100}$
14. 0.77
15. 0.09
16. 0.8
17. $\frac{7}{50}$
18. 0.08
19. $0.\overline{6}$
20. 0.888
21. $\frac{3}{8}$

In Problems 22 through 30, write the given percents as fractions and *as decimals. See Examples 5 and 7.*

22. 87%
23. 11%
24. 50%
25. 8%
26. 98%
27. $22\frac{1}{2}\%$
28. 0.3%
29. $3\frac{1}{3}\%$
30. 125%

In Problems 31 through 39, find the quantity described. See Example 8.

31. 10% of 400
32. 28% of 10,000
33. 80% of 245
34. 5% of 720
35. 99% of 5280
36. 1% of 550
37. $8\frac{1}{2}\%$ of \$50,000
38. $33\frac{1}{3}\%$ of 963
39. 0.7% of 21,200

Practice Exercises

In Problems 40 through 44, write each fraction as a decimal.

40. $\frac{7}{8}$
41. $\frac{9}{4}$
42. $\frac{1}{7}$
43. $\frac{11}{6}$
44. $5\frac{2}{5}$

In Problems 45 through 56, write the given percents as fractions and *as decimals.*

45. 78%
46. 33%
47. 10%
48. 55%
49. 2%
50. 25%
51. 11%
52. 89%
53. $11\frac{1}{2}\%$
54. 0.9%
55. $6\frac{2}{3}\%$
56. 250%

In Problems 57 through 65, write the given numbers as percents.

57. $\frac{97}{100}$
58. 0.99
59. 0.03

60. $\frac{1}{25}$ **61.** $\frac{11}{50}$ **62.** 0.008

63. $0.\overline{2}$ **64.** 0.555 **65.** $\frac{5}{8}$

In Problems 66 through 74, find the quantity described.

66. 15% of 400 **67.** 34% of 10,000 **68.** 80% of 555

69. 3% of 700 **70.** 77% of 5280 **71.** 1% of 470

72. $\frac{1}{2}$% of \$3,000 **73.** $66\frac{2}{3}$% of 963 **74.** 0.4% of 21,200

Perform the operations indicated.

75. $28.91 - 20.07$

76. $34.9 + 123.45$

77. $(23.15)(2.03)$

78. $84.3572 \div 68.03$

79. $(12.5 - 6.2) + 3.8$

80. $4.8(2.1) - 3.2$

81. $12.4 \div 0.062 - 6.3$

82. $(34.25 - 24.5) - (23.01 - 16.7)$

83. $0.8/0.04 + 0.2/0.4$

84. $6.3 - (5.7 - 1.2)$

85. $2.1(2.5 + 6.7)$

86. $(8.8 - 2.5) \div 0.7$

87. $12 - (2 - 1.5) - 8.5$

88. $4(1.2) \div 1.6 - 2.1$

89. $(2.1)^2$

90. $(0.2)^3$

91. $(0.1)^2$

92. $4(1.2)^2 - (0.1)^3$

93. $(2.01)(10.2)^2$

94. $(1.01)^2(2.01)^2$

95. Subtract 3.2345 from the sum of 4.6 and 8.022.

96. Subtract the sum of 12.09 and 5.009 from 25.

97. Find one-half of 48.06 added to one third of 66.009.

Answer each question.

98. Greg has 50 nickels and six \$5 bills. How much money does he have?

99. Jamie has 8 dimes, 3 quarters, and five \$10 bills. How much money does he have?

100. Jennifer has 24 pennies, 33 nickels, 15 dimes, and 7 quarters. How much money does she have?

101. Kristen wanted to buy \$5.35 worth of chicken, \$.89 worth of onions, \$1.89 worth of potatoes, and \$.99 worth of broccoli at the grocery store. She had \$7.50 in her purse. Did she have enough money?

102. Sondra paid phone bills of \$34.24, \$45.89, \$29.00, \$57.90. Her budget allows for \$175. Did she stay within her budget?

103. Suzanne bought a diamond ring for \$5000. She discovered that its value has increased by 18%. What is the ring worth now?

104. A variety store advertised a 10% discount on drills. If the original price of a drill was \$59.95, what is the sale price?

105. Joseph receives an allowance of \$50 per week. If his father increases his allowance by 5%, what will his new allowance be?

106. Ray has 22 nickels and twelve \$5 bills. How much money does he have?

107. Frank has 2 dimes, 4 quarters, and seven \$20 dollar bills. How much money does he have?

108. Joan has 4 pennies, 3 nickels, 12 dimes and 5 quarters. How much money does she have?

109. Deli ham sells for \$2.29 per $\frac{1}{4}$ lb. What is the price of deli ham per pound?

110. Jo Ann bought \$4.19 worth of chicken, \$.79 worth of onions, \$1.49 worth of potatoes, and \$1.19 worth of broccoli at the grocery store. She had \$7.50 in her purse. Did she have enough money?

111. Carolyn is buying hamburger every week. For four weeks she has bought 2.12 lb, 4.65 lb, 3.15 lb, and 5.05 lb. Has she bought more than 15 lb?

112. Sara bought 2.01 lb of grapes, 6.9 lb of apples, 3 lb of pears, 4.33 lb of bananas, and 3.05 lb of nectarines. She cannot carry more than 20 lb because of a bad back. Will she be able to carry all her purchases?

113. A dress was marked down 20%. If the original price was \$59, what was the sale price?

114. Jack decided to sell his house for 7% more than he paid for it. If he paid $62,000 for it, what will his selling price be?

115. John studied his math for 2 hours (hr) every day. He thinks he can make an A if he increases his daily study time by 25%. How much longer each day will he study?

IN YOUR OWN WORDS...

116. Describe how to convert 24% into a fraction.

117. How do we add decimals?

118. How do we multiply decimals?

119. How do we divide decimals?

0.5 SQUARE ROOTS

If we square 3, we get 9. That is,

$$3^2 = 9$$

The reverse of squaring is taking a square root. We write $\sqrt{9}$ to indicate the square root of 9. The square root of 9 is 3 because $3^2 = 9$.

EXAMPLE 1. Find the square roots indicated.

(a) $\sqrt{4}$ (b) $\sqrt{16}$

(c) $\sqrt{25}$ (d) $\sqrt{18^2}$

Solutions:

(a) $\sqrt{4} = 2$ because $2^2 = 4$

(b) $\sqrt{16} = 4$ because $4^2 = 16$

(c) $\sqrt{25} = 5$ because $5^2 = 25$

(d) $\sqrt{18^2} = 18$ because $18^2 = 18^2$ ▭

A number that is the square of a natural number is called a **perfect square.** For example, 9 and 16 are perfect squares because $3^2 = 9$ and $4^2 = 16$.

In Example 1, we found square roots of perfect squares. If we want to find square roots of natural numbers that are not perfect squares, we use a calculator or a square root table to *approximate* them. For example, $\sqrt{2}$ is approximately 1.414. We write $\sqrt{2} \approx 1.414$ to indicate that 1.414 is an approximation.

The $\sqrt{\ }$ symbol is another grouping symbol. That is, we perform operations *inside* the $\sqrt{\ }$ *before* we evaluate it. The next example illustrates this.

EXAMPLE 2. Evaluate each expression.

(a) $\sqrt{144 + 25}$ (b) $\sqrt{144} + \sqrt{25}$

Solutions:

(a) Since $\sqrt{\ }$ is a grouping symbol, we *must* perform the operations inside first.

$$\sqrt{144 + 25} = \sqrt{169}$$

$$= 13 \qquad \text{Because } 13^2 = 169$$

(b) $\sqrt{144} + \sqrt{25} = 12 + 5 \qquad 12^2 = 144 \text{ and } 5^2 = 25$

$$= 17$$

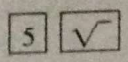
▭

Day: Date:

CALCULATOR BOX

Square Roots with a Calculator

To find $\sqrt{5}$ using a calculator, press the following keys.



Read 2.236067977 on the display.

$\sqrt{5} \approx 2.236$, rounding to the nearest thousandth. It is important to estimate the answer so that we can determine that our answer is reasonable. $\sqrt{5}$ should be a little larger than 2, since $\sqrt{4} = 2$. So, our answer is reasonable.

Calculator Exercises

Approximate the square roots indicated to three decimal places. Be sure to use the $\approx$ symbol to indicate an approximation. Estimate the answer first.

1. $\sqrt{3}$ **2.** $\sqrt{10}$ **3.** $\sqrt{47}$ **4.** $\sqrt{82}$

Answers:

1. Estimate is between 1 and 2; $\sqrt{3} \approx 1.732$

2. Estimate is about 3; $\sqrt{10} \approx 3.162$

3. Estimate is a little less than 7; $\sqrt{47} \approx 6.856$

4. Estimate is a little more than 9; $\sqrt{82} \approx 9.055$

PROBLEM SET 0.5

Warm-ups

In Problems 1 through 8, find the square roots indicated. See Example 1.

1. $\sqrt{25}$ **2.** $\sqrt{36}$ **3.** $\sqrt{100}$ **4.** $\sqrt{400}$

5. $\sqrt{49}$ **6.** $\sqrt{121}$ **7.** $\sqrt{13^2}$ **8.** $\sqrt{15^2}$

Evaluate each expression. See Example 2.

9. $\sqrt{9 + 16}$ **10.** $\sqrt{9} + \sqrt{16}$ **11.** $\sqrt{8^2} + \sqrt{6^2}$

12. $\sqrt{8^2 + 6^2}$ **13.** $\sqrt{289 - 64}$ **14.** $\sqrt{289} - \sqrt{64}$

15. $\sqrt{26^2} - \sqrt{10^2}$ **16.** $\sqrt{26^2 - 10^2}$

Practice Exercises

Evaluate each of the following expressions. If necessary, approximate the square roots indicated to three decimal places. Be sure to use the $\approx$ symbol to indicate an approximation.

17. $\sqrt{64}$ **18.** $\sqrt{81}$ **19.** $\sqrt{900}$

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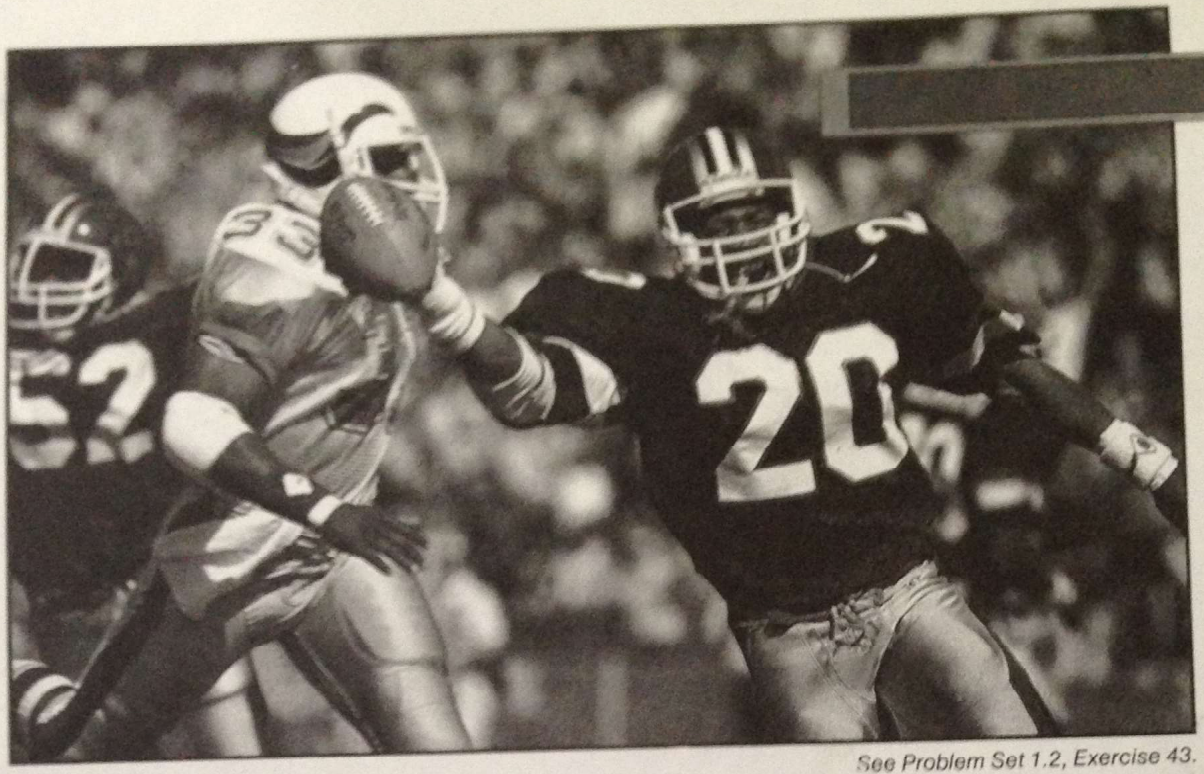
Chapter 1

Real Numbers



See Problem Set 1.2, Exercise 43.

Real Numbers

CONNECTIONS

For over 6000 years, people have been using numbers and the properties of numbers. The need to record information and transmit knowledge from one place to another forced the development of a written system of numeration. Today, most of the world uses the Hindu-Arabic numeral system, named for the Hindus, who probably invented it, and for the Arabs, who introduced it to Western Europe.

At first, small counting numbers like 1, 2, and 3 were sufficient for barter. As society developed, the growth of commerce dictated a need for larger numbers, negative numbers, and fractions. Numbers like $\frac{3}{8}$ were used by merchants to indicate portions of goods, and numbers like -60 were used to indicate debts.

In today's world, stock market gains and losses are reported using signed numbers like $+5\frac{1}{8}$, and -4, and weather forecasts routinely use negative integers to report temperatures. Scientists and engineers use numbers like $\sqrt{2}$ and π in their daily work.

The system of real numbers discussed in this chapter is a natural extension of the arithmetic of natural numbers discussed in Chapter 0 and provides the basic number system of algebra. We begin by learning the arithmetic of real numbers and their properties. The idea of using a letter for a number is introduced and we form expressions and equations. These ideas are essential concepts of algebra which are examined in more detail in later chapters.

1.1 SETS OF NUMBERS

A set of is a collection of objects. The objects are called **elements,** or **members.** We usually use capital letters to name sets and list the elements in braces. For example, $A = \{a, b, c, d, e\}$ is a set containing the first five letters of the alphabet. $B = \{1, 2, 3, \ldots, 10\}$ is a set containing the counting numbers 1 through 10. (The three dots, called an ellipsis, mean "continue on in the pattern that has been established.") $N = \{1, 2, 3, \ldots\}$ is a set containing all the counting numbers.

If the number of elements in a set is a counting number, the set is called a **finite** set. If there are no elements at all in a set, it is the **empty set.** Otherwise it is called an **infinite** set. Of the sets listed above, A and B are finite, and N is infinite.

Be Careful!

The empty set is sometimes called the **null set.** It is written $\{\ \}$ or $\varnothing$. It would be incorrect to write the empty set as $\{\varnothing\}$, as the set $\{\varnothing\}$ contains one element.

The symbol $\in$ is used to mean **is an element of,** and the symbol $\notin$ is used to mean **is not an element of.** If $D = \{a, b, c, d, e\}$, then we see that $a \in D$, but $2 \notin D$.

Sometimes we don't list the elements of the set. Instead, we use a notation called **set-builder notation.** $\{x \mid x \text{ is a vowel}\}$ describes the "set of all x such that x is a vowel." The bar is read, "such that." It is the same set as {a, e, i, o, u}.

Set A is a **subset** of a set B if every element in A is also an element in B. We use the symbol $\subseteq$ to mean **is a subset of.** If $A = \{1, 5, 7\}$ and $B = \{1, 3, 5, 7, 9\}$, then $A \subseteq B$.

Day: Date:

EXAMPLE 1. List all the subsets of $\{a, b, c\}$.

Solution:

The subsets are:

$\{a, b, c\}$ $\{a, b\}$ $\{a\}$ $\varnothing$

$\{a, c\}$ $\{b\}$

$\{b, c\}$ $\{c\}$ □

Notice that the set is a subset of itself and that the empty set is a subset of every set.

Subsets of Every Set

If A is any set, $A \subseteq A$ and $\varnothing \subseteq A$.

Many times we combine sets using the operations of **union** ($\cup$) and **intersection** ($\cap$).

Union of Two Sets

If A and B are sets,

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

That is, the union of two sets is the set of all elements that belong to either of the two sets (or both of them).

Intersection of Two Sets

If A and B are sets,

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

The intersection of two sets is the set of all elements that belong to both of the two sets.

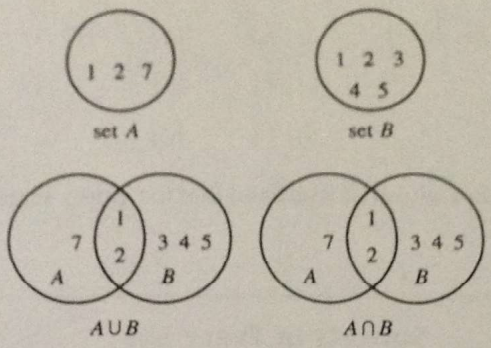
EXAMPLE 2. If $A = \{1, 2, 7\}$ and $B = \{1, 2, 3, 4, 5\}$, find $A \cup B$ and $A \cap B$.

Solutions:

$A \cup B = \{1, 2, 3, 4, 5, 7\}$

$A \cap B = \{1, 2\}$ □

John Venn, an English mathematician, popularized the use of diagrams to help visualize sets. Called **Venn diagrams,** these diagrams use circles to indicate the contents of sets. For example, if circle A indicates the contents of set A and circle B shows the contents of set B, we can use shading to indicate the contents of various sets.



(The numbers in the sets are from Example 2.)

In algebra we are concerned with sets of numbers. We worked with counting numbers, or natural numbers, in Chapter 0, and in the beginning of this section we named the set of counting numbers N.

Natural Numbers, or Counting Numbers

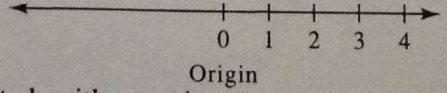
$N = \{1, 2, 3, \ldots\}$

If we put the number 0 into this set, we have a set called the set of whole numbers. We name the set W and write it as follows.

Whole Numbers

$W = \{0, 1, 2, 3, \ldots\}$

It is much easier to visualize sets of numbers if we draw a picture of them. This is done by using a **number line.** We draw a line and pick a starting point called the **origin** and label it with the number 0. We move to the right of 0 and represent natural numbers in increasing order. We space the numbers at equal distances with a convenient unit of measure.

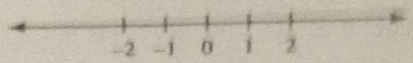


Origin

The number associated with a point on a number line is called the point's **coordinate.**

To **graph** a set of numbers means to locate them on a number line.

There are points to the left of 0 as well as to the right. If we move to the left of 0 with the same unit of measure, we name the points associated with negative numbers. The number one unit to the left of 0 is called -1 (read "negative one"). The number two units to the left of 0 is called -2, and so on.



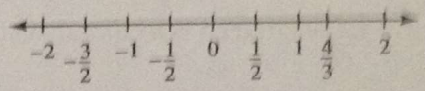
We call this set of numbers the integers and name it Z.

Integers

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The integers 0, 2, 4, 6 and 8 are called **even.** The integers 1, 3, 5, 7 and 9 are called **odd.** An **even integer** is an integer that ends in an even digit, whereas, an **odd integer** is one that ends in an odd digit.

Fractions are placed on the number line using the same unit of measure.



This set of numbers is called the **rational numbers.** We name it Q and write it in set notation as follows.

Rational Numbers

$$Q = \{x \mid x \text{ can be written as } \frac{p}{q}, \text{ where } p \text{ and } q \text{ are integers and } q \neq 0\}$$

Since a rational number is the quotient of two integers, we often refer to rational numbers as fractions. We saw in Chapter 0 that every quotient of natural numbers can be written as a decimal that repeats or terminates. So, every rational number can be written as a fraction or as a terminating or repeating decimal.

There are other points on the number line that we have not named. When written in decimal form, they do not repeat or terminate. They are called the **irrational numbers.** Some examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, and π. In fact, $\sqrt{x}$, where x is a positive integer but not a perfect square, is an irrational number.

Irrational Numbers

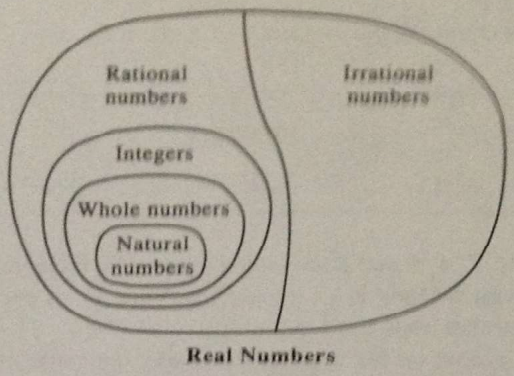
$$I = \{x \mid x \text{ is on the number line but is not a rational number}\}$$

We have now named all the numbers on the number line. We call all these numbers the **real numbers.** We name the set of real numbers R, and with set notation write them as follows.

Real Numbers

$$R = \{x \mid x \text{ is a rational or an irrational number}\}$$

The following diagram shows how these sets are related to each other.



Notice that since $5 = \frac{5}{1}$, the integer 5 is a rational number. Likewise all integers are rational numbers.

EXAMPLE 3. Classify each number as rational or irrational.

(a) $\frac{3}{7}$ (b) $0.\overline{78}$ (c) $\sqrt{7}$ (d) 1.7

Solutions:

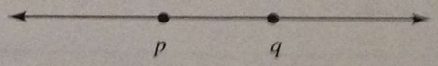
(a) $\frac{3}{7}$ is rational.

(b) $0.\overline{78}$ is a repeating decimal; thus it is rational.

(c) $\sqrt{7}$ is irrational, since 7 is not a perfect square.

(d) 1.7 is rational because $1.7 = \frac{17}{10}$. □

The number line helps us to understand many concepts of numbers. If a number p lies to the left of another number q on a number line, we say that p is **less than** q.



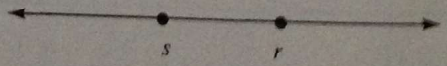
Less Than

If p lies to the left of q on a number line,

$$p < q$$

We read this "p is less than q."

Likewise if a number r lies to the right of a number s on a number line, we say that r is **greater than** s, which we write as $r > s$.



Greater Than

If r lies to the right of s on a number line,

$$r > s$$

We read this "r is greater than s."

Sometimes we combine equals with less than or greater than, using the symbols $\leq$ for **less than or equal to** and $\geq$ for **greater than or equal to.**

Less Than or Equal To–Greater Than or Equal To

$p \leq q$ means that p is less than or equal to q.
$r \geq s$ means that r is greater than or equal to s.

EXAMPLE 4. Indicate whether each statement is true or false.

(a) $-\frac{1}{2} < 0$ (b) $\frac{3}{5} < \frac{2}{3}$

(c) $-1.4 > -1.2$ (d) $5 \leq 5$

(e) $-\frac{1}{2} \leq \frac{1}{2}$

Solutions:

(a) $-\frac{1}{2} < 0$

Because $-\frac{1}{2}$ lies to the left of 0, the statement is true.

(b) $\frac{3}{5} < \frac{2}{3}$

To see where these numbers fall on a number line, we write them with the same denominator.

$$\frac{3}{5} = \frac{9}{15} \quad \text{and} \quad \frac{2}{3} = \frac{10}{15}$$

So, $\frac{9}{15}$ lies to the left of $\frac{10}{15}$. Thus $\frac{3}{5} < \frac{2}{3}$ is a true statement.

(c) $-1.4 > -1.2$

Since -1.4 lies to the left of -1.2 on a number line, the statement is false.

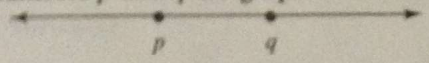
(d) $5 \leq 5$

Since $5 = 5$, the statement is true.

(e) $-\frac{1}{2} \leq \frac{1}{2}$

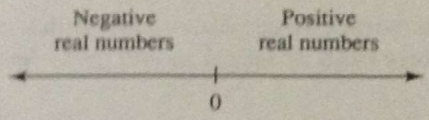
Because $-\frac{1}{2}$ lies to the left of $\frac{1}{2}$ on a number line, the statement is true. □

Suppose that the numbers p and q are graphed on a number line:



We can see that $p < q$ and also that $q > p$. When we write $p < q$ and read it as p is less than q, we are reading from left to right. If we read $p < q$ from right to left, it says q is greater than p. We usually read from left to right.

The numbers to the right of 0 are called the **positive** numbers and the numbers to the left of 0 are called the **negative** numbers. Zero is neither positive nor negative.



In set notation we write these sets as follows.

Positive Real Numbers

$\{x \mid x > 0\}$

Negative Real Numbers

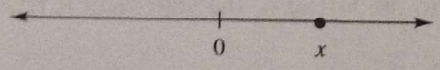
$\{x \mid x < 0\}$

EXAMPLE 5. Locate each number on a number line.

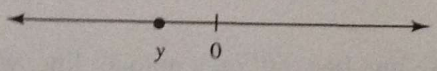
(a) x, where x is a positive real number.
(b) y, where y is a negative real number.
(c) x and y, where $x > 0$ and $y < 0$.

Solutions:

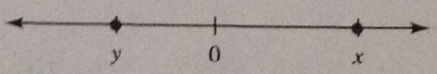
(a)



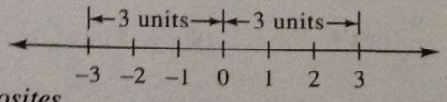
(b)



(c)



The points on the number line whose coordinates are 3 and -3 are on opposite sides of 0 but are the same distance from 0.



We call 3 and -3 *opposites.*

Additive Inverse, or Opposite

If p is a real number, then $-p$ is the real number that is on the opposite side of 0 and the same distance from 0 as p. We call $-p$ the **additive inverse,** or **opposite,** of p.

EXAMPLE 6. Find the additive inverse of each number.

(a) $\frac{2}{3}$ (b) -0.2 (c) 0 (d) x

Solutions:

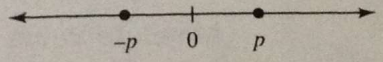
(a) The additive inverse of $\frac{2}{3}$ is $-\frac{2}{3}$.

(b) The additive inverse of -0.2 is 0.2.

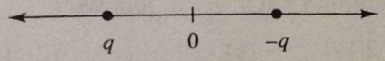
(c) The additive inverse of 0 is 0.

(d) The additive inverse of x is $-x$. □

If p is a positive number, then p is to the right of 0. The opposite of p, which we call $-p$, is to the left of 0.



If q is a negative number, then q lies to the left of 0. The opposite of q, which we call $-q$, is to the right of 0. Notice that $-q$ is a positive number because q is negative.



The opposite of -3 is 3. We can write this by saying

$$-(-3) = 3$$

What is the opposite of the opposite of a number p? That question sounds confusing. However, if we write it in symbols, it is less confusing. The opposite of p is $-p$. So, the opposite of the opposite of p is $-(-p)$. What number lies on the opposite side of 0 from $-p$? It is p.

Double Negative Property

If p is a real number, $-(-p) = p$.

EXAMPLE 7. Simplify each of the following.

(a) $-(-5)$ (b) $-\left(-\frac{3}{4}\right)$ (c) $-(-z)$

Solutions:

(a) $-(-5) = 5$

(b) $-\left(-\frac{3}{4}\right) = \frac{3}{4}$

(c) $-(-z) = z$ □

We have seen three uses of the symbol $-$. We have used it to indicate the *operation of subtraction*, as part of the *name of a number*, and to indicate the *opposite*,

or *additive inverse,* of a number. Sometimes we read it as "minus," or sometimes as "subtract," sometimes as "negative," and sometimes as "the opposite of."

$\boxed{1}\ 6-2 \qquad \boxed{2}\ -5 \qquad \boxed{3}\ -p$

In $\boxed{1}$ the $-$ symbol indicates subtraction and we read it "6 minus 2," or "6 subtract 2." In $\boxed{2}$ it indicates the name of a number, negative 5. In $\boxed{3}$ it indicates the opposite of p and is read "the opposite of p," or "negative p."

A number and its opposite have the common property that each is the same distance from 0. Both 3 and -3 are 3 units from 0. We name this distance from 0 the **absolute value** of a number and indicate it by vertical bars, $|\ |$. $|3|$ is the distance between 0 and 3. So, $|3| = 3$. Likewise, $|-3|$ is the distance between 0 and -3. So, $|-3| = 3$. Thus, we see that $|3| = |-3|$.

Absolute Value

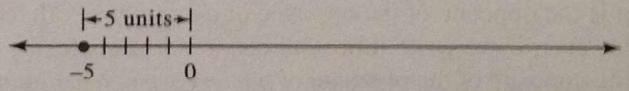
If p is a real number, the absolute value of p, $|p|$, is its distance from 0.

EXAMPLE 8. Simplify each of the following.

(a) $|-5|$ (b) $2|17|$ (c) $-|-7|$ (d) $|-\pi|$

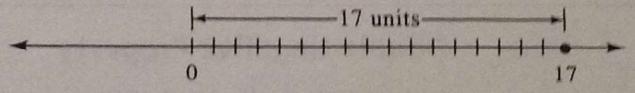
Solutions:

(a) On the number line we see that -5 is 5 units from 0.



Thus $|-5| = 5$.

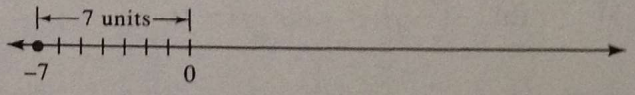
(b) Again we look at the number line.



So $|17| = 17$; then

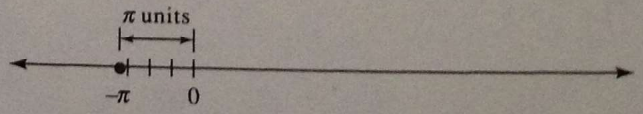
$$2|17| = 2 \cdot 17$$
$$= 34$$

(c)



Thus $|-7| = 7$, and so $-|-7| = -7$.

(d)



And so $|-\pi| = \pi$. □

PROBLEM SET 1.1

Warm-ups

In Problems 1 through 6, $A = \{2, 4, 6, 8\}$, $B = \{1, 4\}$ and $C = \{-2, 0, 2\}$.
In Problems 1 and 2, see Example 1.

1. List all the subsets of B.
2. List all the subsets of C.

In Problems 3 through 6, see Example 2.

3. List the elements in $A \cup B$.
4. List the elements in $A \cup C$.
5. List the elements in $A \cap B$.
6. List the elements in $A \cap C$.

In Problems 7 through 10, classify each number as rational or irrational. See Example 3.

7. 5.41
8. $\frac{3}{8}$
9. $\sqrt{11}$
10. $2.\overline{12}$

In Problems 11 through 16, indicate whether each statement is true or false. See Example 4.

11. $-\frac{1}{3} < -\frac{1}{4}$
12. $1.1 \geq 1.1$
13. $-5 \leq -10$
14. $5.7 > 0$
15. $-1.12 \leq -1.25$
16. $-0.111 > 2.333$

In Problems 17 through 20, graph each number on a number line. Assume that x is a positive *real number and that y is a* negative *real number. See Example 5.*

17. x
18. y
19. $-x$
20. $-y$

In Problems 21 through 24, find the additive inverse (opposite) of each number. See Example 6.

21. -3.5
22. $\sqrt{5}$
23. $\frac{11}{13}$
24. t

Simplify each of the following. See Examples 7 and 8.

25. $-(-6)$
26. $-(-35)$
27. $-(-\pi)$
28. $-(-q)$
29. $|-12|$
30. $|29|$
31. $2|-8|$
32. $3|6|$
33. $-|4|$
34. $-|-5|$
35. $-|-16|$
36. $-|-3|$

Practice Exercises

In Problems 37 through 39, $A = \left\{\frac{1}{2}, 2, \frac{9}{4}, 5\right\}$ and $B = \left\{\frac{9}{4}, 5\right\}$.

37. List the elements in $A \cup B$.
38. List the elements in $A \cap B$.
39. List all the subsets of B.

In Problems 40 through 49, indicate whether each statement is true or false.

40. The set of natural numbers is a subset of the set of whole numbers.
41. Every negative integer is a rational number.
42. $\frac{1}{3}$ is a real number.
43. 0 is a rational number.
44. The positive integers are a subset of the whole numbers.
45. A whole number is a nonnegative integer.
46. $\sqrt{16}$ is an irrational number.
47. The empty set is written as $\{\varnothing\}$.
48. Every rational number is an integer.
49. $\sqrt{11}$ is real.
50. Fill in the chart to indicate the set(s) to which each number belongs. N = natural numbers, W = whole numbers, Z = integers, Q = rational numbers, I = irrational numbers, R = real numbers.

Number	N	W	Z	Q	I	R
$\frac{4}{3}$						
-1						
$\sqrt{7}$						
π						
$\sqrt{25}$						
10						
$1.\overline{32}$						
-2.3						
$\frac{20}{10}$						

In Problems 51 through 54, graph each number on a number line. Assume that c is a positive *real number and that d is a* negative *real number.*

51. c
52. d
53. $-c$
54. $-d$

In Problems 55 through 60, indicate whether each statement is true or false.

55. $-\frac{2}{3} < -\frac{5}{8}$
56. $7 \leq 7$
57. $-5 \leq -1$
58. $-5.7 > 0$
59. $-3.415 < 2.333$
60. $-3.14 \leq -3.18$

In Problems 61 through 72, simplify each.

61. $|-2|$
62. $|29|$
63. $2|-4|$
64. $|9|$
65. $-(-5)$
66. $-(-98)$
67. $-(-3.14)$
68. $-(-k)$
69. $-|8|$
70. $-|-11|$
71. $-|-37|$
72. $-|-6|$
73. Write three consecutive integers if the smallest is 48.
74. Write three consecutive *even* integers if the smallest is 88.
75. Write three consecutive odd integers if the smallest is 37.
76. Write three consecutive integers if the largest is 20.

Challenge Problems

77. Express 0.5 as a quotient of two integers.

78. Express 1.2 as a quotient of two integers.

In Problems 79 through 82, list the elements of A.

79. $A = \{x \mid x \text{ is a natural number less than } 10\}$

80. $A = \{x \mid x \text{ is an integer between } -5 \text{ and } 2\}$

81. $A = \{x \mid x \text{ is a natural number divisible by } 5\}$

82. $A = \{x \mid x \text{ is an integer divisible by } 5\}$

In Problems 83 through 86, write A using set-builder notation.

83. $A = \{2, 4, 6, \ldots\}$

84. $A = \{2, 4, 6, 8, 10, 12\}$

85. $A = \{-3, -2, -1, 0, 1, 2, 3\}$

86. $A = \{3, 4, 5, 6, 7, 8\}$

IN YOUR OWN WORDS...

87. Describe the set of integers.

88. What does it mean to say that the number x is less than 4?

89. What do we mean by the absolute value of a number?

90. What do we mean by the opposite of a number?

1.2 ADDITION

In this section we will study addition of real numbers. In arithmetic we add natural numbers. In algebra we must learn how to add real numbers. We know that $2 + 3 = 5$, but what about $-2 + 3$?

To help us understand how the rules for addition work, imagine that we are statisticians for a football game. We use a positive number to represent yards gained and a negative number to represent yards lost. If two plays gained 2 yd and 3 yd, then the total yardage gained would be 5 yd, which is the sum of 2 and 3. Now suppose that the first play lost 2 yd and the second play gained 3 yd. The total yardage gained would be $-2 + 3$, which is 1 yd. So,

$$-2 + 3 = 1$$

Likewise, a gain of 2 yd followed by a loss of 3 yd would give a total loss of 1 yd. So, $2 + -3$ must equal -1. That is,

$$2 + -3 = -1$$

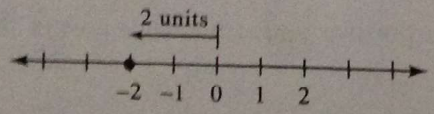
This looks awkward, so we use parentheses around the -3 to separate the two symbols $+$ and $-$.

$$2 + (-3) = -1$$

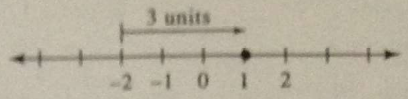
A loss of 2 yd and another loss of 3 yd would give a total loss of 5 yd. So,

$$-2 + (-3) = -5$$

These ideas can be pictured using a number line. We think of moving to the right as a gain and moving to the left as a loss. So, to illustrate that $-2 + 3 = 1$, we start at 0 and move to the left two units. This puts us at -2.



From there we move 3 units to the right.



We arrive at 1.

We have two rules for the addition of real numbers. One rule is for addition of numbers that are both positive or both negative and the other is for numbers that are not both positive or not both negative. We must use the idea of absolute value to write the rules.

Addition of Real Numbers

1. To add two numbers with *like signs* (both positive or both negative), add their absolute values and keep the common sign.
2. To add two numbers with *unlike signs* (one positive and one negative), subtract the smaller absolute value from the larger absolute value and keep the sign of the number with the greater absolute value.

Using the rules is easier than reading the rules. Practicing with many problems is the secret to success.

The next example uses the first rule. Notice how the rule is used. Each problem can be verified using a number line.

EXAMPLE 1. Find the following sums.

(a) $-5 + (-9)$ (b) $-7 + (-12)$

Solutions:

(a) We use the first rule, since both numbers are negative. $|-5| = 5$ and $|-9| = 9$. We add 5 and 9 and keep the negative sign.

$$-5 + (-9) = -14$$

(b) Again, we use the first rule. $|-7| = 7$ and $|-12| = 12$. We add 7 and 12 and keep the negative sign.

$$-7 + (-12) = -19$$ □

The next example uses the second rule.

EXAMPLE 2. Find the sums.

(a) $-5 + 9$ (b) $5 + (-9)$

Solutions:

(a) We use the second rule, since one number is positive and the other is negative.

$$|-5| = 5 \quad \text{and} \quad |9| = 9$$

So, we subtract 5 from 9 and keep the positive sign, since $|9|$ is greater than $|-5|$.

$$-5 + 9 = 4$$

(b) Again we use the second rule.

$$|5| = 5 \quad \text{and} \quad |-9| = 9$$

We subtract 5 from 9 and keep the negative sign, since $|-9|$ is greater than $|5|$.

$$5 + (-9) = -4$$ □

EXAMPLE 3. Find each sum.

(a) $4 + (-8)$ (b) $-4 + (-7)$ (c) $-7 + 5$

Solutions:

(a) $4 + (-8) = -4$

(b) $-4 + (-7) = -11$

(c) $-7 + 5 = -2$ □

When we add 0 to any real number, we get the same number.

Addition Property of Zero

If p is a real number,

$$p + 0 = 0 + p = p$$

The order of operations that we learned in Chapter 0 applies to real numbers as well as natural numbers.

Order of Operations

If grouping symbols are present, perform operations inside them, starting with the innermost grouping symbol, in the following order.

1. Perform any exponentiations.
2. Perform all multiplications and divisions in order from left to right.
3. Perform all additions and subtractions in order from left to right.

If grouping symbols are *not* present, perform operations in the above order.

EXAMPLE 4. Perform the operations indicated.

(a) $5 + (-8 + 6)$ (b) $[5 + (-13)] + [-6 + (-8)]$

(c) $\{-1.2 + [2.7 + (-6.1)]\}$

Solutions:

(a) We work inside the parentheses first.

$$5 + (-8 + 6) = 5 + (-2)$$

$$= 3$$

(continued)

(b) We work inside the brackets first.

$$[5 + (-13)] + [-6 + (-8)] = [-8] + [-14]$$
$$= -22$$

(c) We begin with the brackets.

$$[-1.2 + [2.7 + (-6.1)]] = [-1.2 + (-3.4)]$$
$$= -4.6$$ □

Fraction bars, absolute value bars, and square root symbols are also grouping symbols. That is, we perform operations *over* and *under* the fraction bar, *inside* the absolute value symbols, and *inside* the square root symbol *before* we evaluate further.

EXAMPLE 5. Evaluate the following:

(a) $|-4 + 9|$ (b) $|-4| + |9|$ (c) $\sqrt{25 + 144}$

(d) $\sqrt{25} + \sqrt{144}$ (e) $\dfrac{9 + 5}{9 - 2}$

Solutions:

(a) Since absolute value bars are a grouping symbol, we must perform the operations inside first.

$$|-4 + 9| = |5|$$
$$= 5$$

(b) $|-4| + |9| = 4 + 9$
$$= 13$$

(c) The $\sqrt{\ }$ is another grouping symbol.

$$\sqrt{25 + 144} = \sqrt{169}$$
$$= 13$$

(d) $\sqrt{25} + \sqrt{144} = 5 + 12$
$$= 17$$

(e) The fraction bar is also a grouping symbol. We evaluate both the numerator and the denominator *before* we divide.

$$\frac{9 + 5}{9 - 2} = \frac{14}{7}$$
$$= 2$$ □

EXAMPLE 6. Answer each of the following questions.

(a) What is the sum of 12 and -5, increased by 9?

(b) The low temperature this morning was $-3°$. If the temperature rose $21°$ by noon, what was the noontime temperature?

Solutions:

(a) To increase a number by 9 we add 9 to it.

$$[12 + (-5)] + 9 = 7 + 9$$
$$= 16$$

(b) Since the temperature *rose* 21° from −3°, we must add 21 to −3.

$$-3 + 21 = 18$$

The noontime temperature was 18°. □

PROBLEM SET 1.2

Warm-ups

Perform the operations indicated.
In Problems 1 through 21, see Examples 1, 2, and 3.

1. $5 + 3$
2. $5 + (-8)$
3. $-3 + 7$
4. $6 + 0$
5. $-5 + (-8)$
6. $10 + 1$
7. $0 + (-1)$
8. $-1 + (-1)$
9. $8 + (-4)$
10. $-11 + (-3)$
11. $7 + (-7)$
12. $\frac{1}{4} + \left(-\frac{1}{3}\right)$
13. $-\frac{1}{2} + \left(-\frac{1}{2}\right)$
14. $2 + \left(-\frac{1}{8}\right)$
15. $-\frac{1}{8} + \frac{1}{3}$
16. $-\frac{5}{6} + \frac{5}{6}$
17. $-\frac{2}{3} + (-1)$
18. $-\frac{3}{5} + \left(-\frac{1}{10}\right)$
19. $-1.2 + 3.4$
20. $-9.8 + (-2.7)$
21. $5.3 + 8.7$

In Problems 22 through 35, see Example 4.

22. $-7 + (-5 + 9)$
23. $(-9 + 12) + (-5 + 3)$
24. $\{-9 + [-8 + (-3)]\} - 3$
25. $[13 + (-8)] + [-2 + (-7)]$
26. $\frac{1}{4} + \left(-\frac{1}{2} + \frac{1}{8}\right)$
27. $[1.2 + (-3.6)] + (-2.5)$
28. $7 + \{-12 + [-5 + (-3)]\}$
29. $-3 + \{[-2 + 4] + [5 + (-8)]\}$
30. $-3.6 + [-2.4 + (-3.1)]$
31. $\left(-\frac{2}{3} + \frac{1}{6}\right) + \left(-\frac{3}{8}\right)$
32. $-3 + \{[-5 + (-5)] + [7 + (-13)]\}$
33. $(15 + 4) + \{[-3 + (-8)] + [6 + (-4)]\}$
34. $-\frac{5}{6} + \left[-\frac{3}{4} + \left(-\frac{2}{3}\right)\right]$
35. $-9.9 + [-3.3 + (-6.6)]$

In Problems 36 and 37, evaluate the given expressions. See Example 5.

36. (a) $|-2 + 7|$ (b) $|-2| + |7|$
37. (a) $\sqrt{225 + 64}$ (b) $\sqrt{225} + \sqrt{64}$

In Problems 38 through 44, answer each question. See Example 6.

38. Find the sum of −6 and 3.
39. What is the sum of −9 and 6 increased by 5?
40. Find the sum of 11 and −6 increased by 5.
41. Last night the low temperature was −8°. Tonight's low is predicted to be 12° higher than last night's low. What is the predicted low temperature for tonight?
42. The high temperature for today was −3°. If the temperature dropped 27° to reach the low temperature, find the low temperature for the day.

Day: Date:

43. A football team had the following yardage on its first four downs, a gain of 1 yd, a loss of 5 yd, a loss of 6 yd, and a gain of 15 yd. Did they make a first down? (A first down means that they gained more than 10 yd.)

44. Henry owes Gerald $35. If he pays him $24 today and $9 next week, how much will Henry still owe Gerald?

Practice Exercises

In Problems 45 through 100, perform the operations indicated.

45. $7 + 2$

46. $4 + (-3)$

47. $-2 + 6$

48. $-6 + (-7)$

49. $5 + 0$

50. $3 + (-5)$

51. $-12 + 7$

52. $32 + (-9)$

53. $-9 + 9$

54. $-13 + 5$

55. $21 + (-5)$

56. $-8 + 8$

57. $6 + (-2)$

58. $-3 + (-3)$

59. $8 + (-5)$

60. $5.5 + (-5.8)$

61. $-6.3 + 6.3$

62. $-5.0 + 1.0$

63. $25 + 25$

64. $-30 + 20$

65. $14 + (-17)$

66. $-13 + (-5)$

67. $8 + (-8)$

68. $\frac{1}{2} + \left(-\frac{1}{3}\right)$

69. $-\frac{1}{5} + \left(-\frac{1}{5}\right)$

70. $3 + \left(-\frac{1}{4}\right)$

71. $-\frac{2}{5} + \frac{1}{3}$

72. $-\frac{7}{8} + \frac{7}{8}$

73. $-\frac{3}{4} + (-1)$

74. $-\frac{3}{7} + \left(-\frac{1}{14}\right)$

75. $-1.5 + 2.1$

76. $-7.7 + (-2.3)$

77. $6.8 + 3.7$

78. $6.5 + (-6.8)$

79. $-4.3 + 4.3$

80. $-7.0 + 3.0$

81. $-5 + (-7 + 8)$

82. $(-5 + 11) + (-7 + 2)$

83. $-8 + [-9 + (-2)]$

84. $[17 + (-9)] + [-3 + (-6)]$

85. $\frac{1}{3} + \left(-\frac{1}{2} + \frac{5}{6}\right)$

86. $[7.2 + (-4.6)] + (-8.5)$

87. $9 + \{-10 + [-3 + (-3)]\}$

88. $-1 + \{[-3 + 7] + [1 + (-1)]\}$

89. $[-4 + (-8)] + (-6)$

90. $[2 + (-22)] + (-12)$

91. $-5.6 + [-1.4 + (-3.1)]$

92. $\left(-\frac{2}{3} + \frac{1}{2}\right) + \left(-\frac{5}{8}\right)$

93. $5 + [3 + (-7)]$

94. $[-1 + (-1)] + (-6 + 7)$

95. $-1 + \{[-4 + (-4)] + [7 + (-12)]\}$

96. $[-5 + (-3)] + [3 + (-8)]$

97. $[-8 + (-3)] + \{7 + [1 + (-9)]\}$

98. $-\frac{7}{15} + \left[-\frac{3}{5} + \left(-\frac{2}{3}\right)\right]$

99. $[1 + (-2)] + (-3 + 2)$

100. $-0.9 + [-2.4 + (-4.6)]$

In Problems 101 and 102, evaluate each of the given expressions.

101. (a) $|-6 + 17|$ (b) $|-6| + |17|$

102. (a) $\sqrt{15^2 + 8^2}$ (b) $\sqrt{15^2} + \sqrt{8^2}$

In Problems 103 through 110, answer each question.

103. Find the sum of -11 and -11.

104. Find the sum of 16 and -8 decreased by 9.

105. What is the sum of -25 and -10 increased by 50.

106. The high temperature yesterday was $-10°$. If today's high is 5° more than yesterday's high, what is today's high?

107. If the temperature is $-15°$ at 5 A.M. and rises at the rate of 3° per hour, what will the temperature be at noon?

108. Bill Barber got a raise of \$200 a month. If his federal income tax increased by \$59 and his state income tax rose by \$12, how much was his net raise?

109. Chuck makes a deposit of \$450 and withdrawals of \$235 and \$350. If he started with a balance of \$195, what is his new balance?

110. A football team gains 3 yd, loses 12 yd, gains 9 yd, and gains 5 yd. What is the team's total yardage?

IN YOUR OWN WORDS...

111. How do we add real numbers?

Write in words the meaning of each of the following expressions.

112. $7 + (-3)$

113. $-11 + 8$

114. $-4 + 0$

115. $-2 + (-2)$

1.3 SUBTRACTION

We remember from arithmetic that $5 - 3$ is 2. In the last section we noticed that $5 + (-3)$ is also 2. That is, subtracting 3 is the same as adding the opposite of 3. We use this relationship for our definition of subtraction.

Subtraction of Real Numbers

$$p - q = p + (-q)$$

1. Rewrite the problem as an addition problem by adding the opposite of the number being subtracted.
2. Follow the rules for addition.

EXAMPLE 1. Perform the subtractions indicated.

(a) $7 - 2$ (b) $-7 - 2$ (c) $-7 - (-2)$

Solutions:

(a) $7 - 2 = 7 + (-2)$ Add the opposite of 2.
$= 5$

(b) $-7 - 2 = -7 + (-2)$ Add the opposite of 2.
$= -9$

(c) $-7 - (-2) = -7 + 2$ Add the opposite of -2.
$= -5$ □

EXAMPLE 2. Perform the operations indicated.

(a) $[(9 - 4) - (14 - 18)]$ (b) $\frac{1}{2} - \left(\frac{1}{3} - 2\right)$

(c) $14 - |3 - 8|$ (d) $\sqrt{225 + 8^2} - 8^2$

Solutions:

(a) We begin with the innermost grouping symbol.

$$\begin{aligned}[(9-4)-(14-18)] &= [5-(-4)] \\ &= 5+4 \\ &= 9\end{aligned}$$

(b) $$\begin{aligned}\frac{1}{2}-\left(\frac{1}{3}-2\right) &= \frac{1}{2}-\left(\frac{1}{3}-\frac{6}{3}\right) \\ &= \frac{1}{2}-\left(-\frac{5}{3}\right) \\ &= \frac{1}{2}+\frac{5}{3} \\ &= \frac{3}{6}+\frac{10}{6} \\ &= \frac{13}{6}\end{aligned}$$

(c) Absolute value bars are a grouping symbol.

$$\begin{aligned}14-|3-8| &= 14-|-5| \\ &= 14-5 \\ &= 9\end{aligned}$$

(d) The $\sqrt{\ }$ symbol is another grouping symbol.

$$\begin{aligned}\sqrt{225+8^2}-8^2 &= \sqrt{225+64}-64 \\ &= \sqrt{289}-64 \\ &= 17-64 \\ &= -47\end{aligned}$$

□

EXAMPLE 3. Answer each question.

(a) Subtract 8 from 5.

(b) Subtract -11 from 2.

(c) What is -8 decreased by the sum of 3 and -5?

(d) If today's high temperature is 7° and the temperature falls 24° for the nighttime low, what is the nighttime low?

Solutions:

(a) To subtract 8 from 5, we add the opposite of 8 to 5.

$$\begin{aligned}5-8 &= 5+(-8) \\ &= -3\end{aligned}$$

(b) To subtract -11 from 2, we add the opposite of -11 to 2.

$$\begin{aligned}2-(-11) &= 2+11 \\ &= 13\end{aligned}$$

(c) To *decrease* -8 by a quantity, we subtract the quantity from -8. So our problem is to evaluate,

$$-8 - [3 + (-5)]$$

The order of operations tells us to work inside the brackets first.

$$\begin{aligned} -8 - [3 + (-5)] &= -8 - (-2) \\ &= -8 + 2 \\ &= -6 \end{aligned}$$

(d) If today's high temperature is 7° and the temperature falls 24°, then we must subtract 24 from 7 to obtain the nighttime low.

$$\begin{aligned} 7 - 24 &= 7 + (-24) \\ &= -17 \end{aligned}$$

The nighttime low is $-17°$. □

PROBLEM SET 1.3

Warm-ups

Perform the operations indicated.
In Problems 1 through 24, See Example 1.

1. $5 - 3$
2. $5 - (-8)$
3. $-3 - 7$
4. $-13 - 5$
5. $21 - (-5)$
6. $-8 - 8$
7. $-5 - (-8)$
8. $6 - 0$
9. $0 - 1$
10. $5 - (-1)$
11. $-1 - (-1)$
12. $8 - (-4)$
13. $-15 - 10$
14. $-20 - 10$
15. $18 - (-3)$
16. $-\frac{1}{2} - \left(-\frac{1}{2}\right)$
17. $2 - \left(-\frac{1}{8}\right)$
18. $-\frac{1}{8} - \frac{1}{3}$
19. $-\frac{5}{6} - \frac{5}{6}$
20. $-\frac{2}{3} - (-1)$
21. $-\frac{3}{5} - \left(-\frac{1}{10}\right)$
22. $-1.2 - 3.4$
23. $-9.8 - (-2.7)$
24. $5.3 - 8.7$

In Problems 25 through 38, see Example 2.

25. $-7 - (-5 - 9)$
26. $(-9 - 12) - (-5 - 3)$
27. $-9 - [-8 - (-3)]$
28. $[13 - (-8)] - [-2 - (-7)]$
29. $\frac{1}{4} - \left(-\frac{1}{2} - \frac{1}{8}\right)$
30. $[1.2 - (-3.6)] - (-2.5)$
31. $2 - [(-8 - 7) - (-3 + 1)]$
32. $[-11 + (-3)] - 5$
33. $\frac{2}{5} + \left(\frac{1}{3} - \frac{3}{5}\right)$
34. $(-2.7 + 5.1) - 4.8$
35. $-5 - \sqrt{-3 - (-7)}$
36. $[-6 - 7] - |-6 + 13|$
37. $12 - |7 - 11| - 7$
38. $23 - \sqrt{11 - (3 - 8)}$

In Problems 39 through 44, answer the question in each. See Example 3.

39. Subtract -6 from 6.
40. Subtract 8 from the sum of -11 and -4.
41. What is -12 decreased by 5?
42. What is the sum of 4 and -9, decreased by 1?

43. A grocer has his freezer set at $-5°$. If he decreases the temperature by $10°$, what will the temperature be?

44. The high temperature for today was $15°$. If the temperature dropped $27°$ to reach the low temperature, find the low temperature for the day.

Practice Exercises

Perform the operations indicated.

45. $8 - 5$
46. $-7 - 1$
47. $-8 - (-3)$
48. $-6 - 2$
49. $3 - (-1)$
50. $5 - (-5)$
51. $-5 - (-5)$
52. $4 - (-6)$
53. $-3 - 1$
54. $-9 - (-7)$
55. $-3 - (-3)$
56. $-5 - (-8)$
57. $6 - (-6)$
58. $-6 - (-6)$
59. $6 - (-1)$
60. $-10 - 10$
61. $-12 - (-7)$
62. $13 - 7$
63. $15 - (-12)$
64. $-17 - 17$
65. $-17 - (-17)$
66. $18 - 32$
67. $18 - (-32)$
68. $-18 - 32$
69. $-18 - (-32)$
70. $-32 - 18$
71. $-32 - (-18)$
72. $-\frac{2}{5} - \frac{2}{3}$
73. $\frac{7}{8} - \frac{3}{4}$
74. $-\frac{1}{6} - \left(-\frac{1}{3}\right)$
75. $-\frac{4}{5} - \frac{4}{5}$
76. $3 - \frac{3}{2}$
77. $\frac{1}{2} - 1$
78. $1.2 - 2.4$
79. $-2.5 - 3.4$
80. $-3.3 - (-6.6)$
81. $7 - (6 - 12)$
82. $(-7 - 9) - 3$
83. $[(7 - 14) - (-5 - 8)]$
84. $[11 - (-6 - 6)]$
85. $[-9 - (-6)] - [13 - (-7)]$
86. $-\frac{2}{3} - \left(-\frac{5}{6} - \frac{1}{2}\right)$
87. $(-4 + |-4 - 15|) + \sqrt{16 - (-9)}$
88. $\{[8 + (-12)] - (4 - 7)\} - 8$
89. $12 - [(-3 - 20) + (-33 + 10)]$
90. $\{-11 + [-9 - (-20)]\} - \{9 + [-3 - (-6)]\}$

In Problems 91 through 95, answer each question.

91. Find the difference when -6 is subtracted from 3.
92. Subtract -5 from -2.
93. Subtract -7 from the sum of -6 and -5.
94. Last night the low temperature was $8°$. Tonight's low is predicted to be $12°$ less than last night's low. What is the predicted low temperature for tonight?
95. The temperature was $-3°$ at 7:00 P.M. If the temperature dropped $15°$ by midnight, what was the temperature at midnight?

IN YOUR OWN WORDS...

96. How is subtraction of real numbers done?

Write the meaning of the following expressions.

97. $3 - 7$
98. $7 - (-3)$
99. $-7 - (-3)$
100. $16 - (4 + 10)$

1.4 MULTIPLICATION AND DIVISION

Multiplication and division with real numbers follow two rules. The rules are the same for both operations, since division is just a form of multiplication. Writing $\frac{6}{3}$ is

the same as writing $6 \cdot \frac{1}{3}$. That is, a division problem can be written as a multiplication problem, as long as the divisor is not 0.

Division

$$\frac{p}{q} = p \cdot \frac{1}{q}; \qquad q \neq 0$$

If we look at all the possibilities for multiplying two real numbers, we see that both numbers can be positive, both numbers can be negative, or one number can be positive and the other number negative.

We already know how to multiply two positive numbers. For example,

$$2 \cdot 3 = 6$$

That is, a positive number times a positive number is a positive number.

To understand what $2(-3)$ is, remember that multiplication of -3 by 2 is the same as adding $-3 + (-3)$, which is -6. This gives us the idea that a positive number times a negative number is a negative number.

The product of two negative numbers is a positive number. To gain some understanding why this must be so, consider the following.

$$(-2) \cdot \ 3 = -6$$
$$(-2) \cdot \ 2 = -4$$
$$(-2) \cdot \ 1 = -2$$
$$(-2) \cdot \ 0 = \ \ 0$$
$$(-2) \cdot -1 = \ \ 2$$
$$(-2) \cdot -2 = \ \ 4$$

Notice that the answer increases by 2 each time. So, the product of two negative numbers must be positive.

Multiplication and Division of Real Numbers

1. To multiply or divide two numbers with *like signs* (both positive or both negative), multiply or divide their absolute values. The sign will be positive.
2. To multiply or divide two numbers with *unlike signs* (one positive and one negative), multiply or divide their absolute values. The sign will be negative.

Remember, we **cannot** divide by 0.

EXAMPLE 1. Perform the multiplications indicated.

(a) $(-3)(5)$ (b) $(-4)(-2)$

(c) $\frac{1}{3}\left(-\frac{3}{2}\right)$ (d) $(-2)^2$

Solutions:

(a) $(-3)(5) = -15$ (b) $(-4)(-2) = 8$

(c) $\frac{1}{3}\left(-\frac{3}{2}\right) = -\frac{1}{2}$ (d) $(-2)^2 = (-2)(-2)$
$= 4$

EXAMPLE 2. Perform the divisions indicated.

(a) $-6 \div 3$ (b) $\frac{15}{-5}$ (c) $10/(-2)$

Solutions:

(a) $-6 \div 3 = -2$ (b) $\frac{15}{-5} = -3$ (c) $10/(-2) = -5$ □

The three real numbers 1, -1 and 0 have special properties. If we multiply a real number by 1, we get the real number itself. If we multiply a real number by -1, we get the opposite of the real number. If we multiply a real number by 0, we get 0.

Multiplication With 1, −1 and 0

If p is any real number,

$$1 \cdot p = p \cdot 1 = p$$

$$-1 \cdot p = p(-1) = -p$$

$$0 \cdot p = p \cdot 0 = 0$$

Using 0 in quotients can be troublesome. Remember, we know that $\frac{6}{2} = 3$ because $3 \cdot 2 = 6$. In the same manner, $\frac{0}{6} = 0$ because $0 \cdot 6 = 0$. What about $\frac{6}{0}$? Is $\frac{6}{0} = 0$? This would mean that $0 \cdot 0 = 6$, which is not true. So, it is not 0. If we try any other real number, we see that $\frac{6}{0} = ?$ would mean that $? \cdot 0$ must be 6. This is not possible. Thus, we cannot divide 0 into a real number.

Quotients With Zero

If p and q are real numbers and p is not zero,

1. $\frac{0}{p} = 0$
2. $\frac{q}{0}$ is undefined.

EXAMPLE 3. Perform the operations indicated.

(a) $-1 \cdot 7$ (b) $6 \cdot 0$ (c) $1 \cdot x$

(d) $-1(-4)$ (e) $-1(-x)$ (f) $\frac{-1}{0}$

Solutions:

(a) $-1 \cdot 7 = -7$

(b) $6 \cdot 0 = 0$

(c) $1 \cdot x = x$

(d) $-1(-4) = 4$ Notice 4 is the opposite of -4.

(e) $-1(-x) = x$ Notice x is the opposite of $-x$.

(f) $\frac{-1}{0}$ is undefined. □

Day: Date:

An important property of fractions concerns signs. Notice the signs in the following computations.

1	$\frac{-6}{2} = -3$	(A negative number divided by a positive number)
2	$\frac{6}{-2} = -3$	(A positive number divided by a negative number)
3	$-\frac{6}{2} = -3$	(The opposite of a positive number)

That is, because of the sign rules for multiplication and division, we have,

$$\frac{-6}{2} = \frac{6}{-2} = -\frac{6}{2}$$

This idea holds true in general.

A Sign Property of Fractions

For p and q integers, $q \neq 0$,

$$\frac{-p}{q} = \frac{p}{-q} = -\frac{p}{q}$$

The order of operations still applies as the next example illustrates.

EXAMPLE 4. Perform the operations indicated.

(a) $-3(-2)^2$ (b) $\frac{-4(5)}{10}$ (c) $8/4(-3)$

Solutions:

(a) We must square -2 first.

$$\begin{aligned} -3(-2)^2 &= -3(4) \\ &= -12 \end{aligned}$$

(b) We must multiply in the numerator first.

$$\begin{aligned} \frac{-4(5)}{10} &= \frac{-20}{10} \\ &= -2 \end{aligned}$$

(c) We perform multiplications and divisions in order from left to right. So, first we divide 8 by 4.

$$\begin{aligned} 8/4(-3) &= 2(-3) \\ &= -6 \end{aligned}$$

Day: _____ Date: _____

EXAMPLE 5. A 22-gallon (gal) solution is 10% molasses by volume. How many gallons of molasses does it contain?

Solution:

The solution is 10%, or $\frac{10}{100}$, molasses. Thus it contains $\frac{10}{100} \cdot 22$ gal of molasses.

$$\frac{10}{100} \cdot 22 = \frac{1}{10} \cdot 22$$

$$= \frac{22}{10} = 2.2$$

There are 2.2 gal of molasses in the solution. □

EXAMPLE 6. Eighteen-carat gold is 75% pure gold by weight. How much pure gold is contained in an 18-carat ring that weighs 120 grams (g)?

Solution:

75% of 120 is

$$\frac{75}{100} \cdot 120 = \frac{3}{4} \cdot 120$$

$$= \frac{3 \cdot 120}{4}$$

$$= 3 \cdot 30 = 90$$

The ring contains 90 g of pure gold. □

PROBLEM SET 1.4

Warm-ups

Perform the operations indicated.
In Problems 1 through 12, find each product. See Examples 1 and 3.

1. $(-6)(-1)$

2. $3(0)$

3. $\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$

4. $(-5)(7)$

5. $(1.1)(-10)$

6. $-1(-3)$

7. $(-5)^2$

8. $(-3)^2$

9. $(-5)(-4)$

10. $4(-7)$

11. $0(-4)$

12. $\left(-\frac{3}{4}\right)^2$

In Problems 13 through 28, find each quotient. See Examples 2 and 3.

13. $\frac{-5}{-1}$

14. $9/(-3)$

15. $8 \div 0$

16. $\frac{18}{-9}$

17. $\frac{-35}{7}$

18. $-28/(-4)$

19. $0/(-1)$

20. $12 \div (-4)$

21. $\frac{14}{-7}$

22. $-6/(-6)$

23. $\frac{15}{-1}$

24. $\frac{-63}{-7}$

25. $25 \div (-5)$

26. $\frac{3}{5} \div \left(-\frac{9}{10}\right)$

27. $5/0$

28. $-48 \div 8$

In Problems 29 through 39, see Example 4.

29. $\frac{2(-3)}{3}$ **30.** $2(3)^2$ **31.** $(-2)^2(3)$

32. $(-4)^2/(-8)$ **33.** $\frac{(-7)^2}{7^2}$ **34.** $\frac{(-4)(3)}{2^2}$

35. $(-3)\cdot 2^2$ **36.** $(-1)^2(-6)^2$ **37.** $16(-2)/8$

38. $12/(-2)(3)$ **39.** $(-3)^2/3(2)$

In Problems 40 and 41, answer the question. See Example 5.

40. A gold coin is 95% pure gold by weight. If the coin weighs 2 oz, how much pure gold does it contain?

41. A solution contains 20% acid by volume. How much acid is there in 485 L of the solution?

Practice Exercises

Perform the operations indicated.

42. $(-8)(-1)$ **43.** $7(0)$ **44.** $\left(\frac{2}{3}\right)\left(-\frac{2}{3}\right)$

45. $(-3)(8)$ **46.** $(-2.2)(10)$ **47.** $(-7)^2$

48. $-1(-9)$ **49.** $(-6)^2$ **50.** $1(-2)$

51. $(-8)(-4)$ **52.** $-1\cdot z$ **53.** $(-4)(6)$

54. $(-9)(-8)$ **55.** $-1(-d)$ **56.** $(-8)(8)$

57. $(-10)^2$ **58.** $(-6)(-3)$ **59.** $5(-7)$

60. $0(-8)$ **61.** $\left(-\frac{5}{6}\right)^2$ **62.** $\frac{-7}{-1}$

63. $8/(-4)$ **64.** $7 \div 0$ **65.** $\frac{28}{-7}$

66. $22 \div (-2)$ **67.** $\frac{-45}{5}$ **68.** $-36/(-4)$

69. $0/(-7)$ **70.** $\frac{42}{-7}$ **71.** $-4/(-4)$

72. $\frac{17}{-1}$ **73.** $\frac{-56}{-7}$ **74.** $36 \div (-6)$

75. $\frac{0}{9}$ **76.** $4/0$ **77.** $-72 \div 8$

78. $\frac{-2}{2}$ **79.** $\frac{3}{-3}$ **80.** $-64/(-8)$

81. $\frac{2}{0}$ **82.** $\frac{2(-5)}{5}$ **83.** $2(6)^2$

84. $(-3)^2(3)$ **85.** $(-8)^2/(-4)$ **86.** $\frac{(-9)^2}{9^2}$

87. $\frac{(-4)(5)}{2^2}$ **88.** $(-4)\cdot 2^2$ **89.** $(-1)^2(-5)^2$

90. $(-2)(-9)/(-3)$ **91.** $18/(-6)(4)$ **92.** $45/(-3)/5$

93. If Ivory soap is $99\frac{44}{100}\%$ pure, how much impurity is there in a ton of it? (A ton is 2000 lb.)

Day: Date:

Challenge Problems

94. Are $12/3^2$ and $12/3 \cdot 3$ the same?

95. Write $\frac{14}{5 \cdot 11}$ using the $\div$ symbol.

IN YOUR OWN WORDS...

96. State the rules that govern the *signs* in multiplication and division.

Write in words the meaning of the following expressions.

97. $3(-5)$ **98.** $\frac{-7}{3}$ **99.** $6 \cdot \frac{10}{11} + 5$ **100.** $6\left(\frac{10}{11} + 5\right)$

1.5 EXPRESSIONS

In this section we focus on combining the operations of addition, subtraction, multiplication, and division. Such combinations are called **numerical expressions,** or just **expressions.** We simplify such expressions with the order of operations that we have already learned.

Order of Operations

If grouping symbols are present, perform operations inside them, starting with the innermost grouping symbol, in the following order.

1. Perform any exponentiations.
2. Perform all multiplications and divisions in order from left to right.
3. Perform all additions and subtractions in order from left to right.

If grouping symbols are *not* present, perform operations in the above order.

EXAMPLE 1. Simplify the following expressions.

(a) $(-4)(3) - (5)(-6)$ (b) $-6/(-2) + 2(-3)^2 - 2^2$

Solutions:

(a) We must perform the multiplications first.

$$(-4)(3) - (5)(-6) = -12 - (-30)$$

Next, we do the subtraction.

$$-12 - (-30) = -12 + 30$$
$$= 18$$

(b) We must perform the division and multiplications before the addition.

$$-6/(-2) + 2(-3)^2 - 2^2 = 3 + 2(9) - 4$$
$$= 3 + 18 - 4$$
$$= 17$$

□

We must begin inside grouping symbols, if they are present. Remember that fraction bars, absolute value bars, and square root symbols are grouping symbols.

EXAMPLE 2. Simplify each expression.

(a) $\dfrac{8(-3)-6^2}{8-3(-1)+(-6)}$

(b) $-6-(5-\sqrt{25-16}+1)$

(c) $|5-8| \div (-3) - 4(-2)$

Solutions:

(a) We must simplify the numerator and the denominator before we divide.

$$\frac{8(-3)-6^2}{8-3(-1)+(-6)} = \frac{-24-36}{8-(-3)+(-6)}$$

$$= \frac{-60}{8+3+(-6)}$$

$$= \frac{-60}{5}$$

$$= -12$$

(b) We start with the innermost grouping symbol.

$$-6-(5-\sqrt{25-16}+1) = -6-(5-\sqrt{9}+1)$$

$$= -6-(5-3+1)$$

$$= -6-3$$

$$= -9$$

(c) Remember that absolute value bars are grouping symbols.

$$|5-8| \div (-3) - 4(-2) = |-3| \div (-3) - 4(-2)$$

$$= 3 \div (-3) - 4(-2)$$

$$= -1-(-8)$$

$$= -1+8$$

$$= 7$$ □

It is common in algebra to use letters to name numbers. We write such things as

$$2x + 3$$

to mean

2 times *some number* plus 3

The value of $2x + 3$ depends upon the value of x. This provides us with a way to write many statements with just one expression.

In algebra, as in arithmetic, we perform four operations (addition, subtraction, multiplication, and division) with numbers. The chart below indicates how we read and write these operations with the numbers x and y.

OPERATION	SYMBOL(S)	READ
Addition	$x + y$	x plus y
Subtraction	$x - y$	x minus y
Multiplication	xy	x times y
	$x \cdot y$	
	$x(y)$	
	$(x)y$	
	$(x)(y)$	
Division	$x \div y$	x divided by y
	$\frac{x}{y}$	
	x/y	
	$y\overline{)x}$	

Meaningful collections of sums, differences, products, quotients, exponents, and grouping symbols that contain letters are called **algebraic expressions** or simply **expressions.** For example,

$$6x + 2(x - 3)$$

and

$$\frac{3x}{5 + 4x}$$

are algebraic expressions.

In a sum or difference, the numbers being added or subtracted are called **terms.** In a product the numbers being multiplied are called **factors.** These are *important* words in algebra as they are used very often, and their meanings are quite different.

EXAMPLE 3. In each of the following expressions state if 2 is a *term* or a *factor.*

(a) $5x - 2$ (b) $2x$ (c) $2(x - 5)$

Solutions:

(a) The expression $5x - 2$ is a difference. It has two *terms,* $5x$ and 2. Therefore 2 is a term of this expression.

(b) The expression $2x$ is a product. It has one *term,* namely, $2x$. This term has two *factors,* 2 and x. Thus, 2 is a factor of this expression.

(c) The expression $2(x - 5)$ is a product. It has one *term* containing two *factors,* 2 and $(x - 5)$. Therefore 2 is a factor. □

The order of operations comes up in algebra when working with expressions. Consider the expression $2x - 1$. It has many different values, depending on the

value of x. If x is 2,

$$\begin{aligned} 2x - 1 &= 2(2) - 1 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

So, $2x - 1$ has a value of 3 when x is 2.
If x is 3,

$$\begin{aligned} 2x - 1 &= 2(3) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

Thus, $2x - 1$ has a value of 5 when x is 3.

When we find the value of an expression, we say that we **evaluate** the expression.

EXAMPLE 4. Evaluate the following expressions when x is -2.

(a) $10 + x$ (b) $10 - x$ (c) $\dfrac{10 - 3x}{x}$

(d) $5 + 3x + x^2$ (e) $x\sqrt{7 - x}$ (f) $|3 + 7x| + 21$

Solutions:

(a) It is convenient to think of the expression as

$$10 + x = 10 + (\ )$$

and then fill the parentheses with the number.

$$\begin{aligned} 10 + x &= 10 + (-2) \\ &= 8 \end{aligned}$$

(b) $10 - x$

$$\begin{aligned} 10 - x &= 10 - (\ ) \\ &= 10 - (-2) \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

(c) When replacing a variable in an expression with a number, we must be sure to replace the variable everywhere it occurs.

$$\begin{aligned} \frac{10 - 3x}{x} &= \frac{10 - 3(\ )}{(\ )} \\ &= \frac{10 - 3(-2)}{(-2)} \\ &= \frac{10 + 6}{-2} \end{aligned}$$

(continued)

value of x. If x is 2,

$$\begin{aligned} 2x - 1 &= 2(2) - 1 \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

So, $2x - 1$ has a value of 3 when x is 2.
If x is 3,

$$\begin{aligned} 2x - 1 &= 2(3) - 1 \\ &= 6 - 1 \\ &= 5 \end{aligned}$$

Thus, $2x - 1$ has a value of 5 when x is 3.

When we find the value of an expression, we say that we **evaluate** the expression.

EXAMPLE 4. Evaluate the following expressions when x is -2.

(a) $10 + x$ (b) $10 - x$ (c) $\dfrac{10 - 3x}{x}$

(d) $5 + 3x + x^2$ (e) $x\sqrt{7 - x}$ (f) $|3 + 7x| + 21$

Solutions:

(a) It is convenient to think of the expression as

$$10 + x = 10 + (\ )$$

and then fill the parentheses with the number.

$$\begin{aligned} 10 + x &= 10 + (-2) \\ &= 8 \end{aligned}$$

(b) $10 - x$

$$\begin{aligned} 10 - x &= 10 - (\ ) \\ &= 10 - (-2) \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

(c) When replacing a variable in an expression with a number, we must be sure to replace the variable everywhere it occurs.

$$\begin{aligned} \frac{10 - 3x}{x} &= \frac{10 - 3(\ )}{(\ )} \\ &= \frac{10 - 3(-2)}{(-2)} \\ &= \frac{10 + 6}{-2} \end{aligned}$$

(continued)

$$= \frac{16}{-2}$$

$$= -8$$

(d) $5 + 3x + x^2 = 5 + 3(\ ) + (\ )^2$

$$= 5 + 3(-2) + (-2)^2$$

The order of operations tells us we must evaluate the exponent first.

$$5 + 3x + x^2 = 5 + 3(-2) + 4$$

Then we do multiplication.

$$5 + 3x + x^2 = 5 + (-6) + 4$$

$$= 3$$

(e) Square root symbols are grouping symbols. We must perform operations inside them first.

$$x\sqrt{7 - x} = (-2)\sqrt{7 - (-2)}$$

$$= (-2)\sqrt{7 + 2}$$

$$= (-2)\sqrt{9}$$

$$= (-2)3$$

$$= -6$$

(f) Absolute value bars are also grouping symbols.

$$|3 + 7x| + 21 = |3 + 7(-2)| + 21$$

$$= |3 - 14| + 21$$

$$= |-11| + 21$$

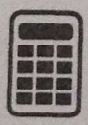
$$= 11 + 21$$

$$= 32$$ □

CALCULATOR BOX

Computations with a Calculator

[+] [−] [×] [÷] [(] [)] [=] [+/−]



Most scientific calculators have the order of operations built in. That is, they will calculate $3 \cdot 5 - 2$ correctly by pressing the keys [3] [×] [5] [−] [2] [=]. If the calculator displays the correct result, [13], it has the order of operations built in. However, the grouping keys, [(] and [)] are needed to preserve the order of operations in a calculation like $\frac{9 - 5}{3 + 1}$. Press [(] [9] [−] [5] [)] [÷] [(] [3] [+] [1] [)] [=] and read [1] on the display.

We must be careful when entering a negative number on a calculator. The [−] key is for *subtraction*. Look for a key marked [+/−] or [CHS]. To calculate $-6 \cdot 8$, press [6] [+/−] [×] [8] [=] and read [redacted] on the display.

Calculator Exercises

Perform the following computations on a calculator and check the result with pencil and paper.

1. $71 - (8 - 3)$
2. $\dfrac{45 - 9}{3 \cdot 4}$
3. $5(18 - 11) - 2(234 - 222)$
4. $1 \div (3 + 9) \cdot 24$
5. $-11(41 - 50)$
6. $\dfrac{-7 - 4}{-2(-3 - 8)}$

Answers:

1. 66 2. 3 3. 11 4. 2 5. 99 6. −0.5

[redacted] PROBLEM SET 1.5

Warm-ups

In Problems 1 through 8, simplify each expression. See Example 1.

1. $3(-2) + (-3)(-1)$
2. $\dfrac{-18}{-2} + \dfrac{6}{-3}$
3. $2^2 - 6/(6 - 3)$
4. $-7 + 3(-3)^2 + 5$
5. $|-3| + (-6)/(-2) - 3$
6. $2(-3)^2 - \dfrac{-60}{15} + (-8)$
7. $-\dfrac{8}{9} \div \dfrac{2}{3} + \left(-\dfrac{5}{6}\right)(-5)(-2)$
8. $(-3)(-2)(-1) - (-8) \div (-8)$

In Problems 9 through 19, simplify each expression. See Example 2.

9. $\dfrac{2(-3) + 4(-2)}{5 \cdot 2 - 3}$
10. $-5 - 2[8 - 2(4 - 6) - 7]$
11. $2|7 - 5| \div (-2 - 2)$
12. $\dfrac{3 - |0 - 3|}{4^2}$
13. $8 - 6 - (6 - 4)$
14. $\dfrac{3(-3) - 3^2 + (-12)}{(-3)^2 - 2(-3)}$
15. $\dfrac{8 - (-5) - (7 - 9)}{3(2^2 - 3^2)}$
16. $\dfrac{\sqrt{64 + 36}}{4}$
17. $14 - [2(4 - 7) - 3(8) - (5 - 11)]$
18. $10 - \sqrt{|16 - 25| + 16}$
19. $5 + 3^2 \div \sqrt{4 + 5}$

We must be careful when entering a negative number on a calculator. The [−] key is for *subtraction*. Look for a key marked [+/−] or [CHS]. To calculate $-6 \cdot 8$, press [6] [+/−] [×] [8] [=] and read [redacted] on the display.

Calculator Exercises

Perform the following computations on a calculator and check the result with pencil and paper.

1. $71 - (8 - 3)$
2. $\dfrac{45 - 9}{3 \cdot 4}$
3. $5(18 - 11) - 2(234 - 222)$
4. $1 \div (3 + 9) \cdot 24$
5. $-11(41 - 50)$
6. $\dfrac{-7 - 4}{-2(-3 - 8)}$

Answers:

1. 66 2. 3 3. 11 4. 2 5. 99 6. −0.5

[redacted] PROBLEM SET 1.5

Warm-ups

In Problems 1 through 8, simplify each expression. See Example 1.

1. $3(-2) + (-3)(-1)$
2. $\dfrac{-18}{-2} + \dfrac{6}{-3}$
3. $2^2 - 6/(6 - 3)$
4. $-7 + 3(-3)^2 + 5$
5. $|-3| + (-6)/(-2) - 3$
6. $2(-3)^2 - \dfrac{-60}{15} + (-8)$
7. $-\dfrac{8}{9} \div \dfrac{2}{3} + \left(-\dfrac{5}{6}\right)(-5)(-2)$
8. $(-3)(-2)(-1) - (-8) \div (-8)$

In Problems 9 through 19, simplify each expression. See Example 2.

9. $\dfrac{2(-3) + 4(-2)}{5 \cdot 2 - 3}$
10. $-5 - 2[8 - 2(4 - 6) - 7]$
11. $2|7 - 5| \div (-2 - 2)$
12. $\dfrac{3 - |0 - 3|}{4^2}$
13. $8 - 6 - (6 - 4)$
14. $\dfrac{3(-3) - 3^2 + (-12)}{(-3)^2 - 2(-3)}$
15. $\dfrac{8 - (-5) - (7 - 9)}{3(2^2 - 3^2)}$
16. $\dfrac{\sqrt{64 + 36}}{4}$
17. $14 - [2(4 - 7) - 3(8) - (5 - 11)]$
18. $10 - \sqrt{|16 - 25| + 16}$
19. $5 + 3^2 \div \sqrt{4 + 5}$

In Problems 20 through 25, state whether 7 is a term *or a* factor. *See Example 3.*

20. $7 + 3x - 5x^3$

21. $8x - 7$

22. $(1 - 3s)7$

23. $x^3 + 13x - 7$

24. $\frac{ax}{by} \cdot 7$

25. $7\sqrt{x + 3}$

In Problems 26 through 31, evaluate each expression when x has the value 3. See Example 4.

26. $15x$

27. $15 + x$

28. $\frac{15}{x}$

29. $\frac{5 + x}{5 - x}$

30. $\sqrt{82 + 6x}$

31. $x - |3x - 15|$

In Problems 32 through 37, evaluate each expression when x has the value of −3. See Example 4.

32. $15x$

33. $15 + x$

34. $\frac{15}{x}$

35. $\frac{5 + x}{5 - x}$

36. $\sqrt{82 + 6x}$

37. $x - |3x - 15|$

Practice Exercises

In Problems 38 through 59, simplify each expression.

38. $3(-5) - (-3)(-1)$

39. $\frac{-28}{4} + \frac{-9}{-3}$

40. $3^2 - 8/(6 - 3)$

41. $-8 + 2(-2)^2 + 7$

42. $|-7| + (-14)/(-2) - 14$

43. $3(-3)^2 - \frac{-30}{-6} + (-8)$

44. $3(-2) - (-8)(-4) - (-3)(-2)$

45. $3|-5| \div (-5) + (-3)^2$

46. $-\frac{6}{7} \div \frac{2}{7} + \left(-\frac{7}{9}\right)(-5)(-9)$

47. $(-4)(-3)(-1) - [(-6) \div (-6)]$

48. $\frac{2[(-3) + 4(2)]}{5(2) - 5}$

49. $-1 - 2[1 - 2(4 - 7) - 8]$

50. $3|5 - 7| \div (-3 - 3)$

51. $\frac{3 + |0 - 3|}{4^2 - 4(5) + 1}$

52. $9 - 4 - (4 - 2)$

53. $\frac{5(-6) + 2 \cdot 3^2 + (-12)}{(-3)^2 - 2 - 3}$

54. $\frac{3}{4}\left(-\frac{4}{3}\right) - \frac{1}{5}(-5) + \frac{5}{7}\left(-\frac{7}{15}\right)$

55. $|6 - 9| - 2(-3) \div (-3)$

56. $\frac{9 - (-7) - (4 - 9)}{3(4^2 - 3^2)}$

57. $4 - [2(4 - 8) - 3(4) - (3 - 7)]$

58. $\sqrt{169 - 25} + 11^2$

59. $\frac{2 + |2 - 8|}{2\sqrt{5^2 - 3^2}}$

In Problems 60 through 65, state whether x is a term *or a* factor.

60. $7 + x$

61. $x - 7y$

62. $x(1 - 3s)$

63. $x - 7$

64. $7ax$

65. $7\sqrt{y + 3} - x$

In Problems 66 through 74, evaluate each expression when x has the value 2 and again when x has the value −2.

66. $21x$ 42;

67. $21 + x$ 23;

68. $\frac{144}{x}$ 72;

69. $5x^2$ 20;

70. $(5x)^2$ 100;

71. $-x^2$ −4;

72. $\frac{1-x}{1+x}$ 73. $6x|x-6|$ 74. $\sqrt{4x+17}-5$

Challenge Problems

Problems 75 through 78 each contain two expressions. Decide whether they are equal or not; then test by replacing x with -6.

75. $|x-3|$ and $|x|-|3|$

76. $|x-3|$ and $|3-x|$

77. $|2x|$ and $|2||x|$

78. $\left|\frac{x}{3}\right|$ and $\frac{|x|}{|3|}$

IN YOUR OWN WORDS...

79. What do the expressions $20 \div 4 + x$ and $20 \div (4 + x)$ mean?

1.6 PROPERTIES OF REAL NUMBERS

In our work with real numbers in this chapter, we have not discussed their properties. Many of them are so natural to us that we completely overlook them. For example, we know that

$$2 + 3 = 3 + 2$$

and that

$$2 \cdot 3 = 3 \cdot 2$$

This illustrates what we call the **Commutative Properties.** If we add or multiply two real numbers, the *order* in which we do the operations does not matter.

Commutative Properties

If p and q are real numbers,

Addition: $p + q = q + p$
Multiplication: $pq = qp$

We also know that

$$(2 + 3) + 4 = 2 + (3 + 4)$$

and that

$$(2 \cdot 3) \cdot 4 = 2 \cdot (3 \cdot 4)$$

This illustrates the **Associative Properties.** If we add or multiply three numbers, we may group the numbers in different ways to add or multiply.

Associative Properties

If p, q, and r are real numbers,

Addition: $(p + q) + r = p + (q + r)$
Multiplication: $(pq)r = p(qr)$

EXAMPLE 1. Name the property used in each of the following.

(a) $3(4) = 4(3)$ (b) $(x + 3) + 4 = x + (3 + 4)$

(c) $x(3 + y) = x(y + 3)$ (d) $5 \cdot (y + z) = (y + z) \cdot 5$

Solutions:

(a) $3(4) = 4(3)$

This is the Commutative Property of Multiplication.

(b) $(x + 3) + 4 = x + (3 + 4)$

This is the Associative Property of Addition.

(c) $x(3 + y) = x(y + 3)$

Notice that the order of 3 and y is all that has changed. This is the Commutative Property of Addition.

(d) $5 \cdot (y + z) = (y + z) \cdot 5$

All that is changed is the order in which the two numbers 5 and $(y + z)$ are written. This is the Commutative Property of Multiplication. □

If we are given more than two numbers to add with no symbols of grouping, the commutative and associative properties allow us to choose which two numbers to add first. Suppose we want to add $2 + 3 + 4$. We may choose which two numbers to add first.

$$2 + 3 + 4 = (2 + 3) + 4 \qquad \text{Associative Property}$$
$$= \quad 5 \quad + 4 = 9$$

or

$$2 + 3 + 4 = 2 + (3 + 4) \qquad \text{Associative Property}$$
$$= 2 + \quad 7 \quad = 9$$

or

$$2 + 3 + 4 = \ 2 + 4 \ + 3 \qquad \text{Commutative Property}$$
$$= (2 + 4) + 3 \qquad \text{Associative Property}$$
$$= \quad 6 \quad + 3 = 9$$

The same idea applies to multiplication.

The Commutative and Associative Properties can be useful for working problems efficiently. Notice how they work in the next example.

EXAMPLE 2. Perform the operations indicated.

(a) $-1345 + 23{,}456 + 1345$ (b) $\frac{1}{8} \cdot 15 \cdot 16$

Solutions:

(a) First we apply the Commutative Property to the last two terms.

$$-1345 + 23{,}456 + 1345 = -1345 + 1345 + 23{,}456$$

Now we use the Associative Property.

$$-1345 + 1345 + 23{,}456 = (-1345 + 1345) + 23{,}456$$
$$= 0 + 23{,}456$$
$$= 23{,}456$$

(b) $\frac{1}{8} \cdot 15 \cdot 16 = \frac{1}{8} \cdot 16 \cdot 15$ Commutative Property

$= \left(\frac{1}{8} \cdot 16\right) \cdot 15$ Associative Property

$= 2 \cdot 15$

$= 30$ □

The Distributive Property combines addition and multiplication. For example,

$$2(3 + 4) = 2 \cdot 7$$
$$= 14$$

Notice also that

$$2 \cdot 3 + 2 \cdot 4 = 6 + 8$$
$$= 14$$

So,

$$2(3 + 4) = 2 \cdot 3 + 2 \cdot 4$$

Distributive Properties

If p, q, and r are real numbers,

$$p(q + r) = p \cdot q + p \cdot r$$
$$p(q - r) = p \cdot q - p \cdot r$$

Example 3 shows how to use the Distributive Property.

EXAMPLE 3. Multiply each of the following using the Distributive Property.

(a) $3(5 + 8)$ (b) $2(x - 3)$ (c) $3\left(\frac{2}{3} + y\right)$

Solutions:

(a) $3(5 + 8) = 3 \cdot 5 + 3 \cdot 8$
$= 15 + 24$
$= 39$

(b) $2(x - 3) = 2 \cdot x - 2 \cdot 3$
$= 2x - 6$

(c) $3\left(\frac{2}{3} + y\right) = 3 \cdot \frac{2}{3} + 3 \cdot y$
$= 2 + 3y$ □

There are two real numbers, 0 and 1, that have special properties. Since the sum of 0 and any real number is the real number itself, we call 0 the identity for addition. Likewise, since 1 times any real number is the real number itself, we call 1 the multiplicative identity.

Identities

If p is a real number, then

1. $p + 0 = 0 + p = p$ Identity for addition
2. $p \cdot 1 = 1 \cdot p = p$ Identity for multiplication

We say that 0 is the **additive identity** and that 1 is the **multiplicative identity.**

If we add opposites, we get the additive identity.

$$3 + (-3) = 0$$

The product of 2 and $\frac{1}{2}$ is the multiplicative identity.

$$2 \cdot \frac{1}{2} = 1$$

Inverses

1. ***Additive Inverse (opposite)***

   If p is a real number, then there is a real number $-p$ such that

   $$p + (-p) = (-p) + p = 0.$$

   $-p$ is called the **additive inverse,** or **opposite,** of p. (We read $-p$ as "negative p" or "the opposite of p.")

2. ***Multiplicative Inverse (reciprocal)***

   If p is a nonzero real number, then there is a real number, $\frac{1}{p}$, such that

   $$p \cdot \frac{1}{p} = \frac{1}{p} \cdot p = 1.$$

   $\frac{1}{p}$ is called the **multiplicative inverse,** or **reciprocal,** of p.

EXAMPLE 4. Give the additive inverse (opposite) and the multiplicative inverse (reciprocal) for each.

(a) -10 (b) $-\frac{3}{4}$ (c) $\frac{2}{x}$

Solutions:

(a) The additive inverse is 10 because $-10 + 10 = 0$.

The multiplicative inverse is $-\frac{1}{10}$ because

$$(-10)\left(-\frac{1}{10}\right) = 1$$

(b) The additive inverse is $\frac{3}{4}$ because $-\frac{3}{4}+\frac{3}{4}=0$.

The multiplicative inverse is $-\frac{4}{3}$ because

$$\left(-\frac{3}{4}\right)\left(-\frac{4}{3}\right)=1$$

(c) The additive inverse is $-\frac{2}{x}$ because

$$\frac{2}{x}+\left(-\frac{2}{x}\right)=0$$

The multiplicative inverse is $\frac{x}{2}$ because

$$\frac{2}{x}\cdot\frac{x}{2}=1$$

□

EXAMPLE 5. Name the property that is illustrated in each.

(a) $4+0=4$ (b) $5+(-5)=0$ (c) $1\cdot x=x$ (d) $\frac{7}{x}\cdot\frac{x}{7}=1$

Solutions:

(a) $4+0=4$	Additive identity
(b) $5+(-5)=0$	Additive inverse
(c) $1\cdot x=x$	Multiplicative identity
(d) $\frac{7}{x}\cdot\frac{x}{7}=1$	Multiplicative inverse

□

PROBLEM SET 1.6

Warm-ups

In Problems 1 through 15, name the property illustrated by each statement. See Examples 1 and 5.

1. $4\cdot 7=7\cdot 4$

2. $(3+7)+4=(7+3)+4$

3. $(x+7)+4=x+(7+4)$

4. $8+n=n+8$

5. $2(5+y)=2\cdot 5+2y$

6. $x+0=x$

7. $-\frac{1}{3}(-3)=1$

8. $-\sqrt{6}+\sqrt{6}=0$

9. $-x+y=y+(-x)$

10. $\frac{1}{t}\cdot 1=\frac{1}{t}$

11. $1\cdot 14=14$

12. $9(7-\sqrt{3})=9\cdot 7-9\cdot\sqrt{3}$

13. $-\frac{1}{2}+\frac{1}{2}=0$

14. $(1+x)+z=1+(x+z)$

15. $\frac{2}{3}\cdot\frac{3}{2}=1$

16. Match expressions that have the same value and name the property that justifies the match.

1. $2 \cdot 5 - 2 \cdot 8$	**a.** $3 + 7$
2. $7 + 3$	**b.** 9
3. $(4 + 5) + 6$	**c.** -5
4. $-5 + 0$	**d.** $2(5 - 8)$
5. $1 \cdot 9$	**e.** $(5 + 4) + 6$

In Problems 17 through 20, simplify using the Associative Property.

17. $5\left(\frac{1}{5}x\right)$ **18.** $\frac{1}{7}(7x)$ **19.** $-3\left(-\frac{1}{3}y\right)$ **20.** $\frac{3}{4}\left(\frac{4}{3}z\right)$

In Problems 21 through 24, use the Commutative and Associative Properties to perform the operations indicated efficiently. See Example 2.

21. $367 + 874 + (-367)$

22. $-12 + 48 + 8 + 4$

23. $-6(4)(1/6)$

24. $\left(\frac{1}{3}\right)\left(\frac{2}{5}\right)(6)(10)$

In Problems 25 through 29, give the additive inverse (opposite) and the multiplicative inverse (reciprocal) of each number. Assume x is not zero. See Example 4.

25. x **26.** $-\frac{2}{7}$ **27.** π **28.** $-5x$ **29.** 1

In Problems 30 through 35, use the Distributive Property to multiply each. See Example 3.

30. $2(1 + x)$

31. $3(x - 4)$

32. $-2(x + 9)$

33. $-4(x - 2)$

34. $x(y + z)$

35. $x(y - z)$

Practice Exercises

In Problems 36 through 49, name the property illustrated by each statement.

36. $2 \cdot 3 \cdot 4 = 3 \cdot 2 \cdot 4$

37. $(3 + 7) + 2 = 3 + (7 + 2)$

38. $(g + 7) + 2 = (7 + g) + 2$

39. $5(6x) = (5 \cdot 6)x$

40. $9(r + 2) = 9r + 9 \cdot 2$

41. $y \cdot 1 = y$

42. $-\frac{1}{3} + \frac{1}{3} = 0$

43. $x \cdot \frac{1}{x} = 1$

44. $-p + p = 0$

45. $\frac{1}{j} + 0 = \frac{1}{j}$

46. $4q + 4 \cdot 7 = 4(q + 7)$

47. $0 + \sqrt{7u} = \sqrt{7u}$

48. $\frac{x}{y} \cdot 1 = \frac{x}{y}$

49. $\frac{1}{2} \cdot 2 = 1$

In Problems 50 through 57, simplify each expression.

50. $9\left(\frac{1}{3}x\right)$

51. $\frac{1}{12}(4x)$

52. $367 + 874 + (-367)$

53. $-52 + 148 + 52$

54. $-14(4)\left(\frac{1}{7}\right)$

55. $\left(\frac{1}{3}\right)\left(\frac{2}{7}\right)(9)(14)$

56. $-5\left(-\frac{1}{15}y\right)$ **57.** $\frac{2}{5}\left(\frac{15}{8}z\right)$

In Problems 58 through 62, give the additive inverse (opposite) and the multiplicative inverse (reciprocal) of each number.

58. 14 **59.** $\frac{5}{3}$ **60.** $-\pi$ **61.** 0 **62.** -1

In Problems 63 through 76, use the distributive property to multiply.

63. $2(4 + x)$ **64.** $3(x - 8)$ **65.** $-2(6 + y)$

66. $-4(y - 1)$ **67.** $x(y + 3)$ **68.** $x(3 + z)$

69. $x(y - 5)$ **70.** $x(4 - y)$ **71.** $t(3 - x)$

72. $x(y + 2z)$ **73.** $n(3 + 2x)$ **74.** $k(x - 9)$

75. $c(-9 + x)$ **76.** $-5(t + x)$

Challenge Problems

77. Is subtraction commutative? (Is $p - q$ the same as $q - p$?)

78. Is division commutative? (Is $p \div q$ the same as $q \div p$?)

In Problems 79 through 82, use the properties to carry out the multiplications indicated.

79. $x(x + 3)$ **80.** $5(x + y + z)$ **81.** $2x \cdot 3x^2$ **82.** $2x(3x^2 + z)$

IN YOUR OWN WORDS...

State the following properties.

83. Commutative Properties

84. Associative Properties

85. Distributive Property

2.1 EXPONENTS

We can use mathematical shorthand to indicate the product of four 2's:

$$2 \cdot 2 \cdot 2 \cdot 2 \quad \text{is abbreviated} \quad 2^4$$

and is called the *fourth power of* 2. The symbol 2^4 is read 2 (*raised*) *to the fourth power*. We call 2 the *base* and 4 the *exponent*. In general, the abbreviation a^n indicates that the number a is used as a factor n times, where n is a positive integer. The expression a^n is called the **nth power of a**, where n is the **exponent** and a is the **base**:

The nth power of a:

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors of } a}$$

If a number does not have an exponent indicated, the exponent is understood to be 1. Thus $5 = 5^1$.

Example 1

a. $6^2 = 6 \cdot 6 = 36$ (the *second power* of 6 or 6 *squared*). The base is 6 and the exponent is 2. Do *not* confuse 6^2 with $6 + 6$:

$$6^2 \neq 6 + 6 \quad \text{because} \quad 36 \neq 12.$$

b. $x^3 = x \cdot x \cdot x$ (*x to the third*, or *x cubed*).

c. $4^2x^4 = 4 \cdot 4 \cdot x \cdot x \cdot x \cdot x$.

d. $(x-2)^1 = x - 2$. The base is $x - 2$.

Example 2

a. $(-1)^4 = (-1)(-1)(-1)(-1) = 1$. In general, *an even power of a negative number is positive.*

b. $(-6)^3 = (-6)(-6)(-6) = -216$. *An odd power of a negative number is negative.*

We emphasize that *an exponent applies only to the quantity immediately to the left and below it.* For example,

Expression	*Base*	*Value*
$(-3)^2$	-3	$(-3)^2 = (-3)(-3) = 9.$
-3^2	3	$-3^2 = -(3 \cdot 3) = -9.$
$(3x)^2$	$3x$	$(3x)^2 = (3x)(3x).$
$3x^2$	x	$3x^2 = 3 \cdot x \cdot x.$

We turn now to the basic rules of exponents. The product $2^3 \cdot 2^2$ is $8 \cdot 4 = 32$. However, we can also write the product as

$$2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5.$$

In the result, notice that the exponent 5 is the sum of the exponents 3 and 2 in the original expression. That is,

$$2^3 2^2 = 2^5 = 2^{3+2}.$$

This illustrates the general rule that *to* ***multiply*** *numbers with the same base, we may* ***add*** *the exponents and keep the base the same*:

1. $a^m a^n = a^{m+n}$.

Example 3

a. $x^5 \cdot x^4 = x^{5+4} = x^9$.

b. $y^8 y^6 = y^{8+6} = y^{14}$.

c. $(x-2)^3(x-2)^5 = (x-2)^8$.

d. $x(x^n) = x^1 x^n = x^{1+n} = x^{n+1}$.

e. $t^2 t^5 t^4 = t^{2+5+4} = t^{11}$. (The pattern of Rule 1 extends to a product of more than two powers having the same base.)

Example 4

a. $\dfrac{x^3x^5}{y^2y^4} = \dfrac{x^{3+5}}{y^{2+4}} = \dfrac{x^8}{y^6}$.

b. $(3x^4)(9x^2) = 3 \cdot x^4 \cdot 9 \cdot x^2 = 27x^{4+2} = 27x^6$.

Keep in mind that Rule 1 requires that both powers have the same base before exponents are added. For example,

$$(-2)^2(-2^4) \neq (-2)^6 \quad \text{but} \quad (-2)^2(-2^4) = 4(-16) = -64.$$

(Base -2; Base 2)

Furthermore, in Rule 1 the base remains the same after exponents are added. Do not multiply the bases! Thus

$$3^4 \cdot 3^2 \neq 9^6 \quad \text{but} \quad 3^4 \cdot 3^2 = 3^6 = 729.$$

Another rule of exponents deals with taking a power of an expression that is itself a power of a number. For example, consider taking the second power of the third power of 2:

$$(2^3)^2 = (8)^2 = 64.$$

On the other hand, we can look at $(2^3)^2$ another way. Because 2^3 is the base for the exponent 2, we have

$$(2^3)^2 = 2^3 \cdot 2^3 = 2^{3+3} = 2^6.$$

Observe that the exponent 6 in the result is the product of the exponents 3 and 2 in the original expression. That is,

$$(2^3)^2 = 2^6 = 2^{3 \cdot 2}.$$

In general, *to find a power of a power, we may* ***multiply*** *the exponents and keep the base the same*:

2. $(a^m)^n = a^{mn}$.

Example 5

a. $(10^2)^6 = 10^{2 \cdot 6} = 10^{12}$.

b. $(x^4)^3 = x^{4 \cdot 3} = x^{12}$.

c. $(y^8)^5 = y^{8 \cdot 5} = y^{40}$.

d. $[(x + 7)^3]^5 = (x + 7)^{15}$.

We caution you not to confuse $(x^4)^3$ with x^4x^3. The first expression is a power of a power, so we *multiply* the exponents. The second expression is a product of powers with the same base, so we *add* the exponents.

$$(x^4)^3 = x^{12}, \quad \text{but} \quad x^4x^3 = x^7.$$

Now let us look at division (here, as elsewhere in this book, we assume that all denominators are nonzero). In the fraction $4^5/4^3$, the numerator has more factors of 4 than does the denominator because $5 > 3$. By cancellation we have

$$\frac{4^5}{4^3} = \frac{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot 4\cdot 4}{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}} = 4^2.$$

Notice that the exponent in the result is the *difference* of the exponents in the original expression: $2 = 5 - 3$. Similarly, in the fraction $4^3/4^5$, the exponent in the denominator is greater than that in the numerator. Cancellation gives two factors of a in the denominator.

$$\frac{4^3}{4^5} = \frac{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}}{\cancel{4}\cdot\cancel{4}\cdot\cancel{4}\cdot 4\cdot 4} = \frac{1}{4^2} = \frac{1}{4^{5-3}}.$$

More generally we have the following rules.

3.	$\dfrac{a^m}{a^n} = a^{m-n}$	for $m > n$,
4.	$\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$	for $n > m$,
5.	$\dfrac{a^n}{a^n} = 1.$	

Example 6

a. $\dfrac{x^{11}}{x^7} = x^{11-7} = x^4.$

b. $\dfrac{t^6}{t^9} = \dfrac{1}{t^{9-6}} = \dfrac{1}{t^3}.$

c. $\dfrac{y^{10}}{y^{10}} = 1.$

d. $\dfrac{-x^8}{x} = -\dfrac{x^8}{x} = -x^{8-1} = -x^7.$

e. $\dfrac{(x^2+1)^4}{(x^2+1)^{12}} = \dfrac{1}{(x^2+1)^{12-4}} = \dfrac{1}{(x^2+1)^8}.$

f. $\dfrac{(-2)^3}{(-2)^4} = \dfrac{1}{(-2)^{4-3}} = \dfrac{1}{(-2)^1} = \dfrac{1}{-2} = -\dfrac{1}{2}$.

We now turn to a power of a product, such as raising the product $2 \cdot 3$ to the second power:

$$(2 \cdot 3)^2 = 6^2 = 36.$$

However, we can also write

$$(2 \cdot 3)^2 = (2 \cdot 3)(2 \cdot 3) = (2 \cdot 2)(3 \cdot 3) = 2^2 \cdot 3^2.$$

Notice that the result is the product of the second powers of the original factors 2 and 3. That is,

$$(2 \cdot 3)^2 = 2^2 \cdot 3^2 = 4 \cdot 9 = 36.$$

There is a similar pattern for a power of a quotient:

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \cdot \frac{2}{3} = \frac{2 \cdot 2}{3 \cdot 3} = \frac{2^2}{3^2}.$$

More generally, *to raise a product to a power, we may raise each factor of the product to that power. To raise a quotient to a power, we may raise both the numerator and denominator to that power.*

6. $(ab)^n = a^n b^n$.

7. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

Example 7

a. $(xy)^4 = x^4 y^4$.

b. $(abc)^6 = a^6 b^6 c^6$.

c. $\left(\dfrac{x}{y}\right)^{12} = \dfrac{x^{12}}{y^{12}}$.

d. $\left(\dfrac{2}{z}\right)^4 = \dfrac{2^4}{z^4} = \dfrac{16}{z^4}$.

e. $(3x)^3 = 3^3 x^3 = 27x^3$. Note that $(3x)^3 \neq 3x^3$.

f. $\dfrac{20^3}{5^3} = \left(\dfrac{20}{5}\right)^3 = 4^3 = 64$, by Rule 7.

Rule 6 applies to a power of a *product*, not the power of a *sum*. For example,

$$(3 + 4)^2 \neq 3^2 + 4^2 \quad \text{because} \quad 49 \neq 25.$$

The next examples show various ways in which Rules 1–7 can be used. These rules are now summarized.

1. $a^m a^n = a^{m+n}$.
2. $(a^m)^n = a^{mn}$.
3. $\dfrac{a^m}{a^n} = a^{m-n}$ for $m > n$.
4. $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$ for $n > m$.
5. $\dfrac{a^n}{a^n} = 1$.
6. $(ab)^n = a^n b^n$.
7. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

Example 8

Find $(x^4y^2)^6$.

Solution The expression is a power of a product, so we first use the rule $(ab)^n = a^n b^n$ with x^4 replacing a and y^2 replacing b.

$$\begin{aligned}(x^4y^2)^6 &= (x^4)^6(y^2)^6 \\ &= x^{24}y^{12} \qquad \text{[Rule 2]}.\end{aligned}$$

Example 9

In each of the following, the numbers to the right refer to the rules used.

a. $\dfrac{(x^6)^3}{(x^3)^4} = \dfrac{x^{18}}{x^{12}} = x^6$ [2, 3].

b. $\dfrac{y^2(y^6)^3}{y^{24}} = \dfrac{y^2(y^{18})}{y^{24}} = \dfrac{y^{20}}{y^{24}} = \dfrac{1}{y^4}$ [2, 1, 4].

c. $(2s^2t^3)^3 = 2^3(s^2)^3(t^3)^3 = 8s^6t^9$ [6, 2].

d. $\left(\dfrac{x}{y^4}\right)^2 = \dfrac{x^2}{(y^4)^2} = \dfrac{x^2}{y^8}$ [7, 2].

c. $\left(\frac{2b^2}{c^3d^4}\right)^6 = \frac{(2b^2)^6}{(c^3d^4)^6} = \frac{2^6(b^2)^6}{(c^3)^6(d^4)^6} = \frac{64b^{12}}{c^{18}d^{24}}$ [7, 6, 2].

The next example shows you how to deal with minus signs involved with powers.

Example 10

a. $(-x^2y)^9 = [(-1)x^2y]^9 = (-1)^9(x^2)^9y^9 = (-1)x^{18}y^9 = -x^{18}y^9$.
(Note the use of brackets for grouping.)

b. $(-x)^{10} = [(-1)x]^{10} = (-1)^{10}x^{10} = 1 \cdot x^{10} = x^{10}$.

Example 11

The energy $\mathcal{E}$ stored in the electric field of a charged capacitor is given by the formula

$$\mathcal{E} = \tfrac{1}{2}CV^2,$$

where C is the capacitance and V is the voltage across the terminals of the capacitor. What happens to the energy if the voltage is doubled?

Solution The formula states that when the voltage is V, the energy $\mathcal{E}$ is $\frac{1}{2}CV^2$. Let $\mathcal{E}_1$ be the energy when the voltage is doubled, that is, after the voltage is changed from V to $2V$. Then, replacing $\mathcal{E}$ by $\mathcal{E}_1$ and V by $2V$ in the preceding formula gives

$$\mathcal{E}_1 = \tfrac{1}{2}C(2V)^2 = \tfrac{1}{2}C(4V^2) = 4(\tfrac{1}{2}CV^2).$$

But the factor $\frac{1}{2}CV^2$ is $\mathcal{E}$, the original energy. Thus

$$\mathcal{E}_1 = 4\mathcal{E},$$

so the energy increases to four times its original value. That is, doubling the voltage increases the energy by a factor of 4.

In the hierarchy of operations, raising to powers has priority over multiplication and division. Thus

$$4 \times 2^3 = 4 \times (2^3) = 4 \times 8 = 32.$$

To calculate powers with a calculator, two keys are available. The square key $\boxed{x^2}$ is used to compute the square of a number x (see Sec. A.4).

Example	*Keystroke Sequence*	*Display*
$(6.5)^2$	$\boxed{6.5}$ $\boxed{x^2}$	42.25

The power key $\boxed{y^x}$ is used to compute the xth power of y (see Sec. A.5). First the base y is keyed in, followed by $\boxed{y^x}$, the exponent x, and $\boxed{=}$.

Example	*Keystroke Sequence*	*Display*
$(4.56)^2$	$\boxed{4.56}\ \boxed{y^x}\ \boxed{2}\ \boxed{=}$	20.7936
$(2.6)^4$	$\boxed{2.6}\ \boxed{y^x}\ \boxed{4}\ \boxed{=}$	45.6976

Problem Set 2.1

In Problems **1–12**, *evaluate the numbers without using a calculator.*

1. 2^3.
2. $(-3)^2$.
3. $2^4 - 2^5$.
4. $(-7)^1$.
5. $-(-2)^4$.
6. $\dfrac{-2^2}{(-2)^3}$.
7. $(-2^3)(-3)^2$.
8. $(3-5)^2$.
9. $\dfrac{-(-3)^2}{(-3)^3}$.
10. $2^3 \cdot 3^2 - 6^2$.
11. $(10^3)^2$.
12. $(-2+3^2)^2$.

In Problems **13–82**, *simplify.*

13. x^3x^8.
14. x^4x^4.
15. y^5y^4.
16. t^2t.
17. x^2x^4x.
18. x^ax^b.
19. $(x-2)^5(x-2)^3$.
20. $y^9y^{91}y^2$.
21. $\dfrac{x^5x^2}{y^2y^3y^4}$.
22. $\dfrac{x(x^2)}{y^2y^3}$.
23. $(2x^2)(7x^6)$.
24. $x^4(3x^2)$.
25. $(-3x)(4x^3)$.
26. $(-2x^5)(-3x^4)$.
27. $(x^8)^2$.
28. $(x^4)^3$.
29. $(x^3)^3$.
30. $(x^5)^7$.
31. $(t^2)^n$.
32. $(x^b)^c$.
33. $(x^4)^2(x^3)^7$.
34. $x^6(x^4)^2$.
35. $\dfrac{x^7}{x^3}$.
36. $\dfrac{x^8}{x^{12}}$.
37. $\dfrac{x^{21}}{x^{22}}$.
38. $\dfrac{(a+b)^{16}}{(a+b)^{12}}$.
39. $\dfrac{y^{14}}{-y^8}$.
40. $\dfrac{-y^2}{-y^5}$.
41. $\dfrac{x^2x^8}{x^{16}}$.
42. $\dfrac{x^{18}}{x^{10}x^{10}}$.
43. $\dfrac{(x^5)^3}{x^2}$.
44. $\dfrac{(x^4)^2}{(x^5)^3}$.
45. $\dfrac{(x^4)^2}{x(x^6)}$.
46. $\dfrac{t^{12}(t^6)}{(w^5)^3}$.
47. $\dfrac{(x^2)^4(x^4)^2}{(x^3)^7}$.
48. $\dfrac{1}{(x^4)^5}(x^4)^5$.
49. $(ab)^6$.
50. $(xy)^4$.
51. $(2x)^4$.
52. $(3x)^3$.
53. $(2x^4y^2)^4$.
54. $(x^2yz^3)^2$.

55. $(3y)^3(5y^4)$.

56. $(3y^2)(2y^2)^4$.

57. $\left(\frac{a}{b}\right)^3$.

58. $\left(\frac{x}{2}\right)^4$.

59. $\left(\frac{3}{x}\right)^4$.

60. $\left(\frac{x}{y}\right)^2$.

61. $(xy^2)^4$.

62. $(3x^2)^3$.

63. $\left(\frac{2y}{z}\right)^3$.

64. $\left(\frac{1}{x^2y^3}\right)^5$.

65. $\left(\frac{2}{3}a^2b^3c^6\right)^2$.

66. $(-4)(2x^2)^2$.

67. $\left(\frac{x^2}{y^5}\right)^3$.

68. $\left(\frac{2x^2}{y^2}\right)^4$.

69. $\left(\frac{x^2y^3}{2z^4}\right)^4$.

70. $\left(\frac{2x^4}{5y^2}\right)^3$.

71. $(-x)^{13}$.

72. $(-3x)^4$.

73. $(-2x^2y)^4$.

74. $(-3)^2(-x)^3$.

75. $\frac{(-xy)^5}{(-t)^4}$.

76. $\frac{-y^3}{(-z)^2}$.

77. $(-3x)^3(-x)^7$.

78. $\frac{5^{100}}{5^{99}}$.

79. $\frac{2^6 2^{11}}{(2^5)^3}$.

80. $\frac{(80)^6}{(40)^6}$.

81. $(x^a y^b)^c$.

82. $\left(\frac{x^a}{y^b}\right)^a$.

In Problems **83–92**, *perform the indicated operations. Round your results to three significant digits.*

83. $(5.64)^2$.

84. $(0.00456)^2$.

85. $(17.6)^3$.

86. $(1.05)^4$.

87. $\frac{1}{(1.06)^5}$.

88. $\left(\frac{5.89}{2.57}\right)^4$.

89. $(0.0287)^2(46.3)^3$.

90. $(2.86^3)^2$.

91. $\frac{(34.57)^3}{(23.9)^4}$.

92. $\frac{(1.27)^{12}(310.6)}{(1.895)^8(228)}$.

93. The power P (in watts) dissipated in a resistance R (in ohms) is given by $P = I^2R$, where I is the current (in amperes). Find the power in each of the following cases.
   (a) $I = 3$ amperes and $R = 2$ ohms.
   (b) $I = -2$ amperes and $R = 2$ ohms.
   (c) $I = 0.1$ ampere and $R = 5$ ohms.

94. Under certain conditions, the pressure p of a vacuum system is given by the formula

$$p = \frac{ah^2}{V_0}.$$

Find p if (a) $h = 3d$, and (b) $h = \frac{2}{3}d$.

95. In addition to the formula in Problem 93, another formula for the power P dissipated in a resistance R is $P = V^2/R$, where V is the voltage across the resistor terminals. What happens to the power if the voltage is (a) doubled, (b) tripled, and (c) halved?

96. Compute the value of

$$\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + (x_3 - \bar{x})^2 + (x_4 - \bar{x})^2 + (x_5 - \bar{x})^2}{4}$$

if $x_1 = 6$, $x_2 = 8$, $x_3 = 5$, $x_4 = 3$, $x_5 = 3$, and $\bar{x} = 5$.

97. Repeat Problem 96 if $x_1 = 3.45$, $x_2 = 4.81$, $x_3 = 9.84$, $x_4 = 7.11$, $x_5 = 3.84$, and $\bar{x} = 5.81$. Give your answer to two decimal places.

98. The volume V of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$. What is the effect of doubling the radius?

99. The volume V of a cylinder with radius r and height h is given by $V = \pi r^2 h$.
   (a) Find the volume (in cubic meters) of a cylinder with $r = 0.850$ meters and $h = 1.50$ meters. (Use the $\boxed{\pi}$ key on your calculator and round your result to three significant digits.)
   (b) If both the radius and height of a cylinder are doubled, by what factor does the volume increase?

2.2 ZERO AND NEGATIVE EXPONENTS

Up to now we have worked only with exponents that are positive integers. Now we shall attach a meaning to a^0, where $a \neq 0$ (the symbol 0^0 is not defined). We want to define a^0 so that the rules for exponents hold. For instance, by the rule $a^m a^n = a^{m+n}$, we must have

$$a^m \cdot a^0 = a^{m+0} = a^m.$$

The result is identical to the first factor in the original expression, so multiplying by a^0 should have the same effect in multiplication as does the number 1. Thus it is reasonable to define a^0 to be 1.

8. $a^0 = 1$ if $a \neq 0$.

With this definition, the other rules of exponents also hold if an exponent is 0. To illustrate, for the rule $(ab)^n = a^n b^n$, we have

$$(ab)^0 = 1 = 1 \cdot 1 = a^0 b^0.$$

Example 1

a. $2^0 = 1$.

b. $(-3)^0 = 1$.

c. $(x^2 + 3)^0 = 1$.

d. $5(\frac{3}{2})^0 = 5(1) = 5$.

We now give meaning to a negative exponent, as in a^{-n}, where n is positive. Again we want the rules of exponents to hold. From the rule $a^m a^n = a^{m+n}$ we must have

$$a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1.$$

Thus the effect of multiplying a^n by a^{-n} is the same as multiplying a^n by its reciprocal, because the product of a number and its reciprocal is 1. Thus we are motivated to make the following definition.

9. $a^{-n} = \dfrac{1}{a^n}.$

For example,

$$3^{-4} = \frac{1}{3^4} = \frac{1}{81}, \qquad x^{-6} = \frac{1}{x^6}, \quad \text{and} \quad y^{-1} = \frac{1}{y}.$$

If we apply Rule 9 to $\dfrac{1}{a^{-n}}$, we get

$$\frac{1}{a^{-n}} = \frac{1}{\dfrac{1}{a^n}} = 1 \cdot \frac{a^n}{1} = a^n.$$

Thus we have the following rule:

10. $\dfrac{1}{a^{-n}} = a^n.$

For example,

$$\frac{1}{3^{-2}} = 3^2 = 9 \quad \text{and} \quad \frac{1}{x^{-5}} = x^5.$$

Example 2

a. $-x^{-6} = -(x^{-6}) = -\dfrac{1}{x^6}.$

b. $\dfrac{1}{x^2} = x^{-2}$, by Rule 9.

c. $2x^{-3} = 2\left(\dfrac{1}{x^3}\right) = \dfrac{2}{x^3}$, but $(2x)^{-3} = \dfrac{1}{(2x)^3} = \dfrac{1}{8x^3}.$

d. $\dfrac{1}{(-2)^{-3}} = (-2)^3 = -8$. Note that we changed only the sign of the exponent; we should not change the sign of the base. That is,

$$\frac{1}{(-2)^{-3}} \neq (2)^3.$$

e. $x^{-2} + y^{-2} = \dfrac{1}{x^2} + \dfrac{1}{y^2}$. *Note:* $x^{-2} + y^{-2} \neq \dfrac{1}{x^2 + y^2}$.

We can obtain a rule for a negative power of a fraction. Observe that

$$\left(\frac{a}{b}\right)^{-2} = \frac{1}{\left(\frac{a}{b}\right)^2} = \frac{1}{\frac{a^2}{b^2}} = 1 \cdot \frac{b^2}{a^2} = \frac{b^2}{a^2} = \left(\frac{b}{a}\right)^2.$$

Generalizing this result, we have Rule 11.

11. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n.$

For example, $(\frac{2}{3})^{-4} = (\frac{3}{2})^4 = \frac{81}{16}$.

Using Rules 8–10, we can replace Rules 3–5 by the following rule, which holds regardless of the values of m and n.

12. $\dfrac{a^m}{a^n} = a^{m-n} = \dfrac{1}{a^{n-m}}.$

For example,

$$\frac{x^3}{x^7} = x^{3-7} = x^{-4} = \frac{1}{x^4}.$$

But, more directly, we have

$$\frac{x^3}{x^7} = \frac{1}{x^{7-3}} = \frac{1}{x^4}.$$

Using exponents we can manipulate factors in a fraction. Observe that

$$\frac{x^2y^{-3}}{z^{-4}} = x^2 \cdot y^{-3} \cdot \frac{1}{z^{-4}} = x^2 \cdot \frac{1}{y^3} \cdot z^4 = \frac{x^2z^4}{y^3}.$$

Compare the factors in the first and last fractions. You can see that a nonzero *factor* of the numerator (or denominator) of a fraction may be equivalently expressed as a

factor of the denominator (or numerator) if the sign of its exponent is *changed*. For example,

$$\frac{x^{-2}y^3}{2z^{-2}} = \frac{y^3z^2}{2x^2}.$$

The word *factor* in the preceding statement is crucial! For example,

$$\frac{1}{2^2 + 3^2} \neq 2^{-2} + 3^{-2} \quad \text{but} \quad \frac{1}{2^2 3^2} = 2^{-2}3^{-2}.$$

Example 3

Simplify and give all answers with positive exponents.

a. $x^{-2}y^{-2} = \dfrac{1}{x^2y^2}$.

b. $\dfrac{16x^{-4}}{32x^7} = \dfrac{1}{2x^7x^4} = \dfrac{1}{2x^{11}}$. Note that we canceled the common factor 16.

c. $\dfrac{-x^{-2}}{y^{-3}z^2} = -\dfrac{y^3}{x^2z^2}$.

d. $\dfrac{x^{-7}y^6}{x^9y^{-2}} = \dfrac{y^6y^2}{x^9x^7} = \dfrac{y^8}{x^{16}}$.

Although the rules of exponents stated in Sec. 2.1 assumed the exponents to be positive integers, *the rules of exponents are equally true for any exponents.* Henceforth, we assume this fact, as Example 4 shows.

Example 4

Perform the operations and write the answers with positive exponents only.

a. $x^5x^{-2} = x^{5+(-2)} = x^{5-2} = x^3$.

b. $2x^{-2}x^{-3} = 2x^{-2-3} = 2x^{-5} = \dfrac{2}{x^5}$.

c. $(2x^3y^{-6})^2 = 2^2(x^3)^2(y^{-6})^2 = 4x^6y^{-12} = \dfrac{4x^6}{y^{12}}$.

d. $(x^2y^{-4})^{-3} = (x^2)^{-3}(y^{-4})^{-3} = x^{-6}y^{12} = \dfrac{y^{12}}{x^6}$.

Alternatively,

$$(x^2y^{-4})^{-3} = \frac{1}{(x^2y^{-4})^3} = \frac{1}{x^6y^{-12}} = \frac{y^{12}}{x^6}.$$

e. $\left(\frac{x^{-2}y^3}{z^{-4}}\right)^{-6} = \frac{(x^{-2}y^3)^{-6}}{(z^{-4})^{-6}} = \frac{x^{12}y^{-18}}{z^{24}} = \frac{x^{12}}{y^{18}z^{24}}$.

Another way to do this problem is

$$\left(\frac{x^{-2}y^3}{z^{-4}}\right)^{-6} = \left(\frac{y^3z^4}{x^2}\right)^{-6} = \left(\frac{x^2}{y^3z^4}\right)^{6} = \frac{x^{12}}{y^{18}z^{24}}.$$

A negative power of a number may be found with a calculator, as illustrated here.

Example	*Keystroke Sequence*	*Display*
4^{-3}	4 y^x 3 +/− =	0.015625

Problem Set 2.2

In Problems **1–17**, *find the values of the numbers without using a calculator.*

1. 3^0.
2. $\left(\frac{3}{4}\right)^0$.
3. 2^{-3}.
4. 3^{-2}.
5. $\frac{1}{3^{-3}}$.
6. $\frac{3}{4^{-2}}$.
7. $2x^0 + (2x)^0$.
8. $2(-3)^0$.
9. $\frac{-1^0}{4^{-1}}$.
10. $-5^{-2}(25)$.
11. $\frac{1}{(-3)^{-3}}$.
12. $2^{-1} + 3^{-1}$.
13. $\frac{6^{-4}}{6^{-2}}$.
14. $3^{-2}(2^{-2})$.
15. $3^{-2} + 4(2^{-2})$.
16. $\frac{(3^{-2})^0}{1^{-1}}$.
17. $\left(\frac{2}{5}\right)^{-2}$.

In Problems **18–32**, *write each expression by using positive exponents only. Simplify.*

18. x^{-2}.
19. x^{-6}.
20. $2^{-1}x$.
21. $\frac{1}{x^{-3}}$.
22. $\frac{1}{3x^{-2}}$.
23. $3y^{-4}$.
24. $2^{-2}x^{-4}$.
25. $\frac{x}{4^{-2}}$.
26. $\frac{7^0}{x^{-1}yz^{-2}}$.
27. $x^{-5}y^{-7}$.
28. $x^{-1}y^{-2}z^4$.
29. $\frac{2a^2b^{-4}}{c^{-5}}$.
30. $\frac{a^5b^{-4}}{c^{-3}d}$.
31. $\frac{x^9y^{-12}}{w^2z^{-4}}$.
32. $\frac{(x^2 + 4x^4)^0}{x^{-2}}$.

In Problems **33–60**, *perform the operations and simplify. Give all answers with positive exponents only.*

33. x^8x^{-7}.
34. $x^{-7}x$.
35. $x^{-2}x^{-3}$.

36. $x^2x^{-4}x^9$.
37. $(2x^{-5})(4x)$.
38. $(5x^{-5})(2x^{-4})$.
39. $(xy^{-5})^{-4}$.
40. $(2x^2y^{-1})^2$.
41. $2(x^{-1}y^2)^2$.
42. $(x^{-5}y^6)^{-1}$.
43. $(3t)^{-2}$.
44. $(x^{-4}y^{-4})^4$.
45. $(x^{-5}y^5z)^{-3}$.
46. $\dfrac{10x^6}{x^{-2}}$.
47. $\dfrac{t^{-8}}{t^{-12}}$.
48. $\dfrac{-2b^{-30}}{b^{-5}}$.
49. $\dfrac{x^{-2}y^4}{x^6y^{-1}}$.
50. $\dfrac{x^{-6}x^{10}}{x^3x^{-4}}$.
51. $\dfrac{x^{-3}}{(x^{-3})^2}$.
52. $\dfrac{x^3y^3}{x^{-2}y^{-2}}$.
53. $\left(\dfrac{x}{y}\right)^{-4}$.
54. $\left(\dfrac{y}{z^{-1}}\right)^{-1}$.
55. $\left(\dfrac{8x^2}{5y^2}\right)^{-1}$.
56. $\dfrac{1}{(3x^{-1})^{-1}}$.
57. $\left(-\dfrac{z^{-1}}{x}\right)^{-1}$.
58. $\left(\dfrac{x^{-1}y^4}{z^2}\right)^{-5}$.
59. $\left(\dfrac{y^{-6}z^2}{2x}\right)^{-2}$.
60. $\left[\left(\dfrac{x}{y}\right)^{-2}\right]^{-4}$.

In Problems **61–66**, *evaluate. Round your answers to three significant digits.*

61. $(7.28)^{-2}$.
62. $(12.3)^{-4}$.
63. $\dfrac{(4.75)^{-1}}{(8.21)^{-2}}$.
64. $\left(\dfrac{4.62}{8.14}\right)^{-3}$.
65. $\dfrac{(3.24)^{-2}(1.11)^2}{(3.13-1.24)^4}$.
66. $\dfrac{(2.67+6.04)^{-3}}{(1.73)^2}$.

2.3 SCIENTIFIC NOTATION

In science and technology, we often have to deal with numbers that are either very small or very large. Writing such numbers in our usual notation may be inconvenient. For example, the longest wavelength of visible red light is about 0.00000076 meter. The speed at which that light travels through air is about 300,000,000 meters per second, and the frequency of the light is 390,000,000,000,000 hertz. To avoid having to write so many zeros to locate the decimal points in such numbers, we may express the numbers in a compact form called *scientific notation.*

Scientific notation is based on powers of 10, some of which are as follows:

$$10 = 10^1, \qquad 0.1 = \frac{1}{10} = 10^{-1},$$

$$100 = 10^2, \qquad 0.01 = \frac{1}{100} = 10^{-2},$$

$$1000 = 10^3, \qquad 0.001 = \frac{1}{1000} = 10^{-3},$$

$$10{,}000 = 10^4, \qquad 0.0001 = \frac{1}{10{,}000} = 10^{-4}.$$

Day: Date:

2.3 Scientific notation is based on powers of 10 and powers of 10 are the decimal place values. The following table will help you recall the place values.

PLACE VALUES FOR DECIMALS

Number	Words	Product Form	Exponential Form
1,000,000	One million	10 x 10 x 10 x 10 x10 x 10	10^6
100,000	One hundred thousand	10 x 10 x 10 x 10 x10	10^5
10,000	Ten thousand	10 x 10 x 10 x 10	10^4
1,000	One thousand	10 x 10 x 10	10^3
100	One hundred	10 x 10	10^2
10	Ten	10	10^1
1	One	1	10^0
0.1	One tenth	1/10	$(1/10)^1$ or 10^{-1}
0.01	One hundredth	1/10 x 1/10	$(1/10)^2$ or 10^{-2}
0.001	One thousandth	1/10 x 1/10 x 1/10	$(1/10)^3$ or 10^{-3}
0.0001	One ten-thousandth	1/10 x 1/10 x 1/10 x 1/10	$(1/10)^4$ or 10^{-4}
0.00001	One hundred-thousandth	1/10 x 1/10 x 1/10 x 1/10 x 1/10	$(1/10)^5$ or 10^{-5}
0.000001	One millionth	1/10 x 1/10 x 1/10 x 1/10 x 1/10 x 1/10	$(1/10)^6$ or 10^{-6}

You may recall that to multiply a decimal by 10, we need only move the decimal point one position to the *right*. Thus

$$7.34 \times 10 = 73.4.$$

Similarly, we move the decimal point two and three places to the *right* to multiply by 100 (or 10^2) and 1000 (or 10^3), respectively.

$$7.34 \times 10^2 = 734 \qquad [10^2 \rightarrow 2 \text{ places to right}],$$

$$7.34 \times 10^3 = 7340 \qquad [10^3 \rightarrow 3 \text{ places to right}].$$

In each case the exponent for the power of 10 corresponds to the number of places the decimal point must be moved from its original position.

To divide a decimal by 10, 100, or 1000, we need only move the decimal point one, two, or three places to the *left*, respectively. For example,

$$\frac{7.34}{10} = 0.734.$$

Similarly, by expressing division by a positive power of 10 in terms of multiplication by a negative power of 10, we have

$$7.34 \times 10^{-1} = 0.734 \qquad [10^{-1} \rightarrow 1 \text{ place to left}],$$

$$7.34 \times 10^{-2} = 0.0734 \qquad [10^{-2} \rightarrow 2 \text{ places to left}],$$

$$7.34 \times 10^{-3} = 0.00734 \qquad [10^{-3} \rightarrow 3 \text{ places to left}].$$

We describe scientific notation as follows.

> A number N is in **scientific notation** when it is expressed as the product of a decimal number between 1 and 10 and some integer power of 10. That is, $N = a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

For example, the numbers 2×10^3 and 3.4×10^{-4} are in scientific notation.

Example 1

Write each number in scientific notation.

a. 2575.

Solution We want to move the decimal point three places to the left so that 2575 becomes 2.575 (which is a number between 1 and 10). To move the decimal point, we multiply by 10^{-3}. We adjust for this by multiplying by 10^3. In short, we multiply by $10^{-3} \times 10^3$, which is 1.

$$2575 = (2575 \times 10^{-3}) \times 10^3$$

$$= 2.575 \times 10^3.$$

b. 0.063.

Solution Here we want to move the decimal point two places to the right so that 0.063 becomes 6.3, which is a number between 1 and 10. To do this we multiply by $10^2 \times 10^{-2}$:

$$0.063 = (0.063 \times 10^2) \times 10^{-2}$$
$$= 6.3 \times 10^{-2}.$$

In each answer of Example 1, observe that the power to which 10 is raised corresponds to the number of places the decimal point must be moved from its original position. Moreover:

1. If the decimal point is moved to the *left*, the exponent is *positive*;
2. If it is moved to the *right*, the exponent is *negative*.

For example,

$18{,}000{,}000. = 1.8 \times 10^7$ ← Positive 7

7 places to left

and

$0.00027 = 2.7 \times 10^{-4}.$ ← Negative 4

4 places to right

Use this technique to verify the results in Example 2.

Example 2

Write each number in scientific notation.

a. $0.000624 = 6.24 \times 10^{-4}$.

b. $21{,}000{,}000 = 2.1 \times 10^7$.

c. $0.000000000000409 = 4.09 \times 10^{-13}$.

d. $6170 = 6.17 \times 10^3$.

e. $16.2 = 1.62 \times 10$.

f. $0.005930 = 5.930 \times 10^{-3}$.

g. The wavelength and frequency given at the beginning of this section can be written 7.6×10^{-7} meter and 3.9×10^{14} hertz, respectively.

To change a number from scientific notation to ordinary decimal notation, we use the exponent of the power of 10 to help us locate the decimal point. If the

exponent is *positive*, the decimal point is moved to the *right*; if it is *negative*, the decimal point is moved to the *left*. This is illustrated in Example 3.

Example 3

Write each number as a decimal.

a. 6.23×10^5.

Solution The exponent is *positive* 5, so we move the decimal point five places to the *right*. Thus $6.23 \times 10^5 = 623{,}000$.

b. 4.536×10^{-4}.

Solution The exponent is *negative* 4, so we move the decimal point four places to the *left*. Thus $4.536 \times 10^{-4} = 0.0004536$.

c. $6.24 \times 10^{-3} = 0.00624$.

d. $2.613 \times 10^8 = 261{,}300{,}000$.

e. $-7.0030 \times 10^{-5} = -0.000070030$.

Example 4

Evaluate by using scientific notation and give the answer in scientific notation.

a. $(0.0004)^3$.

Solution

$$(0.0004)^3 = (4 \times 10^{-4})^3 = 4^3 \times (10^{-4})^3 = 64 \times 10^{-12}.$$

Writing the result in scientific notation, we have

$$64 \times 10^{-12} = (6.4 \times 10) \times 10^{-12} = 6.4 \times 10^{1-12}$$
$$= 6.4 \times 10^{-11}.$$

b. (30,000)(50,000,000).

Solution

$$(30{,}000)(50{,}000{,}000) = (3 \times 10^4)(5 \times 10^7)$$
$$= (3 \times 5)(10^4 \times 10^7)$$
$$= 15 \times 10^{11} = (1.5 \times 10) \times 10^{11}$$
$$= 1.5 \times 10^{12}.$$

Observe how we separated the operations involving powers of 10 from the operations involving the other numbers.

c. $\dfrac{0.00762}{0.00003}$.

Solution

$$\frac{0.00762}{0.00003} = \frac{7.62 \times 10^{-3}}{3 \times 10^{-5}} = \frac{7.62}{3} \times \frac{10^{-3}}{10^{-5}} = 2.54 \times 10^{2}.$$

Example 5

Coulomb's law states that the force F (in newtons) acting between two small objects having electric charges q_1 and q_2 (in coulombs) is given by

$$F = \frac{(9 \times 10^{9})q_1 q_2}{r^2},$$

where r is the distance between the objects (in meters). Given that in the hydrogen atom the electron and proton each have a charge of 1.6×10^{-19} coulomb and are separated by a distance of 5.3×10^{-11} meter, compute the force between them. Give the answer in scientific notation.

Solution Here we treat the powers of 10 and the other factors separately:

$$F = \frac{(9 \times 10^{9})(1.6 \times 10^{-19})(1.6 \times 10^{-19})}{(5.3 \times 10^{-11})^2}$$

$$= \frac{(9 \times 1.6 \times 1.6)(10^{9} \times 10^{-19} \times 10^{-19})}{(5.3)^2(10^{-22})}$$

$$= \left[\frac{9 \times 1.6 \times 1.6}{(5.3)^2}\right] 10^{9-19-19+22}.$$

The first factor is approximately 0.82 and the second factor is 10^{-7}. Thus

$$F = 0.82 \times 10^{-7}.$$

(The use of the equals sign here is a convenience that we shall adopt throughout the text. More precisely, we should write $F \approx 0.82 \times 10^{-7}$.) In scientific notation we have

$$F = (8.2 \times 10^{-1}) \times 10^{-7} = 8.2 \times 10^{-8} \quad \text{newton}.$$

In Example 5, the separation of powers of 10 and the other factors illustrates the mathematics of the computation. When calculators are used, such a separation may or may not be necessary. For example, a calculator that allows us to enter only numbers that are not more than 8 to 10 digits long could not directly handle this calculation without a scientific-notation capability. This is a good time for you to see if your calculator allows entry of a number in scientific notation and how such a number is displayed.

If your calculator has this capability, you enter a number in scientific notation by using the exponential (or enter-exponent) key, designated [EXP], [EEX], or [EE] (see Sec. A.2). The exponent for 10 is entered *after* the [EXP] key is pressed.

Enter	*Keystroke Sequence*	*Display*
2.35×10^{7}	[2.35] [EXP] [7]	2.35 07
6×10^{-8}	[6] [EXP] [8] [+/−]	6 −08
10^{22}	[1] [EXP] [22]	1 22

To compute

$$(2.7 \times 10^{20}) \times (5.8 \times 10^{-9}),$$

we have:

Keystroke Sequence	*Display*
[2.7] [EXP] [20] [×] [5.8] [EXP] [9] [+/−] [=]	1.566 12

Problem Set 2.3

In Problems **1–12**, *write each number in scientific notation.*

1. 0.000006021.
2. 213,146,100,000,000.
3. 10.4.
4. 0.0624.
5. 26,245.1.
6. 2,600,000.
7. 142.
8. 0.0071.
9. 0.76.
10. 0.1.
11. 0.0348×10^{2}.
12. $20{,}300 \times 10^{-2}$.

In Problems **13–20**, *write each number in decimal form.*

13. 2.62×10^{8}.
14. 1.234×10^{-8}.
15. 6.24×10^{-10}.
16. 1.006×10^{4}.
17. 2.020×10^{-1}.
18. 6.0411×10^{12}.
19. 7.611×10^{5}.
20. 2.0×10.

21. The mass of the earth, in kilograms, is usually taken to be

5,983,000,000,000,000,000,000,000.

Express this mass in scientific notation.

22. The speed of light in a vacuum is taken to be 300,000,000 meters per second. Express this speed in scientific notation.

23. A numerical value of the gravitational constant is

0.0000000000667.

Express this constant in scientific notation.

24. The Milky Way galaxy is estimated to contain about 100 billion stars. Express this number in scientific notation. (*Note*: One billion is 1,000,000,000.)

In Problems **25–32**, *perform the indicated operations and give your answer in scientific notation.*

25. $(1.3 \times 10^{-4})(2.0 \times 10^{6})$.

26. $\dfrac{9.3 \times 10^{-1}}{3.1 \times 10^{5}}$.

27. $\dfrac{(3.0 \times 10^{11})(4.2 \times 10^{-4})}{2 \times 10^{13}}$.

28. $\dfrac{(4.8 \times 10^{-1})(5.0 \times 10^{-2})}{(3.2 \times 10^{-3})(3.0 \times 10^{-4})}$.

29. $\dfrac{(1.0 \times 10^{4})^{3}}{2.5 \times 10^{5}}$.

30. $\dfrac{(1.2 \times 10^{-2})^{2}}{2.88 \times 10}$.

31. $\dfrac{(35{,}600{,}000)(5{,}870{,}000)}{(12{,}600)(567{,}000)(1230)}$.

32. $\dfrac{(576 \times 10^{-4})(0.00000562)}{(0.00891)(4730)(0.938)}$.

33. A current I of 10^{4} amperes is produced in a conductor, with length L of 4 meters, at a point where the earth's magnetic field B is 5×10^{-5} teslas at right angles to the conductor. The force F, in newtons, on the conductor is given by

$$F = IBL$$
$$= (10^{4})(5 \times 10^{-5})(4).$$

Evaluate F.

34. The electric potential at a distance of 1.73×10^{-4} meter from a point charge of 4.62×10^{-2} coulomb is

$$\frac{(9 \times 10^{6})(4.62 \times 10^{-2})}{1.73 \times 10^{-4}} \text{ volts.}$$

Evaluate this potential.

35. The volume V of a sphere is given by $V = \frac{4}{3}\pi r^{3}$, where r is the radius. Find the volume of the earth (in cubic meters) if the earth's radius is 6.37×10^{6} meters.

36. An equilibrium constant, K, that relates to the solubility of lead iodide is given by

$$K = [Pb^{++}][I^{-}]^{2}.$$

Calculate K if $[Pb^{++}]$ is 1×10^{-2} and $[I^{-}]$ is 1.2×10^{-3}.

37. Two small objects have electric charges of 3.3×10^{-6} and 4×10^{-6} coulomb, respectively. Find the force between the objects when they are placed 2 meters apart. (Refer to Example 5.)

38. In calibrating the scale of a voltmeter, the following formula is used.

$$R_{x} = R_{T}\left(\frac{I_{M} - I_{x}}{I_{x}}\right).$$

Find the value of R_x (in ohms) if $R_T = 2000$ ohms, $I_M = 10^{-3}$ ampere, and $I_x = 7.5 \times 10^{-4}$ ampere.

39. The resistance R (in ohms) of a wire of length L (in meters) and cross-sectional area A (in square meters) is given by $R = \rho L/A$, where ρ (the Greek letter rho) is the resistivity of the material (in ohm · meters). Find the resistance of a copper wire 5 meters long with a

cross-sectional area of 0.0000003 square meter. The resistivity of copper is 0.000000017 ohm · meter.

40. While the universe is about 10^{10} years old, humans have existed only for about 10^6 years. Determine the number of years the universe existed before humans, and express that number in scientific notation.

41. The effective capacitance C of capacitors C_1 and C_2 connected in series is given by

$$C = \frac{C_1 C_2}{C_1 + C_2}.$$

Find C (in farads) if $C_1 = 2.0 \times 10^{-6}$ farad and $C_2 = 6.0 \times 10^{-6}$ farad.

42. If the mass of a molecule of a certain substance is 2.5×10^{-25} kilogram, find the number of molecules that have a total mass of 1 kilogram.

43. If the mass (in kilograms) of a proton is 1.673×10^{-27} and that of an electron is 9.109×10^{-31}, how many times larger is the mass of a proton compared to the mass of an electron?

2.4 RADICALS

If a and b are real numbers and $b^2 = a$, then b is called a **square root** of a. For example, $3^2 = 9$, so 3 is a square root of 9. In fact, because $(-3)^2 = 9$, the negative number -3 is also a square root of 9. There is a way to distinguish between the positive and negative square roots of a number. The *positive* square root is called the **principal square root**. Thus 3 is the principal square root of 9. For convenience we usually omit the word *principal* and simply say that 3 is *the* square root of 9.

We denote the (principal) square root of a nonnegative number a by the symbol $\sqrt{a}$, where a is called the **radicand**, the symbol $\sqrt{\ }$ is the **radical sign**, and the symbol $\sqrt{a}$ is itself called a **radical**.

$$\boxed{\sqrt{a} = b \quad \text{means} \quad b^2 = a, \text{ where } b \geq 0.}$$

Thus $\sqrt{9} = 3$, not -3. We define $\sqrt{0}$ to be 0.

You will find the appearance of radicals in many of your other courses, for they are found in virtually every area of technology. To illustrate, the expressions

$$\frac{1}{2\pi\sqrt{LC}}, \qquad \sqrt{\frac{\beta}{d}}, \quad \text{and} \quad \sqrt{\frac{M}{2\rho_0 N_0}}$$

are, respectively, the resonant frequency of an alternating current circuit, the velocity of a compressional wave, and the distance between ions in a cubic crystal.

Example 1

a. $\sqrt{25} = 5$ (the square root of 25 is 5) because $5^2 = 25$ and 5 is positive. The radicand is 25.

b. $\sqrt{0.01} = 0.1$ (the square root of 0.01 is 0.1) because $(0.1)^2 = 0.01$ and 0.1 is positive.

c. $-\sqrt{81} = -(\sqrt{81}) = -(9) = -9$ (the *negative* of the square root of 81 is -9).

d. $\sqrt{\frac{49}{100}} = \frac{7}{10}$ because $\left(\frac{7}{10}\right)^2 = \frac{49}{100}$.

We wish to mention two things. First, troubles arise with square roots of negative numbers. For example, there is no real number whose square is -4 because the square of a real number is never negative. This means that the symbol $\sqrt{-4}$ does not represent a real number. For the present we shall not concern ourselves with square roots of negative numbers. (Such roots are called *imaginary numbers* and are discussed in Chapter 13.)

Second, a square root of a number may be irrational. One example is $\sqrt{2}$, which is often approximated by 1.414. In some problems we shall find it convenient to leave an answer in radical form instead of using a decimal approximation obtained with a calculator.

There are four properties of square roots that will be useful in our future work. To begin with, note that since $\sqrt{9}$ is a number whose square is 9, we have $(\sqrt{9})^2 = 9$. Similarly,

$$(\sqrt{6})^2 = 6.$$

A more general rule is:

1. $(\sqrt{a})^2 = a.$

Moreover, because the square of a is a^2, a is a square root of a^2. If a is nonnegative, a general rule is:*

2. $\sqrt{a^2} = a.$

For example, $\sqrt{3^2} = 3$.

The next two rules state that the square root of a product (or quotient) is the product (or quotient) of the square roots:

3. $\sqrt{ab} = \sqrt{a}\sqrt{b}.$

4. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$

* For any real number a, the rule is $\sqrt{a^2} = |a|$. For example, $\sqrt{(-6)^2} = |-6| = 6$. Rules 3 and 4 assume a and b are nonnegative.

These rules allow us to simplify many radicals by writing them as a product or quotient of radicals so that one of them is easy to compute. For example, to simplify $\sqrt{24}$, we write 24 as a product of factors so that one factor is a perfect square. Then we use Rule 3.

$$\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\,\sqrt{6} = 2\sqrt{6}.$$

We say that the factor 4 has been *removed* from the radicand, and $2\sqrt{6}$ is considered to be the *simplified form* of $\sqrt{24}$.

When using this procedure, it is best to rewrite the radicand so that one of the factors is the *largest* whose square root can be taken easily. For example, to simplify $\sqrt{48}$ we should write 48 as $16 \cdot 3$. Writing it as $4 \cdot 12$ requires more work:

$$\begin{aligned}\sqrt{48} &= \sqrt{4 \cdot 12} = \sqrt{4}\,\sqrt{12} = 2\sqrt{4 \cdot 3} \\ &= 2(\sqrt{4}\,\sqrt{3}) = 2(2)\sqrt{3} = 4\sqrt{3}.\end{aligned}$$

More simply,

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16}\,\sqrt{3} = 4\sqrt{3}.$$

Example 2

Simplify each radical.

a. $\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\,\sqrt{2} = 3\sqrt{2}.$

b. $\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4}\,\sqrt{5} = 2\sqrt{5}.$

c. $\sqrt{\dfrac{11}{25}} = \dfrac{\sqrt{11}}{\sqrt{25}} = \dfrac{\sqrt{11}}{5}.$

d. $\dfrac{\sqrt{20}}{\sqrt{5}} = \sqrt{\dfrac{20}{5}} = \sqrt{4} = 2.$

When a fraction has a radical in its denominator, as with $2/\sqrt{3}$, it is sometimes convenient to rewrite the fraction in an equivalent form that does not have a radical in the denominator. This procedure is called **rationalizing the denominator**. To rationalize the denominator of

$$\frac{2}{\sqrt{3}},$$

we note that if the denominator were $(\sqrt{3})^2$, it would simplify by the rule $(\sqrt{a})^2 = a$. We can change $\sqrt{3}$ to $(\sqrt{3})^2$ with the use of the fundamental principle of fractions by multiplying both the numerator and denominator by $\sqrt{3}$.

$$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})^2} = \frac{2\sqrt{3}}{3}.$$

Actually, multiplying the numerator and denominator by $\sqrt{3}$ is equivalent to multiplying the given fraction by $\sqrt{3}/\sqrt{3}$. That is,

$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{(\sqrt{3})^2} = \frac{2\sqrt{3}}{3}.$$

Example 3

Rationalize the denominator.

a. $\dfrac{1}{\sqrt{5}} = \dfrac{1}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{\sqrt{5}}{(\sqrt{5})^2} = \dfrac{\sqrt{5}}{5}.$

b. $\sqrt{\dfrac{5}{3}} = \sqrt{\dfrac{5}{3} \cdot \dfrac{3}{3}} = \sqrt{\dfrac{15}{3^2}} = \dfrac{\sqrt{15}}{\sqrt{3^2}} = \dfrac{\sqrt{15}}{3}.$

The notion of square root can be extended to other roots. Since $2^4 = 16$, we say that 2 is a *fourth root* of 16. Similarly, -2 is a fourth root of 16. More generally, if n is a positive integer,

> b is an ***n*th** root of a if $b^n = a$.

Thus 5 is a **cube** (or third) **root** of 125 because $5^3 = 125$. Whenever a number a has a *positive* nth root, we call that root the **principal *n*th root** of a. Thus 2 is the (principal) fourth root of 16.

If a number has a *negative* nth root but does *not* have a positive nth root, we refer to that negative root as the principal nth root. For example, the (principal) cube root of -8 is -2 because $(-2)^3 = -8$ and there are no positive cube roots of -8. We define any root of 0 to be 0.

We denote the principal nth root of a by the symbol $\sqrt[n]{a}$, where n is called the **index** of the radical. Thus $\sqrt[4]{16} = 2$ (the index is 4) and $\sqrt[3]{-8} = -2$. For a principal square root, we omit the index 2. Thus $\sqrt[2]{9} = \sqrt{9} = 3$. We define $\sqrt[n]{0}$ to be 0.

PRINCIPAL *n*th ROOT OF *a*

$\sqrt[n]{a}$

Example 4

a. $\sqrt[3]{-1} = -1$ because $(-1)^3 = -1$ and no positive number has -1 for its cube.

b. $\sqrt[3]{\frac{1}{125}} = \frac{1}{5}$ because $\left(\frac{1}{5}\right)^3 = \frac{1}{125}$.

c. $\sqrt[5]{0} = 0$.

Corresponding to the previous rules for square roots, we have the following rules for nth roots.*

1. $(\sqrt[n]{a})^n = a$.
2. $\sqrt[n]{a^n} = a$.
3. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$.
4. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Example 5

a. $(\sqrt[5]{3})^5 = 3$ and $\sqrt[5]{3^5} = 3$.

b. $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27}\,\sqrt[3]{2} = 3\sqrt[3]{2}$.

c. $\sqrt[3]{-16} = \sqrt[3]{(-8)(2)} = \sqrt[3]{-8}\,\sqrt[3]{2} = -2\sqrt[3]{2}$.

d. $\sqrt[3]{2}\,\sqrt[3]{4} = \sqrt[3]{2 \cdot 4} = \sqrt[3]{8} = 2$.

Example 6

Given the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

find d if $x_1 = 1$, $x_2 = -7$, $y_1 = 2$, and $y_2 = 8$.

Solution

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-7 - 1)^2 + (8 - 2)^2} \\ &= \sqrt{(-8)^2 + (6)^2} = \sqrt{64 + 36} = \sqrt{100} \\ &= 10. \end{aligned}$$

This formula, called the *distance formula*, is used in analytic geometry.

* In each rule we assume that each expression represents a real number.

In Example 6 we have $\sqrt{64 + 36} = \sqrt{100} = 10$. We emphasize that

$$\sqrt{64 + 36} \neq \sqrt{64} + \sqrt{36} = 8 + 6 = 14.$$

That is, in general the square root of a *sum* is **not** the sum of the square roots:

$$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}.$$

Moral: Don't try fancy tricks of your own. Stick to basics.

Example 7

The current I (in amperes) in a resistance R (in ohms) in which the power dissipation is P (in watts) is given by

$$I = \sqrt{\frac{P}{R}}.$$

Find the current if the resistance is 3×10^3 ohms and the power dissipation is 3×10^{-3} watts.

Solution

$$I = \sqrt{\frac{3 \times 10^{-3}}{3 \times 10^3}} = \sqrt{10^{-3}(10^{-3})} = 10^{-3} \text{ ampere.}$$

To find roots of numbers with a calculator, use either the square-root key $\boxed{\sqrt{x}}$ (see Sec. A.4) or the root key $\boxed{\sqrt[x]{y}}$ (see Sec. A.5), which calculates the xth root of y. (The equals key is not necessary with the square-root key.) With the root key, the radicand y is entered first, then the root key is pressed, and finally the index x is entered.

Example	*Keystroke Sequence*	*Display*
$\sqrt{75}$	$\boxed{75}$ $\boxed{\sqrt{x}}$	8.66025
$\sqrt[5]{75}$	$\boxed{75}$ $\boxed{\sqrt[x]{y}}$ $\boxed{5}$ $\boxed{=}$	2.37144

Example 8

The impedance Z of a series R-L circuit is given by

$$Z = \sqrt{X_L^2 + R^2},$$

where X_L is the inductive reactance, R is the resistance, and all quantities are measured in ohms. Find the impedance of such a circuit in which $X_L = 10$ ohms and $R = 12$ ohms.

Solution

$$Z = \sqrt{(10)^2 + (12)^2} = \sqrt{100 + 144} = \sqrt{244} = 15.6 \text{ ohms.}$$

The calculator keystroke sequence is

[10] [x^2] [+] [12] [x^2] [=] [$\sqrt{x}$] Display: 15.620499

Problem Set 2.4

In Problems **1–38**, *compute the numbers without the aid of a calculator.*

1. $\sqrt{36}$.
2. $\sqrt{81}$.
3. $\sqrt[3]{8}$.
4. $\sqrt[3]{125}$.
5. $\sqrt{49}$.
6. $\sqrt{100}$.
7. $\sqrt[3]{-27}$.
8. $\sqrt[3]{-8}$.
9. $-\sqrt[3]{-64}$.
10. $\sqrt[4]{1}$.
11. $\sqrt[4]{16}$.
12. $\sqrt[5]{-1}$.
13. $\sqrt[4]{0}$.
14. $\sqrt[3]{64}$.
15. $\sqrt[3]{-125}$.
16. $\sqrt[5]{32}$.
17. $\sqrt[6]{64}$.
18. $\sqrt{12 \cdot 3}$.
19. $\sqrt{0.04}$.
20. $\sqrt{0.25}$.
21. $\sqrt{\frac{1}{16}}$.
22. $\sqrt{\frac{1}{100}}$.
23. $-\sqrt{25}$.
24. $\sqrt[3]{-1}$.
25. $\sqrt{81} - \sqrt[3]{-8}$.
26. $\sqrt{64} - \sqrt[3]{-8}$.
27. $\dfrac{\sqrt{64} + \sqrt[3]{-64}}{\sqrt{81} + \sqrt[4]{81}}$.
28. $\sqrt[3]{|-8|}$.
29. $\sqrt{5} \cdot \sqrt{5}$.
30. $\sqrt[3]{7} \cdot \sqrt[3]{7} \cdot \sqrt[3]{7}$.
31. $(\sqrt[4]{4})^4 + \sqrt[4]{4^4}$.
32. $(\sqrt{3})^2 + \sqrt[5]{2^5}$.
33. $\sqrt{(-3)^2} - \sqrt{3^2}$.
34. $\dfrac{\sqrt{49} - \sqrt{36}}{\sqrt{100} - \sqrt[3]{1}}$.
35. $\dfrac{(-1)^2 - (3\sqrt{9})}{|-4| + \sqrt{16}}$.
36. $\dfrac{(-2)^2 + (-3)^4}{\sqrt[5]{1} + |2 - 3|}$.
37. $\sqrt{0.01} + \sqrt{0.0025}$.
38. $2(\sqrt[3]{0.008}) - (0.01)^2$.

In Problems **39–63**, *simplify the numbers by using properties of radicals, as in Examples 2, 3, and 5.*

39. $\sqrt{50}$.
40. $\sqrt{75}$.
41. $\sqrt{12}$.
42. $\sqrt{32}$.
43. $\sqrt{8}$.
44. $\sqrt{400{,}000}$.
45. $\sqrt{14{,}400}$.
46. $\sqrt[3]{24}$.
47. $\sqrt[4]{48}$.
48. $\sqrt[4]{162}$.
49. $\sqrt[5]{64}$.
50. $\sqrt[3]{-54}$.
51. $\sqrt[3]{-500}$.
52. $\sqrt{\frac{5}{4}}$.
53. $\sqrt{\frac{14}{9}}$.

54. $\sqrt{\frac{2}{25}}$.

55. $\sqrt[3]{\frac{10}{27}}$.

56. $\frac{\sqrt{90}}{\sqrt{10}}$.

57. $\frac{\sqrt{50}}{\sqrt{2}}$.

58. $\frac{\sqrt[4]{64}}{\sqrt[4]{4}}$.

59. $\frac{\sqrt[3]{-2}}{\sqrt[3]{16}}$.

60. $\frac{\sqrt[5]{-4}}{\sqrt[5]{-128}}$.

61. $\sqrt{8} \cdot \sqrt{2}$.

62. $\sqrt{27} \cdot \sqrt{3}$.

63. $\sqrt[3]{4} \cdot \sqrt[3]{16}$.

In Problems **64–71**, *rationalize the denominators.*

64. $\frac{5}{\sqrt{6}}$.

65. $\frac{3}{\sqrt{2}}$.

66. $\frac{2.4}{\sqrt{6}}$.

67. $\frac{1}{\sqrt{7}}$.

68. $\frac{3}{2\sqrt{5}}$.

69. $\frac{4}{3\sqrt{2}}$.

70. $\sqrt{\frac{1}{15}}$.

71. $\sqrt{\frac{3}{2}}$.

In Problems **72–76**, *compute the numbers. Round your results to four significant digits.*

72. $\sqrt{84.26}$.

73. $\sqrt{0.02864}$.

74. $\sqrt[3]{0.0009264}$.

75. $\sqrt[6]{59{,}270}$.

76. $\sqrt[4]{6.083 \times 10^5}$.

In Problems **77** *and* **78**, *use the distance formula in Example 6 to find d with the given information*

77. $x_1 = -1$, $y_1 = 1$, $x_2 = -6$, $y_2 = 13$.

78. $x_1 = 10$, $y_1 = 2$, $x_2 = 7$, $y_2 = -2$.

79. The radius of a circle with area A is given by

$$r = \sqrt{\frac{A}{\pi}}.$$

Find the radius of a circle if the area is (a) 4π square meters, and (b) 15 square centimeters.

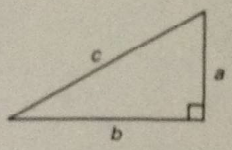
80. The relationship between the molecular velocity (v) and molecular weight (M) of two gases A and B at the same temperature is given by Graham's diffusion law:

$$\frac{v_A}{v_B} = \sqrt{\frac{M_B}{M_A}}.$$

Given that the molecular weight of hydrogen (H_2) is 2 and the molecular weight of oxygen (O_2) is 32, find the value of

$$\frac{v_{H_2}}{v_{O_2}}.$$

The Pythagorean theorem allows one side of a right triangle to be found when the other two sides are known. The relationships to be used are given in Fig. 2.1. In Problems **81–84**, *find the length of the indicated side.*



$a = \sqrt{c^2 - b^2}$
$b = \sqrt{c^2 - a^2}$
$c = \sqrt{a^2 + b^2}$

FIGURE 2.1

81. a if $c = 5$ meters and $b = 4$ meters.

82. a if $c = 25$ centimeters and $b = 24$ centimeters.

83. b if $c = 5 \times 10^{-2}$ meter and $a = 4 \times 10^{-2}$ meter.

84. b if $c = 7.2 \times 10^{-2}$ meter and $a = 3.1 \times 10^{-2}$ meter.

85. In an R-L series circuit, $X_L = 5$ ohms and $R = 12$ ohms. Find the circuit impedance. (Refer to Example 8.)

86. The filament of a 100-watt light bulb has a resistance of 144 ohms. Find the current through the filament. (Refer to Example 7.)

The expression $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ *is used in solving a certain type of equation. In Problems* **87–90**, *evaluate this expression for the given values of a, b, and c.*

87. $a = 1, b = 2, c = -15$.

88. $a = 4, b = 5, c = 0$.

89. $a = 4, b = -12, c = 9$.

90. $a = 1, b = -9, c = 14$.

91. In statistics, the standard deviation of the numbers x_1 and x_2 is given by

$$\sqrt{\frac{(\bar{x} - x_1)^2 + (\bar{x} - x_2)^2}{2}},$$

where $\bar{x} = \dfrac{x_1 + x_2}{2}$. Find the standard deviation of $x_1 = 0$ and $x_2 = 4$.

92. Compute y' (read y *prime*), where

$$y' = \frac{\dfrac{x + 5}{2\sqrt{x + 1}} - \sqrt{x + 1}}{(x + 5)^2},$$

if $x = 3$.

93. The speed v (in meters per second) for an artificial satellite to be in circular orbit around the earth is given by

$$v = \sqrt{\frac{9.8R^2}{r}},$$

where R is the radius of the earth (in meters) and r is the radius of the orbit (in meters). If $R = 6.37 \times 10^6$ meters, find the orbital speed for a satellite 3.59×10^7 meters above the surface of the earth. Round your answer to 3 significant figures.

2.5 RATIONAL EXPONENTS

In this section we will give a meaning to a rational exponent, such as in $5^{1/3}$. We want to define $5^{1/3}$ so that the rules of exponents hold. For example, by the rule $(a^m)^n = a^{mn}$, the cube of $5^{1/3}$ must be

$$(5^{1/3})^3 = 5^{3/3} = 5^1 = 5.$$

But recall that 5 is also the cube of $\sqrt[3]{5}$:

$$(\sqrt[3]{5})^3 = 5.$$

Thus we want the cube of $5^{1/3}$ to be the same as the cube of $\sqrt[3]{5}$. For this reason we define $5^{1/3}$ to be $\sqrt[3]{5}$. More generally, we define $a^{1/n}$ (where n is a positive integer*) to be the principal nth root of a:

1. $a^{1/n} = \sqrt[n]{a}$.

Thus

$$2^{1/4} = \sqrt[4]{2} \quad \text{and} \quad \sqrt{x+5} = (x+5)^{1/2}.$$

With our definition of $a^{1/n}$, the other rules of exponents hold whether the exponents are integers or fractions.

Example 1

Write each of the following in radical form and find the value.

a. $4^{1/2} = \sqrt[2]{4} = \sqrt{4} = 2.$

b. $27^{1/3} = \sqrt[3]{27} = 3.$

c. $(-32)^{1/5} = \sqrt[5]{-32} = -2.$

d. $16^{-(1/2)} = \dfrac{1}{16^{1/2}} = \dfrac{1}{\sqrt{16}} = \dfrac{1}{4}.$

* In Rule 1 we exclude the case where a is negative and n is even. For example, $(-4)^{1/2} = \sqrt{-4}$ is not a real number. Also, in Rule 2 we assume that $\sqrt[n]{a}$ is a real number.

Example 2

Write each of the following with fractional exponents.

a. $\sqrt{7} = 7^{1/2}$.

b. $\sqrt[4]{3x^2 + 4y} = (3x^2 + 4y)^{1/4}$.

c. $x\sqrt{y} + \sqrt{xy} = xy^{1/2} + (xy)^{1/2}$.

Rule 1 can be made more general to include other rational exponents. We define $a^{m/n}$, where m and n are integers and n is positive, as follows.

$$2. \quad a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$

That is, the denominator of m/n is the index of the corresponding radical, and the numerator is the exponent for the root or the exponent in the radicand. In terms of exponents, we have $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$. By Rule 2 we may look at $x^{4/3}$ in two ways: first, as the fourth power of the cube root of x, $(\sqrt[3]{x})^4$; and second, as the cube root of the fourth power of x, $\sqrt[3]{x^4}$. To compute $a^{m/n}$ it is often easier first to take the nth root of a and then raise the result to the mth power.

Example 3

Find the value of each of the following.

a. $8^{2/3} = (\sqrt[3]{8})^2 = 2^2 = 4$. Alternatively, we have $8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$.

b. $\left(\frac{1}{4}\right)^{3/2} = \left(\sqrt{\frac{1}{4}}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

c. $(-27)^{4/3} = (\sqrt[3]{-27})^4 = (-3)^4 = 81$.

d. $27^{-2/3} = \frac{1}{27^{2/3}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{(3)^2} = \frac{1}{9}$.

In Rule 2 we must require that $a \geq 0$ if n is even. Otherwise, the statement $\sqrt{x^2} = x^{2/2} = x$ is not always true. To see why, let $x = 6$. Then $\sqrt{x^2} = \sqrt{36} = 6 = x$. But if $x = -6$, then $\sqrt{x^2} = \sqrt{36} = 6 = -(-6) = -x$. Thus $\sqrt{x^2}$ is x if x is positive, and is $-x$ if x is negative. (Remember that if x is negative, then $-x$ is positive.) More simply this means that

$$\sqrt{x^2} = |x|.$$

Thus

$$\sqrt{6^2} = |6| = 6 \quad \text{and} \quad \sqrt{(-6)^2} = |-6| = 6.$$

Unless otherwise stated, we shall assume that all literal numbers appearing in radicals are positive. As a result, we are free to write $\sqrt{x^2} = x$.

Keep in mind that *the rules for exponents are true for all types of exponents.* These rules will be applied to fractional exponents in Examples 4 and 5.

Example 4

Write each of the following with fractional exponents and simplify if possible.

a. $\sqrt[5]{x^4} = x^{4/5}$.

b. $x\sqrt[6]{y^5} = xy^{5/6}$.

c. $(\sqrt{3})^4 = (3^{1/2})^4 = 3^{4/2} = 3^2 = 9$.

d. $\sqrt[4]{x^8} = x^{8/4} = x^2$.

e. $\sqrt[4]{a^4b^8} = (a^4b^8)^{1/4} = (a^4)^{1/4}(b^8)^{1/4} = a^{4/4}b^{8/4} = ab^2$.

Example 5

Perform the indicated operations and give the answer with positive exponents only.

a. $x^{1/2}x^{1/3} = x^{(1/2)+(1/3)} = x^{5/6}$.

b. $\dfrac{2x}{x^{1/4}} = \dfrac{2x^1}{x^{1/4}} = 2x^{1-(1/4)} = 2x^{3/4}$.

c. $\dfrac{y^{2/3}}{x^{2/3}y^{1/3}} = \dfrac{y^{1/3}}{x^{2/3}}$.

d. $(8a^3)^{2/3} = 8^{2/3}(a^3)^{2/3} = (\sqrt[3]{8})^2a^2 = 2^2a^2 = 4a^2$.

e. $(15^{-1/2})^4 = 15^{-4/2} = 15^{-2} = \dfrac{1}{15^2} = \dfrac{1}{225}$.

f. $(x^{2/9}y^{-4/3})^{1/2} = (x^{2/9})^{1/2}(y^{-4/3})^{1/2} = x^{1/9}y^{-2/3} = \dfrac{x^{1/9}}{y^{2/3}}$.

g. $\left(\dfrac{x^{12}}{y^6}\right)^{-1/3} = \dfrac{(x^{12})^{-1/3}}{(y^6)^{-1/3}} = \dfrac{x^{-4}}{y^{-2}} = \dfrac{y^2}{x^4}$.

h. $\left(\dfrac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^{15} = \dfrac{(x^{1/5}y^{6/5})^{15}}{(z^{2/5})^{15}} = \dfrac{x^3y^{18}}{z^6}$.

Day: Date:

Here are some examples of using a calculator to evaluate numbers involving fractional exponents.

Example	*Keystroke Sequence*	*Display*
$(48.5)^{1/4}$	48.5 y^x 4 1/x =	2.63898
$27^{2/3}$	27 y^x (2 ÷ 3) =	9
$(9.5)^{-2.4}$	9.5 y^x 2.4 +/− =	0.00450

Problem Set 2.5

In Problems **1–15**, *find the value of each expression without the use of a calculator.*

1. $25^{1/2}$.
2. $125^{1/3}$.
3. $81^{-1/2}$.
4. $4^{-1/2}$.
5. $27^{2/3}$.
6. $(\frac{1}{9})^{3/2}$.
7. $27^{-2/3}$.
8. $9^{3/2}$.
9. $(-8)^{4/3}$.
10. $(-8)^{-1/3}$.
11. $16^{3/4}$.
12. $(64^{1/2})^{1/3}$.
13. $(-\frac{1}{32})^{-1/5}$.
14. $(\frac{1}{27})^{-2/3}$.
15. $(4^3)^{2/3}$.

In Problems **16–30**, *rewrite each expression by using positive fractional exponents.*

16. $\sqrt[3]{y}$.
17. $\sqrt{x}$.
18. $\sqrt[4]{x^9}$.
19. $\sqrt[3]{x^2}$.
20. $\dfrac{\sqrt{y}}{\sqrt[4]{x}}$.
21. $\sqrt[4]{x^3y^5}$.
22. $\sqrt[4]{x}\,\sqrt[5]{y}$.
23. $\sqrt[6]{x^5y^{12}}$.
24. $\dfrac{1}{\sqrt{2}}$.
25. $x^2\sqrt[4]{x}$.
26. $\dfrac{2}{\sqrt[3]{x-5}}$.
27. $\dfrac{3}{\sqrt{x}}$.
28. $\sqrt[3]{7-x}$.
29. $\sqrt[3]{(x^2-5x)^2}$.
30. $\dfrac{1}{\sqrt[6]{(x^2-2x)^5}}$.

In Problems **31–64**, *perform the operations and simplify. Write your answers with positive exponents only.*

31. $x^{1/2}x^{3/2}$.
32. $x^{2/3}x^{4/3}$.
33. $x^{4/3}x^{-1/3}$.
34. $x^{-1/2}x^{3/2}$.
35. $x^{-2}x^{7/2}x^{5/2}$.
36. $x^3x^{-3/8}x^{-5/8}$.
37. $x^{1/2}x^{1/4}$.
38. $x^{3/5}x^{1/3}$.
39. $x^{1/2}(3x^{1/2}y)$.
40. $(xy^{-1/2})y^{5/2}$.
41. $(x^{1/2})^3$.
42. $(y^{-1/3})^6$.
43. $(y^6)^{1/3}$.
44. $(t^{10})^{4/5}$.
45. $2(x^{-2/3})^6$.
46. $(-y^{1/3})^6$.
47. $(2x^{-2}y^{1/3})^3$.
48. $(3x^3y^{1/2})^4$.
49. $(ab^2c^3)^{3/4}$.
50. $(x^2y)^{1/3}$.
51. $(27x^{15})^{-1/3}$.
52. $(8x^{-6})^{1/3}$.
53. $(-8x^{-6})^{1/3}$.
54. $(27^{-1}x^{15})^{-1/3}$.
55. $\dfrac{x^{4/9}y^{2/5}}{x^{1/9}y^{7/5}}$.
56. $\dfrac{x^{4/7}y^{3/20}}{x^{3/7}y^{17/20}}$.
57. $\dfrac{x^{2/3}y^{-9/4}}{x^{-4/3}y^{-1/4}}$.

58. $\dfrac{xy^{2/5}}{x^{1/2}y^{2/5}z^{-1/3}}$.

59. $\left(\dfrac{x^{-4/3}}{y^{-2/3}}\right)^{-3}$.

60. $\left(\dfrac{3^{1/4}x^{3/4}}{y^{1/2}}\right)^{4}$.

61. $\left(\dfrac{x^{-1}}{x^{1/3}}\right)^{2}$.

62. $\left(\dfrac{y^{2/3}}{x^{3/4}}\right)^{2/3}$.

63. $\left(\dfrac{2^{1/3}x^{2/3}}{x^{1/3}}\right)^{6}$.

64. $\left(\dfrac{x^{3/2}}{x^{-1}}\right)^{-4}$.

65. Experimental data indicate that the radius R of a nucleus of an isotope of mass number A is given by

$$R = r_0 A^{1/3}.$$

The value of r_0, the radius parameter, can be taken to be 1.5×10^{-13} centimeter. Determine the radius of a naturally occurring isotope of zinc whose mass number is 64.

66. In studies of the effect of an electrical current in a ferromagnetic material, the equation $W = KB^{8/5}$ occurs. If $B = 0.60$, express W in terms of K.

67. When 20 cubic meters of oxygen at a temperature of 285 K and a pressure of 10^5 newtons per square meter is adiabatically compressed to 0.5 cubic meter, its new pressure p and temperature T are given by

$$p = (10)^5\left(\frac{20}{0.5}\right)^{7/5}$$

and

$$T = (285)\left(\frac{20}{0.5}\right)^{7/5}.$$

Find p and T.

68. If the gas in Problem 67 is helium, the expressions for p and T have a fractional exponent of $\frac{5}{3}$ instead of $\frac{7}{5}$. Find the values of p and T if the gas is helium.

69. The rate R (in cubic meters per second) at which water discharges over a 90° V-notch wier is given by $R = 1.37h^{5/2}$, where h is the height (in meters) of the water above the wier crest. Find R if $h = 2.4$ meters.

2.6 REVIEW

Important Terms and Symbols

Section 2.1	Base, exponent, a^n, nth power.
Section 2.2	Zero exponent, a^0, negative exponent, a^{-n}.
Section 2.3	Scientific notation.
Section 2.4	Square root, principal square root, $\sqrt{a}$, radicand, radical sign, radical, rationalizing the denominator, nth root, principal nth root, $\sqrt[n]{a}$, index.
Section 2.5	Rational exponent, $a^{1/n}$, $a^{m/n}$.

Day: _____ Date: _____

Formula Summary

LAWS OF EXPONENTS

$a^0 = 1.$ $\qquad a^{-n} = \dfrac{1}{a^n}.$

$a^m a^n = a^{m+n}.$ $\qquad (a^m)^n = a^{mn}.$

$\dfrac{a^m}{a^n} = a^{m-n} = \dfrac{1}{a^{n-m}}.$ $\qquad (ab)^n = a^n b^n.$

$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}.$ $\qquad \left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n.$

LAWS OF RADICALS

$(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a.$ $\qquad \sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}.$

$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}.$ $\qquad a^{1/n} = \sqrt[n]{a}.$

$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}.$

Review Questions

1. In $(3x)^2$ the base is ___(a)___, but in $3x^2$ it is ___(b)___.
2. The cube of $2x$ is equal to _____.
3. Although $2^2 = 4$ and $(-2)^2 = 4$, by the symbol $\sqrt{4}$ we mean ___(2)(−2)___.
4. The principal square root of 16 is _____.
5. In scientific notation, the number 0.026 is written as _____.
6. $x^2(x^3) =$ ___(a)___ and $(x^3)^2 =$ ___(b)___.
7. The expression x^2/x^3 may be written as x raised to what negative power? _____
8. $7^0 =$ ___(a)___ but 0^0 is ___(b)___.
9. $64^{1/3} = \sqrt[n]{64}$, where $n =$ _____.
10. $x^{5/4} = \sqrt[n]{x^m}$, where $m =$ _____ and $n =$ _____.
11. $x^{2/3} = (\sqrt[n]{x})^m$, where $m =$ _____ and $n =$ _____.
12. $\sqrt[7]{x^5} = x^{m/n}$, where $m =$ _____ and $n =$ _____.
13. $(\sqrt[4]{x})^9 = x^{m/n}$, where $m =$ _____ and $n =$ _____.
14. $\sqrt[4]{x^4} =$ _____; $\sqrt[3]{z^6} = \sqrt[3]{(z^2)^3} =$ _____.
15. $\sqrt{32} = \sqrt{(_____)\cdot 2} = \sqrt{_____}\cdot\sqrt{2} =$ _____ $\sqrt{2}$.

16. $\sqrt{\frac{5}{36}} = \frac{\sqrt{5}}{\sqrt{\quad}} = \frac{\sqrt{5}}{(\quad)}$.

Answers to Review Questions

1. (a) $3x$, (b) x. 2. $8x^3$. 3. 2. 4. 4. 5. 2.6×10^{-2}. 6. (a) x^5, (b) x^6. 7. -1. 8. (a) 1, (b) Not defined. 9. 3. 10. 5, 4. 11. 2, 3. 12. 5, 7. 13. 9, 4. 14. x, z^2. 15. 16, 16, 4. 16. 36, 6.

Review Problems

In Problems **1–24**, *determine whether each statement is, in general, true or false.*

1. $a^3a^2 = a^6$.
2. $\left(\frac{1}{x}\right)^3 = \frac{1}{x^3}$.
3. $2^2 + 3^2 = 5^2$.
4. $(-a)^4 = a^4$.
5. $3 + 3 + 3 = 3^3$.
6. $(x^2y)^2 = x^4y$.
7. $(2x^2)^3 = 2^3x^5$.
8. $\frac{x^5}{x^3} = x^2$.
9. $(-y)^3 = -y^3$.
10. $2^2 \cdot 2^4 = 4^8$.
11. $3x = xxx$.
12. $(-3)^3 = -3^3$.
13. $(-2)^2 = -2^2$.
14. $(2 + 5)^2 = 2^2 + 5^2$.
15. $a^3a^7 = a^7a^3$.
16. $(x^2)^y = x^{2y}$.
17. $(2x)^2 = 2x^2$.
18. $\sqrt{16} = -4$.
19. $\sqrt{5}\sqrt{5} = \sqrt{25}$.
20. $\sqrt{4 \cdot 2} = 4\sqrt{2}$.
21. $\sqrt[3]{-\frac{1}{8}} = -\frac{1}{2}$.
22. $\sqrt[3]{0} = 0$.
23. $\sqrt{1} + \sqrt{1} = \sqrt{2}$.
24. $(\sqrt[3]{-1})^2 = 1$.

In Problems **25–38**, *evaluate the expressions without the use of a calculator.*

25. 3^0.
26. $2\left(-\frac{2}{3}\right)^0$.
27. 5^{-1}.
28. $(-3)^{-1}$.
29. $4\left(-\frac{2}{3}\right)^{-2}$.
30. $\frac{2^{-1}}{4^{-2}}$.
31. $100^{1/2}$.
32. $64^{1/3}$.
33. $4^{3/2}$.
34. $(25)^{-3/2}$.
35. $(32)^{-2/5}$.
36. $\left(\frac{9}{100}\right)^{3/2}$.
37. $\left(\frac{1}{16}\right)^{5/4}$.
38. $\left(-\frac{27}{64}\right)^{2/3}$.

In Problems **39–87**, *simplify. Use only positive exponents in your answers.*

39. $x^6x^4x^3$.
40. $\frac{x^2x^{20}}{y^5y^2}$.
41. $\frac{(x^2)^5}{(y^5)^{10}}$.
42. $(5x^2)(2x^3)$.
43. $(-2xy^4)^5$.
44. $\left(\frac{2xy^3}{z^2}\right)^2$.

45. $\dfrac{(x^3)^6}{x(x^5)}$.

46. $\dfrac{(xy^2)^2}{(t^3w)^3}$.

47. $(-x)^2(-x)^3$.

48. $\dfrac{(x^2)^3(x^4)^5}{(x^5)^8}$.

49. $\dfrac{(5^7)^2}{(5^4)^4}$.

50. $\dfrac{2^6 2^{11}}{(2^5)^3}$.

51. $-\sqrt{36}+\sqrt{81}$.

52. $\dfrac{-\sqrt{144}}{|-12|}$.

53. $|-2|-\sqrt[4]{16}$.

54. $(\sqrt[3]{2})^3$.

55. $\sqrt{13}\cdot\sqrt{13}$.

56. $\dfrac{(-1)^2-3\sqrt{9}}{|-4|+\sqrt{16}}$.

57. $\dfrac{1}{3}\sqrt[3]{\dfrac{1}{27}}$.

58. $\dfrac{\sqrt[5]{-32}}{\sqrt{9}}$.

59. $\sqrt{0.01}+\sqrt{0.0025}$.

60. $\dfrac{\sqrt{4}-\sqrt[3]{8}}{\sqrt{6}}$.

61. $\sqrt{45}$.

62. $\sqrt{72}$.

63. $\sqrt[4]{810}$.

64. $\sqrt[3]{40}$.

65. $\sqrt{\dfrac{7}{81}}$.

66. $\sqrt{\dfrac{10}{9}}$.

67. $\dfrac{\sqrt[3]{24}}{\sqrt[3]{3}}$.

68. $\dfrac{\sqrt{72}}{\sqrt{2}}$.

69. $\sqrt{18}\cdot\sqrt{2}$.

70. $\sqrt[3]{8}\cdot\sqrt[3]{16}$.

71. $2x^{-1}x^{-3}$.

72. xy^{-1}.

73. $(4t^2)^{-2}$.

74. $(-2z)^{-3}$.

75. $(x^{-2}y^2)^3$.

76. $(ab^2c^3)^{3/4}$.

77. $\dfrac{x^{-2}y^{-6}z^2}{xy^{-1}}$.

78. $(2x^{-1}y)^{-2}$.

79. $\dfrac{x^3y^{-2}}{x^5z^2}$.

80. $2x^{1/2}y^{-3}x^{1/3}$.

81. $(-3x^{1/2}y^{2/3})^3$.

82. $\dfrac{2^0}{(2^{-2}x^{1/2}y^{-2})^3}$.

83. $\left(\dfrac{x^{-2}}{w^2y^{-2}}\right)^{-4}$.

84. $(9z^6)^{1/2}$.

85. $(16y^8)^{3/4}$.

86. $\left(\dfrac{27t^3}{8}\right)^{2/3}$.

87. $\left(\dfrac{1000}{a^9}\right)^{-2/3}$.

88. Evaluate

$$\frac{\sqrt[4]{(0.00593)^2(23.5)^4}}{\sqrt{(4.71\times10^{-4})(503)^{0.34}}}.$$

In Problems **89–92**, *rationalize the denominator.*

89. $\dfrac{6}{\sqrt{3}}$.

90. $\dfrac{2}{\sqrt{8}}$.

91. $\sqrt{\dfrac{8}{11}}$.

92. $\sqrt{\dfrac{1}{3}}$.

In Problems **93–96**, *write the numbers in scientific notation.*

93. 0.0000564.

94. 45,030,000,000.

95. 28,000,000.

96. 0.007007.

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97. The elastic potential energy PE (in joules) of a spring is given by

$$PE = \tfrac{1}{2}kx^2,$$

where k is the spring constant (in newtons per meter) and x is the displacement of the spring (in meters) from the equilibrium position ($x = 0$). Find the potential energy of a spring in each of the following cases.

(a) $k = 3.2$, $x = 0.2$.
(b) $k = 4$, $x = -0.2$,
(c) $k = 2$, $x = -1.2$.

98. The area A of a circle of radius r is given by $A = \pi r^2$. By what factor does the area increase if the radius is (a) tripled, and (b) halved?

99. The frequency of vibration of a string of length L, fixed at both ends and vibrating in its fundamental mode, is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}},$$

where f is frequency, μ (mu) is mass per unit length, and T is the tension in the string. If the tension in the string is quadrupled, what happens to the frequency?

100. The resistivity of silver is 0.000000016 ohm · meter. A coil is to be made by using 2500 meters of wire. If the wire has a circular cross section of radius 5×10^{-2} meter, find the resistance of the coil. (Refer to Problem 39 in Sec. 2.3.)

In this chapter and the one that follows, the *language of algebra* is introduced. Like any written language, algebra uses many special symbols and technical words. It is very important that you learn the names of these symbols and become familiar with the vocabulary of algebra. Once this is done, you will then be ready to solve equations and work with the *techniques of algebra.*

4–1 VARIABLES, SUBSCRIPTS, AND PRIMES

Variables

As your understanding of algebra grows, so will your appreciation of how algebraic methods are used to complement arithmetic methods for solving problems. The effectiveness of algebra comes from its use of letters to stand in for numbers. These letters are called **variables.** *Variable* is the technical term for a letter which is used in place of a number. The number represented by a letter may be known or unknown, it may be constant or changing. The important idea is that the letter represents a number and obeys the laws of numbers.

The letters most commonly used as variables in electronic problems are taken from the letters of the English and Greek alphabets. Table 4–1 is a partial listing of the Greek alphabet. It contains the letters commonly associated with electronic applications. Since these letters occur in electronics, you should learn their names and how to write them. Appendix A contains a complete listing of the Greek alphabet.

TABLE 4–1 Greek Letters Commonly Associated with Electronic Applications

Name	Capital	Lowercase	Used to Designate:
Alpha		α	Angles, temperature coefficient, current transfer ratio
Beta		β	Angles, current transfer ratio
Delta	Δ		Increment, determinant of coefficients
Eta		η	Efficiency
Theta		θ	Phase angle, thermal impedance
Lambda		λ	Wavelength
Mu		μ	Permeability, micro, amplification factor
Pi		π	3.1416 (ratio of circumference of circle to diameter)
Rho		ρ	Specific resistance
Sigma	Σ	σ	Summation (capital), conductivity (lowercase)
Phi	Φ	ϕ	Magnetic flux (capital), phase angles and angles (lowercase)
Psi		ψ	Electric flux
Omega	Ω	ω	Ohms (capital), angular velocity (lowercase)

Subscripts

In naming variables for electronics use, sometimes there is a need for the same letter to be used more than once. In this case, a subscript or a prime is used. A **subscript** is a number, a letter, or a group of letters written to the right and below a variable.

EXAMPLE 4–1 Using the letter V for voltage, select variables to represent three different voltages.

Solution Since each voltage is different, a subscript will be used with each variable. The numbers 1, 2, 3, or the letters *a, b, c* could be used as subscripts. Each is shown:

V_1 read "V one" or "V sub one"
V_2 read "V two" or "V sub two"
V_3 read "V three" or "V sub three"
V_a read "V a" or "V sub a"
V_b read "V b" or "V sub b"
V_c read "V c" or "V sub c"

Primes

A **prime** is an apostrophe mark made to the *right* and *above* a variable. It is placed in the same position as an exponent. It is used to distinguish variables with the same name.

EXAMPLE 4–2 Using the Greek letter β, select variables to represent two corresponding angles in two triangles.

Solution Use β to represent the angle in the first triangle. Use β', "beta prime," for the angle in the second triangle. See Figure 4–1.

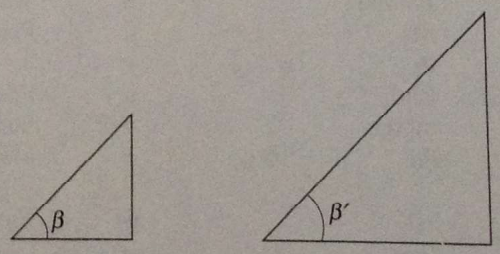


FIGURE 4–1 The two corresponding angles are labeled β and β'.

EXERCISE 4–1

Read each of the following variables.

1. L_1	**2.** I_{max}	**3.** G_m	**4.** k'
5. R_p	**6.** ω_0	**7.** R'	**8.** β_2
9. μ	**10.** m'	**11.** R_{ac}	**12.** i_p
13. e'	**14.** C_{cb}	**15.** λ_t	**16.** A'_v

4–2 INDICATING MULTIPLICATION

In arithmetic, multiplication was indicated by the multiplication sign ($\times$). When we indicate multiplication in algebra, we may use the multiplication sign ($A \times B$), a dot ($A \cdot B$), parentheses (A)(B), or no sign at all (AB). These four ways of indicating multiplication are summarized in Table 4–2. Note that we cannot omit the multiplication sign when multiplying numbers because they would be misunderstood. For example, we would understand 32 (*three two*) as the number thirty-two and not as three times two.

When a multiplication sign ($\times$) is used to indicate multiplication, care must be taken to make it look different from the letter x. When using the dot to indicate multiplication, be careful to place it above the writing line so that it does not appear as a decimal point. For example, $2 \cdot B$, not $2 . B$.

TABLE 4–2 Examples Indicating Multiplication

Symbol	Possible Ways of Expression		
$\times$	3×2	$2 \times b$	$\rho \times \omega$
$\cdot$	$3 \cdot 2$	$2 \cdot b$	$\rho \cdot \omega$
()	(3)(2)	(2)(b)	$(\rho)(\omega)$
None	None	$2b$	$\rho\omega$

EXERCISE 4–2

Write the following in two other ways.

1. 2×3	**2.** $x \cdot z$	**3.** 7×2	**4.** $5b$
5. $(b)c$	**6.** $m \cdot n$	**7.** $\alpha\beta$	**8.** $3h$
9. $\Delta \cdot \Sigma$	**10.** $9 \times R$	**11.** $\lambda\eta$	**12.** (4)(2)
13. (6)(c)	**14.** $r \cdot s$	**15.** $\theta \times \phi$	**16.** $3h_{oe}$

4–3 GENERAL NUMBERS

Numbers that are represented by letters are called **literal numbers** or **general numbers.** By using general numbers rather than words, electrical formulas can be easily stated. Ohm's law, an important formula of electronics, is easily expressed as $E = IR$. To express this same idea in words, the following statement would be needed. One volt of electromotive force (E) causes one ampere of current (I) to flow through a resistance (R) of one ohm. As you can see, written expressions are cumbersome and more easily misunderstood, whereas formulas written with general numbers are concise and less likely to be misunderstood.

4–4 ALGEBRAIC EXPRESSIONS

An **algebraic expression** is a collection of variables, numbers, and exponents connected by operators ($+, -, \times, /, \sqrt{\ }$) and signs of grouping. Table 4–3 shows several examples of algebraic expressions.

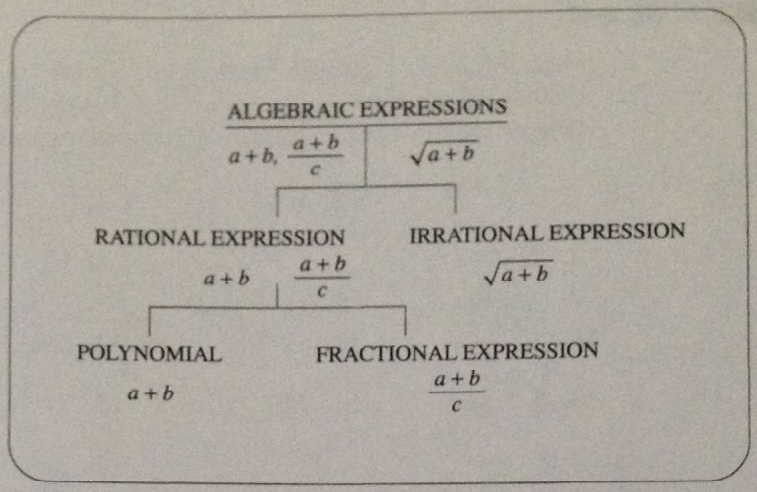
TABLE 4–3 Examples of Algebraic Expressions

(1) $\sqrt{8y}$	(2) $8y + 5$	(3) $(x + 2)(x - 3)$
(4) $x^{3.5}$	(5) $2/y$	(6) $\frac{1}{2}$
(7) $\sqrt{2x + y} + 29$	(8) $3x$	(9) $\frac{R_1 + 2}{3}$
(10) $5x^2 - \frac{1}{3}x - 5$	(11) 3	(12) $\frac{R_b + R}{\alpha + \beta'} + \theta$

Algebraic expressions are classified by mathematicians in several ways. One way is to describe them as *rational expressions* or *irrational expressions.* A **rational expression** is an algebraic expression that contains no radical signs and has only whole numbers as exponents. An **irrational expression** is an algebraic expression that contains a fractional exponent or a radical sign. In Table 4–3, Examples (1), (4), and (7) are irrational expressions; the rest are rational expressions.

Rational expressions are further classified as either polynomial or fractional expressions. A rational expression that has no variable as a divisor (no division with a variable) is called a **rational integral expression,** or simply a **polynomial.** A rational expression that has a variable as a divisor (division with a variable) is called a **fractional expression.** In Table 4–3, examples (2), (3), (6), and (8) through (11) are polynomials. Examples (5) and (12) are fractional expressions. Figure 4–2 summarizes the classification of algebraic expressions.

FIGURE 4–2 Classification of algebraic expressions.



EXAMPLE 4–3 In the following algebraic expressions, list all the terms that apply to each. Are they irrational, rational, polynomial, or fractional expressions?

(a) $3x + y/2$ (b) $\dfrac{1}{x+y}$ (c) $\sqrt{4x-3}$

Solution

(a) $3x + y/2$ rational, polynomial expression

(b) $\dfrac{1}{x+y}$ rational, fractional expression

(c) $\sqrt{4x-3}$ irrational expression

Now that you have an understanding of algebraic expressions, the *general number* can be fully defined. Every number, variable, rational expression, and irrational expression is a **general number.** Because each of these (number, variable, rational expression, and irrational expression) is a numerical quantity, each follows all the rules for numbers.

EXAMPLE 4–4 List several general numbers.

Solution Because every number, variable, rational expression, and irrational expression is a general number, all entries in Table 4–3 are general numbers. Examples would be 6.32, x, $5/x$, and $\sqrt{3/5}$.

EXERCISE 4–3

In the following algebraic expressions, list all the terms that apply to each. Describe whether each is a rational or an irrational expression. If rational, then further indicate whether each is a polynomial or a fractional expression.

1. 12.7	**2.** $\sqrt{5x}$	**3.** $\dfrac{a^2 + 3b}{5y}$
4. $5x + \sqrt{6y}$	**5.** $1 + \sqrt{y}$	**6.** $1/x$
7. $\eta_o + 2$	**8.** $7\pi + (x/3)$	**9.** $\sqrt{3x}$
10. $\sqrt{x + y}$	**11.** $4x^{0.5}$	**12.** $(3/x) + \sqrt{5y}$
13. $5 + 3\mu$	**14.** 7.0	**15.** $5x^2 + \dfrac{1}{3x}$

4–5 PRODUCTS, FACTORS, AND COEFFICIENTS

A *product* is formed when two or more general numbers (numbers, variables, rational expressions) are multiplied together. Thus, $3y$ is a product; it is the result of multiplying 3 (a number) times y (a variable).

The general numbers that are used to form a product are called the *factors* of the product. Since $3 \times y = 3y$, 3 and y are factors of $3y$. Three is called the *numerical factor; y* is called the *literal factor.*

A **coefficient of a product** is any factor of a product. Each factor is the coefficient of the other factor. In the product $3y$, the 3 is the coefficient of y and y is the coefficient of 3. The numerical factor of a product is called the *numerical coefficient.* If a variable has no written numerical coefficient, then it has an **implied coefficient** of 1. For example, T means $1T$ and rs means $1rs$.

EXAMPLE 4–5 In the product $5ab$, state the coefficient of:

(a) ab (b) 5 (c) a

Solution

(a) The coefficient of ab is 5.
(b) The coefficient of 5 is ab.
(c) The coefficient of a is $5b$.

EXERCISE 4–4

State the numerical coefficient of each product.

1. $3x$	**2.** $\frac{2}{5}\mu$	**3.** ab
4. $\frac{1}{2}zy$	**5.** b	**6.** $0.3c$

7. $2y\alpha$ 8. 0.8ω 9. $\frac{1}{3}s$

10. 1.2λ 11. $\frac{7}{4}\beta$ 12. 16η

State the coefficient of π in each of the following problems.

13. 2π 14. πb 15. πr^2

16. $2\pi fL$ 17. $5b\pi$ 18. $\sqrt{\pi S}$

19. $16\pi d^2$ 20. π 21. $7.4\pi\omega$

4–6 COMBINING LIKE TERMS

An algebraic expression made up of a number, a variable, a product, or a quotient is called a **term.** The algebraic expression $5y$ is a term, as are 7, $-2cd$, $b(2 + c)$, x^2, β, $3a/5$, $1/x$, and $\sqrt{5x}$. Notice that both rational and irrational algebraic expressions may be terms.

Like Terms

Terms that have the same literal factors, but different numerical coefficients, are called **like terms.** Examples are $5a$ and $2a$, 8μ and 2μ, $2x^2y$ and $0.5x^2y$. Like terms can be added and subtracted. For example, $2R + 5R = 7R$.

RULE 4–1. Combining Like Terms

To combine *like terms:*

1. Combine the numerical coefficients.
2. Form the new term from the combined numerical coefficients and the literal factor.

EXAMPLE 4–6 Simplify $2a + 5a$ by combining like terms.

Solution Use the steps of Rule 4–1.

Step 1: Combine numerical coefficients:

$$2 + 5 = 7$$

Step 2: Form new term:

7a

$\therefore$ $2a + 5a = 7a$

EXAMPLE 4–7 Simplify $3a - 4a$ by combining like terms.

Solution Follow the steps of Rule 4–1.

Step 1: Combine numerical coefficients:

$$3 - 4 = -1$$

Step 2: Form new term:

$$-1a$$

$\therefore$ $3a - 4a = -a$

EXERCISE 4–5

Simplify each of the following algebraic expressions by combining like terms.

1. $2a + 3a$
2. $7\omega - 3\omega$
3. $9x + 3x$
4. $-ab - 6ab$
5. $-c - 6c$
6. $7y - (-8y)$
7. $5b - (+7b)$
8. $2\mu - 3\mu + 7\mu$
9. $4h + 3h - 2h$
10. $4k - 9k - k$
11. $4A + 3B - B + 6A$
12. $16x + 5y - 4x$
13. $c^2b - 4c^2b$
14. $x^2y^2 - 3x^2y^2$
15. $5x^2 - 3x - 7 - 2x^2 + 6x + 9$
16. $5y^2 + 3y - 8y^2 + 2y$
17. $2x - 3x^3 + 2x^2 - 4x - x^2 + x^3$
18. $9x^2y - 2xy^2 - 3x^2y$
19. $3xy^2 - 5x^2y - 3xy^2$
20. $2y - 3y^2 - 4y^3 - 2y + 4y^3 - y^2$

4–7 POLYNOMIALS

Polynomials are sums of products. Certain polynomials have special names. If the polynomial has only one term, it is called a **monomial.** If the polynomial has two terms, it is called a **binomial.** A three-term polynomial is called a **trinomial.** Special names are not used for polynomials with more than three terms. Table 4–4 is a list of sample polynomials.

TABLE 4–4 Sample Polynomials

Special Name	Example
Monomial	$3y^2$
Binomial	$7y^2 + 4$
Trinomial	$m^2 - mn + n^2$

EXERCISE 4–6

Of the following rational expressions, which can be classified as (a) monomials, (b) binomials, (c) trinomials, (d) fractional expressions?

1. $5h$
2. $2a^2 + 3a + 5$
3. $\pi + 2\lambda$
4. $27y^3$
5. $\dfrac{e + ir}{4x}$
6. 28
7. $E_1 + E_2 + IR$
8. $\frac{5}{8}$
9. $x + 5y$
10. $\dfrac{\mu + e_o}{e_i}$
11. $\dfrac{\eta_1 + \alpha_0}{\Sigma}$
12. $3\psi^3 + 2\psi^2 + \psi$

4–8 ADDING POLYNOMIALS

When adding polynomials, there are two important concepts to keep in mind. First, two terms may be added in either order. Thus, the sum $2x + 3x$ is $5x$; also the sum $3x + 2x$ is $5x$. This concept is called the **commutative law of addition.** Second, three terms may be grouped in either manner to indicate which addition is to be performed first. Thus, $(5x + 3x) + 4x = 5x + (3x + 4x)$. This concept is called the **associative law of addition.** In summary, **addition is independent of order and of grouping.**

EXAMPLE 4–8 Simplify $6x^2 + 5 + 3x + 5x + 2x^2 + 4$.

Solution Apply the commutative and associative laws in order to group like terms:

$$(6x^2 + 2x^2) + (5 + 4) + (3x + 5x)$$

Combine like terms:

$$8x^2 + 9 + 8x$$

$$\therefore \quad 6x^2 + 5 + 3x + 5x + 2x^2 + 4 = 8x^2 + 9 + 8x$$

RULE 4–2. Adding Polynomials

To add polynomials:

1. Arrange the terms in both polynomials in the same order.
2. Write like terms under each other.
3. Add each column, connecting the sums with their correct sign.

EXAMPLE 4–9 Add $6x^2 + 5 + 3x$ and $5x + 2x^2 + 4$.

Solution Follow the steps of Rule 4–2.

Step 1: Arrange terms:

$$(6x^2 + 3x + 5) + (2x^2 + 5x + 4)$$

Step 2: Place like terms in columns:

$$\begin{array}{r} 6x^2 + 3x + 5 \\ 2x^2 + 5x + 4 \\ \hline \text{Add: } 8x^2 + 8x + 9 \end{array}$$

Step 3:

$\therefore$ $6x^2 + 5 + 3x + 5x + 2x^2 + 4 = 8x^2 + 8x + 9$

EXAMPLE 4–10 Add $3m + 6n - 4m^2$ and $2 + 6m^2 - 8n$.

Solution Use the steps of Rule 4–2.

Step 1: Arrange terms:

$$(-4m^2 + 3m + 6n) + (6m^2 - 8n + 2)$$

Step 2: Place like terms in columns:

$$\begin{array}{r} -4m^2 + 3m + 6n \\ 6m^2 \quad - 8n + 2 \\ \hline \text{Add: } 2m^2 + 3m - 2n + 2 \end{array}$$

Step 3:

$\therefore$ $3m + 6n - 4m^2 + 2 + 6m^2 - 8n = 2m^2 + 3m - 2n + 2$

EXAMPLE 4–11 Add $-\theta^3 - 7 + \theta^2$ and $-\theta^2 + \theta^3 + 5$ and $-2 - \theta^3 + \theta^2$.

Solution Use the steps of Rule 4–2.

Step 1: Arrange terms:

$$(-\theta^3 + \theta^2 - 7) + (\theta^3 - \theta^2 + 5) + (-\theta^3 + \theta^2 - 2)$$

Step 2: Place like terms in columns:

$$\begin{array}{r} -\theta^3 + \theta^2 - 7 \\ \theta^3 - \theta^2 + 5 \\ -\theta^3 + \theta^2 - 2 \\ \hline \text{Add: } -\theta^3 + \theta^2 - 4 \end{array}$$

Step 3:

$\therefore$ $-\theta^3 - 7 + \theta^2 + -\theta^2 + \theta^3 + 5 + -2 - \theta^3 + \theta^2 = -\theta^3 + \theta^2 - 4$

EXERCISE 4–7

Read each of the following polynomials, and state the terms in descending powers of the variables.

1. $3x^2 + 2 + 2x$
2. $5y^2 + 4 + 8y$
3. $9 + 2x^2 + 4x$
4. $6 + 3\phi^2 + 13\phi$
5. $3n + 2n^2 - 11$
6. $5\eta - 2 + 2\eta^2$
7. $-8V + 2 - V^2 + 2V^3$
8. $-5 + 3\alpha^3 - \alpha^2 + 4\alpha$
9. $x^2 + 3y^2 + 2xy$
10. $\omega^3 + 2\omega^2\pi + 2\pi^2 + 3 + \omega\pi^2$

Add the following.

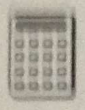
11. $2y + 3$
    $4y + 6$
12. $3x + 5$
    $2x - 4$
13. $x^2 + 3$
    $x^2 - 3$
14. $\mu - \beta^2$
    $\mu + \beta^2$
15. $2x + 4y + z$
    $x - 2y + 3z$
16. $3\psi^2 + 4\psi - 1$
    $\psi^2 + 1$
17. $V^3 + 2V^2 - V$
    $V^2 + V + 5$
18. $c^3 - 2c^2 + c$
    $-2c^3 + 3c - 9$
19. $2x + y - 3z$
    $-2y + 4z$
    $-3x - z$
20. $-\alpha^3 + \alpha^2 - 7\alpha$
    $2\alpha^3 - \alpha^2 + 4\alpha$
    $-\alpha^3 + 2\alpha^2 - 2\alpha$
21. $n + 5$ and $2n + 3$
22. $2\omega - 7$ and $3\omega + 7$
23. $3x^3 - x^2$ and $x^3 - 4x$
24. $5\mu^4 + 3\mu^2$ and $\mu^3 - 8\mu^2$
25. $4m - 2n$ and $3m + 3n$
26. $-3t + S$ and $2t + 4S$
27. $3x^2 - 2x$ and $-x^2 - 3x$
28. $5y^2 - 2xy$ and $-10y^2 + 14xy$
29. $7.2\omega - 3.1\rho$ and $0.8\omega + 5.7\rho$
30. $-4.2V + 0.5W$ and $3.9W - 1.8V$

Simplify each expression by adding like terms.

31. $(x^3 - 4x^2 - 7x + 6) + (2x^3 - 5x + 2)$
32. $(m^2 - mn + n^2) + (4n^2 + m^2 - 3mn)$
33. $(6a^3 - 5a) + (7a - 11a^3 + 5b^2)$
34. $(4x^2 - 2x^3 + 3x + 1) + (x^3 + 3x^2 - 1 - 5x)$
35. $(-3t^2 + t^2 - 2 + 5t) + (-5t - t^2 + 2 + 3t^2)$
36. $(2x + 4y - 12) + (-5y + 7x - 2) + (-1 - 3y + 4x)$
37. $(7\beta - 9 + \theta) + (-5\alpha + \theta) + (3\alpha - 2\theta + 8\beta)$
38. $(-3y^2 - 3y + 2) + (-3y^2 + 7) + (-7y - 5y^2)$

39. $(x - y - z) + (2z - 5x + 3y) + (2y - 7x + 2z)$

40. $(7\lambda^2 - 6 + 7\lambda) + (7\lambda - 4 + \lambda^2) + (-1 + 2\lambda^2)$



Calculator Drill

Work the following problems and express the coefficients in the answers in decimal notation to the appropriate precision. All numbers are approximate.

41. $(9.26x^2 - 4.25x + 8.25) + (-3.81x^2 + 5.01x + 9.88)$

42. $(30.1x^2 + 22.8x - 14.3) + (5.11x^2 - 24.3x - 13.4)$

43. $(10.9x^2 - 7.65x + 4.33) + (18.88x^2 - 7.52x + 7.99)$

44. $(52.4x^2 - 14.2x - 46.5) + (-12.3x^2 - 5.53x + 18.3)$

SELECTED TERMS

associative law of addition Three terms may be grouped in either way to indicate which addition is performed first; thus, $(a + b) + c = a + (b + c)$.

coefficient of a product Any factor of a product; each factor is the coefficient of the other factor.

commutative law of addition Two terms may be added in either order; thus, $a + b = b + a$.

fractional expression Algebraic expression having a variable in the denominator of one or more terms.

irrational expression Algebraic expression containing a fractional exponent or a radical sign.

like terms Having the same literal factors but different numerical coefficients.

polynomial Rational expression having no variable as a divisor.

rational expression Algebraic expression having whole numbers as exponents and containing no radical signs.

variable Symbol for a number that may be constant or varying in an algebraic expression or equation.

INTRODUCTION

The properties that allow us to change the way algebraic expressions look without changing their values or their characteristics are very important. Expressions often need a different format so that we can simplify them or enter them into a calculator or a computer.

In this chapter we will study factoring, simplifying algebraic fractions, and solving equations with algebraic fractions.

Objectives

- In an algebraic expression, find the greatest common factor.
- Identify and factor the difference of two squares.
- Identify and factor the square of a binomial.
- Factor trinomials.
- Factor using the method of grouping.
- Identify and factor the sum or difference of two cubes.
- Add, subtract, multiply, and divide algebraic fractions.
- Solve equations containing algebraic fractions.

5.1 SPECIAL PRODUCTS

In Chapter 2 we introduced certain fundamental algebraic concepts and operations. Before proceeding, we must develop some additional algebraic techniques that are necessary in the development of later topics.

Certain types of algebraic products occur so often that we must be able to recognize them on sight. We must also be able to do the multiplications quickly and mentally.

To illustrate the first two types of special algebraic products, we give the general form of each followed by two examples.

$$\boxed{a(x + y + z) = ax + ay + az}$$

EXAMPLE 1

$$3x(x^2 + 2x - 4) = 3x^3 + 6x^2 - 12x$$

EXAMPLE 2

$$4a^2b^3(7a^4b^3 - 5a^2b^3 - ab + 3a^2) = 28a^6b^6 - 20a^4b^6 - 4a^3b^4 + 12a^4b^3$$

$$\boxed{(x + y)(x - y) = x^2 - y^2}$$

EXAMPLE 3

$$(3a + 4)(3a - 4) = 9a^2 - 16$$

EXAMPLE 4

$$(6a^2 + 5xy^3)(6a^2 - 5xy^3) = 36a^4 - 25x^2y^6$$

There are two general forms of the *square of a binomial.*

$$\boxed{(x + y)^2 = x^2 + 2xy + y^2}$$

EXAMPLE 5

$$(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$$

EXAMPLE 6

$$(5ab^2 + 4c)^2 = 25a^2b^4 + 40ab^2c + 16c^2$$

$$\boxed{(x - y)^2 = x^2 - 2xy + y^2}$$

EXAMPLE 7

$$(3x - 4)^2 = 9x^2 - 24x + 16$$

EXAMPLE 8

$$(2a^2 - 5y^3)^2 = 4a^4 - 20a^2y^3 + 25y^6$$

The *product of two binomials* may be found mentally using the FOIL method discussed in Section 2.3.

EXAMPLE 9

$$(x + 4)(x + 6) = x^2 + 10x + 24$$

EXAMPLE 10

$$(x + 3)(x - 5) = x^2 - 2x - 15$$

EXAMPLE 11

$$(2x + 3)(5x + 4) = 10x^2 + 23x + 12$$

EXAMPLE 12

$$(3x - 4)(-2x + 7) = -6x^2 + 29x - 28$$

The general form of the *square of a trinomial* is as follows.

$$\boxed{(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz}$$

EXAMPLE 13

$$(4x + y + 3z)^2 = 16x^2 + y^2 + 9z^2 + 8xy + 24xz + 6yz$$

The *square of a trinomial* may also be found using grouping and the square of a binomial as follows for Example 13:

$$\begin{aligned}(4x + y + 3z)^2 &= [(4x + y) + 3z]^2 \\ &= (4x + y)^2 + 2(4x + y)(3z) + (3z)^2 \\ &= 16x^2 + 8xy + y^2 + 24xz + 6yz + 9z^2\end{aligned}$$

EXAMPLE 14

$$(3a - b^2 - 2c^3)^2 = 9a^2 + b^4 + 4c^6 - 6ab^2 - 12ac^3 + 4b^2c^3$$

Or

$$\begin{aligned}(3a - b^2 - 2c^3)^2 &= [(3a - b^2) - 2c^3]^2 \\ &= (3a - b^2)^2 - 2(3a - b^2)(2c^3) + (2c^3)^2 \\ &= 9a^2 - 6ab^2 + b^4 - 12ac^3 + 4b^2c^3 + 4c^6\end{aligned}$$

There are two general forms of the *cube of a binomial.*

$$\boxed{(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3}$$

EXAMPLE 15

$$\begin{aligned}(2a + 5b)^3 &= (2a)^3 + 3(2a)^2(5b) + 3(2a)(5b)^2 + (5b)^3 \\ &= 8a^3 + 60a^2b + 150ab^2 + 125b^3\end{aligned}$$

EXAMPLE 16

$$\begin{aligned}(4x + y)^3 &= (4x)^3 + 3(4x)^2(y) + 3(4x)(y)^2 + (y)^3 \\ &= 64x^3 + 48x^2y + 12xy^2 + y^3\end{aligned}$$

$$\boxed{(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3}$$

EXAMPLE 17

$$(4a - b)^3 = (4a)^3 - 3(4a)^2(b) + 3(4a)(b)^2 - (b)^3$$
$$= 64a^3 - 48a^2b + 12ab^2 - b^3$$

EXAMPLE 18

$$(3x - 2y)^3 = (3x)^3 - 3(3x)^2(2y) + 3(3x)(2y)^2 - (2y)^3$$
$$= 27x^3 - 54x^2y + 36xy^2 - 8y^3$$

Exercises 5.1

Find each product mentally.

1. $-8x^2(5x^3 - 9x^2 + 10x)$
2. $5x^2(3x^5 - 4x^3 + 2x^2 - 6x + 7)$
3. $6x^3y^5(4xy - 7x^3y^2 + 9x^4)$
4. $6a^2b^4(-2a^2b^2 + 5ab^2 - 9b^3)$
5. $(2x + 7)(2x - 7)$
6. $(3x^2 + 8yz^2)(3x^2 - 8yz^2)$
7. $(7x^2 + 2y)^2$
8. $(8a^2b + 5c)^2$
9. $(2a^2 - 3b)^2$
10. $(9x - 4y^3)^2$
11. $(x + 4)(x + 12)$
12. $(x - 9)(x + 8)$
13. $(2x + 5)(-3x - 8)$
14. $(4x - 7)(5x - 6)$
15. $(2a - b + 3c)^2$
16. $(-3a + 2b - c)^2$
17. $(2a + b)^3$
18. $(4x + 5y)^3$
19. $(5 - 2x^2)^3$
20. $(2x - 4y^3)^3$
21. $(6a + 5b)(7a - 3b)$
22. $(5x - y)(4x - 8y)$
23. $(4x + 7y)^2$
24. $(2x - 10y)^2$
25. $(2a + 9b)(6a - 11b)$
26. $(2x^2y + 7z^3)^2$
27. $(3a^3 + 10b^2)^3$
28. $(8 - a^2b^3)^2$
29. $-4a^2b^3(6a^3b^4 - 9a^5b^7 + 20a^5)$
30. $(2 + a)(4 - 2a)$
31. $(9a^3 - 12b^2)^2$
32. $(12a - 5b)(15a + 2b)$
33. $(x - 3y - 5z)^2$
34. $(-3x + 7)(7x + 3)$
35. $(9a - 4b)(6a - 10b)$
36. $(80 - a^5)(80 + a^5)$
37. $(6x^3 + 4)(5x^3 - 4)$
38. $(4x^2 - 5)(3x^2 + 7)$
39. $(3xy + 9yz)(5xy - 8yz)$
40. $(-5y + z)^2$
41. $(5abc + 6)(5abc - 6)$
42. $(1 - a^2)^2$
43. $(3x^2 - \frac{1}{2}y)(x^2 + \frac{2}{3}y)$
44. $(2x^2 + \frac{1}{4}y^3)(2x^2 + \frac{1}{2}y^3)$
45. $(x - \frac{2}{3}y)^2$
46. $(4x^2 - \frac{1}{2}y)^2$
47. $x^2(1 + x^2)^2$
48. $x^2(1 - x^2)^3$

5.2 FACTORING ALGEBRAIC EXPRESSIONS

Factoring is the process of writing an algebraic expression as a product of two or more factors*—usually prime factors. The process is the opposite of multiplying algebraic expressions; that is, the reverse of the product forms in the previous section.

*Recall that in the product ab, a and b are called **factors.**

The first type of factoring involves finding the largest monomial factor that divides each factor "evenly," †dividing each term by this factor, and then writing the result as a product.

GREATEST COMMON FACTOR

$$ax + ay + az = a(x + y + z)$$

EXAMPLE 1

Factor $4x + 8y$.

$$4x + 8y = 4(x + 2y)$$

EXAMPLE 2

Factor $21x^3 - 14x^2 + 28x$.

$$21x^3 - 14x^2 + 28x = 7x(3x^2 - 2x + 4)$$

EXAMPLE 3

Factor $50xy^2z - 80xyz - 32xz$.

$$50xy^2z - 80xyz - 32xz = 2xz(25y^2 - 40y - 16)$$

Note: You should always try this type of factoring first, because many times it simplifies the remaining factor into something that factors more easily.

The difference of two perfect squares is the product of the square root of the first *plus* the square root of the second times the square root of the first *minus* the square root of the second.

DIFFERENCE OF TWO PERFECT SQUARES

$$x^2 - y^2 = (x + y)(x - y)$$

EXAMPLE 4

Factor $9a^2 - b^2$.

$$9a^2 - b^2 = (3a + b)(3a - b)$$

EXAMPLE 5

Factor $36x^2 - 49y^4$.

$$36x^2 - 49y^4 = (6x + 7y^2)(6x - 7y^2)$$

EXAMPLE 6

Factor $x^2 - 4a^2b^2$.

$$x^2 - 4a^2b^2 = (x + 2ab)(x - 2ab)$$

†The remainder is zero after dividing.

Note: The sum of two perfect squares, $x^2 + y^2$, cannot be factored.

EXAMPLE 7

Factor $81x^4 - y^8$.

$$81x^4 - y^8 = (9x^2 + y^4)(9x^2 - y^4)$$
$$= (9x^2 + y^4)(3x + y^2)(3x - y^2)$$

Note: To factor this expression completely, we must also factor $9x^2 - y^4$.

The factors of a general trinomial are often binomial factors. To find these binomial factors, you "undo" the FOIL multiplication.

EXAMPLE 8

Factor $x^2 + 7x + 10$.

This type of factoring involves some trial and error. Here we need two numbers whose sum is 7 and whose product is 10. They are $+2$ and $+5$.

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

EXAMPLE 9

Factor $x^2 - 9x + 18$.

We need two numbers whose sum is -9 and whose product is 18: -3 and -6.

$$x^2 - 9x + 18 = (x - 3)(x - 6)$$

EXAMPLE 10

Factor $x^2 - 3x - 28$.

We need two numbers whose sum is -3 and whose product is -28: $+4$ and -7.

$$x^2 - 3x - 28 = (x + 4)(x - 7)$$

A short summary about the signs in trinomials might be helpful. If the trinomial to be factored is one of the following forms, use the corresponding sign patterns.

1. $x^2 + px + q = (x + \quad)(x + \quad)$
2. $x^2 - px + q = (x - \quad)(x - \quad)$
3. $x^2 + px - q = (x + \quad)(x - \quad)$
4. $x^2 - px - q = (x + \quad)(x - \quad)$

EXAMPLE 11

Factor $x^2 + 10x + 24$.

We need two numbers whose sum is 10 and whose product is 24: $+4$ and $+6$.

$$x^2 + 10x + 24 = (x + 4)(x + 6)$$

EXAMPLE 12

Factor $x^2 - 15x + 56$.

We need two numbers whose sum is -15 and whose product is 56: -7 and -8.

$$x^2 - 15x + 56 = (x - 7)(x - 8)$$

EXAMPLE 13

Factor $x^2 + 3x - 40$.

We need two numbers whose sum is 3 and whose product is -40: $+8$ and -5.

$$x^2 + 3x - 40 = (x + 8)(x - 5)$$

EXAMPLE 14

Factor $x^2 - 7x - 18$.

We need two numbers whose sum is -7 and whose product is -18: $+2$ and -9.

$$x^2 - 7x - 18 = (x + 2)(x - 9)$$

EXAMPLE 15

Factor $8x^2 + 14x + 3$.

When the coefficient of the squared term is not 1, the factoring involves more trial and error. Here, we need the following:

1. Pairs of factors whose product is $8x^2$. There are two possible pairs:

$$(8x \quad)(x \quad) \text{ and } (2x \quad)(4x \quad)$$

2. Pairs of factors whose product is 3. There is only one pair: $3 \cdot 1$.
3. To arrange these factors so that the middle term is $14x$. If the coefficient of the squared term is positive, the signs in the factors take the same forms as discussed in the summary after Example 10. The possibilities are as follows.

Factors	*Middle term*	
$(8x + 1)(x + 3)$	$25x$	
$(8x + 3)(x + 1)$	$11x$	
$(2x + 1)(4x + 3)$	$10x$	
$(2x + 3)(4x + 1)$	$14x$	(the correct combination)

Therefore, $8x^2 + 14x + 3 = (2x + 3)(4x + 1)$.

EXAMPLE 16

Factor $12x^2 - 28x + 15$.

Step 1: The pairs of factors of $12x^2$ are

$$(12x \quad)(x \quad), \quad (2x \quad)(6x \quad), \text{ and } (3x \quad)(4x \quad)$$

Step 2: The pairs of factors of 15 are $1 \cdot 15$ and $3 \cdot 5$.

Step 3: Arrange these factors so that the middle term is $-28x$. The possibilities are as follows:

Factors	*Middle term*	
$(12x - 1)(x - 15)$	$-181x$	
$(12x - 15)(x - 1)$	$-27x$	
$(12x - 3)(x - 5)$	$-63x$	
$(12x - 5)(x - 3)$	$-41x$	
$(2x - 1)(6x - 15)$	$-36x$	
$(2x - 15)(6x - 1)$	$-92x$	
$(2x - 3)(6x - 5)$	$-28x$	(the correct combination)
$(2x - 5)(6x - 3)$	$-36x$	

Factors	*Middle term*
$(3x - 1)(4x - 15)$	$-49x$
$(3x - 15)(4x - 1)$	$-63x$
$(3x - 3)(4x - 5)$	$-27x$
$(3x - 5)(4x - 3)$	$-29x$

Note: Both signs inside the factors must be negative. Therefore,

$$12x^2 - 28x + 15 = (2x - 3)(6x - 5)$$

Of course, we do not go through this long listing process for every factoring. The work is often done mentally, and we stop when we have found the combination of factors that gives the desired middle term.

EXAMPLE 17

Factor $12x^2 - 7x - 12$.

$$12x^2 - 7x - 12 = (4x + 3)(3x - 4)$$

EXAMPLE 18

Factor $30x^2 - 59x + 9$.

$$30x^2 - 59x + 9 = (5x - 9)(6x - 1)$$

EXAMPLE 19

Factor $10x^2 + 27xy - 9y^2$.

$$10x^2 + 27xy - 9y^2 = (10x - 3y)(x + 3y)$$

EXAMPLE 20

Factor $4x^2 + 28xy^2 + 45y^4$.

$$4x^2 + 28xy^2 + 45y^4 = (2x + 5y^2)(2x + 9y^2)$$

EXAMPLE 21

Factor $-6x^2 - 11x + 10$.

$$-6x^2 - 11x + 10 = (-3x + 2)(2x + 5)$$

An alternate approach to factoring a trinomial whose first term is negative involves dividing each term first by -1 and then proceeding as usual. Following this approach in this example leads to the following:

$$\begin{aligned} -6x^2 - 11x + 10 &= -(6x^2 + 11x - 10) \\ &= -(3x - 2)(2x + 5) \end{aligned}$$

Do you see that this result is equivalent to our previous result?

EXAMPLE 22

Factor $60x^2 + 20x - 105$.

$$\begin{aligned} 60x^2 + 20x - 105 &= 5(12x^2 + 4x - 21) \\ &= 5(2x + 3)(6x - 7) \end{aligned}$$

Day: Date:

Note: Dividing each term by the common monomial factor, 5, simplified the remaining trinomial factor.

EXAMPLE 23

Factor $180x^2 + 279x + 108$.

$$180x^2 + 279x + 108 = 9(20x^2 + 31x + 12)$$
$$= 9(5x + 4)(4x + 3)$$

EXAMPLE 24

Factor $9x^2 + 24xy + 16y^2$.

$$9x^2 + 24xy + 16y^2 = (3x + 4y)(3x + 4y) = (3x + 4y)^2$$

EXAMPLE 25

Factor $16x^2 - 8xy + y^2$.

$$16x^2 - 8xy + y^2 = (4x - y)(4x - y) = (4x - y)^2$$

When the factors of a trinomial are the same two binomial factors, the trinomial is called a **perfect square trinomial.** The general forms are as follows.

PERFECT SQUARE TRINOMIALS

$$x^2 + 2xy + y^2 = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)^2$$

Exercises 5.2

Factor each expression completely.

1. $6x + 9y$
2. $15x - 20$
3. $10x + 25y - 45z$
4. $8x^2 - 12xy + 10xz$
5. $12x^2 - 30xy + 6xz$
6. $10x^2 + 25x$
7. $8x^3y^2 - 6xy^3 + 12xy^2z$
8. $15x^7y^3 - 6x^6y^2 + 12x^3yz$
9. $x^2 - 16$
10. $x^2 - 64$
11. $9x^2 - 25y^2$
12. $49x^2 - 100y^2$
13. $2e^2 - 72$
14. $20x^2 - 45$
15. $16d^2 - 100$
16. $64x^2 - 4y^2$
17. $4R^2 - 4r^2$
18. $49 - a^4$
19. $x^2 + 6x + 8$
20. $x^2 + 8x + 15$
21. $b^2 + 11b + 24$
22. $m^2 + 11m + 30$
23. $x^2 - 9x + 18$
24. $x^2 - 9x + 14$
25. $a^2 - 18a + 32$
26. $c^2 + 22c + 40$
27. $x^2 - 2x - 35$
28. $x^2 + 4x - 12$
29. $a^2 + 3a - 4$
30. $q^2 - 3q - 28$
31. $2x^2 - x - 6$
32. $5x^2 + 32x + 12$

33. $15x^2 - 31x + 14$
34. $8x^2 + 26x - 45$
35. $45y^2 + 59y + 6$
36. $8m^2 - 14m + 3$
37. $35a^2 + 2a - 1$
38. $16g^2 + 8g + 1$
39. $9c^2 - 24c + 16$
40. $6m^2 - 13m + 5$
41. $25b^2 + 60b + 20$
42. $15t^2 + 69t - 30$
43. $35t^2 - 4ts - 15s^2$
44. $12a^2 + 8ab - 15b^2$
45. $30k^2 - 95kt + 50t^2$
46. $24m^2 + 34m + 10$
47. $54x^2 + 27x - 42$
48. $8x^2 - 24xy - 1440y^2$
49. $4a^2 + 20a + 25$
50. $9x^2 + 42x + 49$
51. $9x^2 - 48xy^2 + 64y^4$
52. $4a^4 + 12a^2b + 9b^2$
53. $25x^2 + 10xy^3 + y^6$
54. $9c^4 - 12c^2y^2 + 4y^4$
55. $x^4 - 81$
56. $x^4 - 16$
57. $a^4 - 3a^2 - 40$
58. $b^4 + 21b^2 - 100$
59. $t^4 - 13t^2 + 36$
60. $x^4 - x^2 - 12$

5.3 OTHER FORMS OF FACTORING

Some algebraic expressions may be factored by grouping their terms so that they are of the types we have already studied. We shall consider those which can be grouped in one of the following ways:

1. With terms having a common binomial factor.
2. As the difference of two squares.
3. In the form of a trinomial.

Factoring out a common binomial factor is similar to factoring out a common monomial factor.

EXAMPLE 1

Factor $ax + bx$.

$$ax + bx = x(a + b)$$

EXAMPLE 2

Factor $a(x - y) + b(x - y)$.

$$a(x - y) + b(x - y) = (x - y)(a + b)$$

EXAMPLE 3

Factor $h(x - 1) - 2(x - 1)$.

$$h(x - 1) - 2(x - 1) = (x - 1)(h - 2)$$

In most cases, however, the algebraic expressions are not grouped so that the common binomial factors are so obvious.

EXAMPLE 4

Factor $3x - 3y + ax - ay$.

Group the first two terms and the last two terms, factor out common monomial factors, and then factor out the resulting binomial factors.

$$\begin{aligned} 3x - 3y + ax - ay &= 3(x - y) + a(x - y) \\ &= (x - y)(3 + a) \end{aligned}$$

EXAMPLE 5

Factor $a^2x - x - 3a^2 + 3$.

$$\begin{aligned} a^2x - x - 3a^2 + 3 &= x(a^2 - 1) - 3(a^2 - 1) \\ &= (a^2 - 1)(x - 3) \\ &= (a + 1)(a - 1)(x - 3) \end{aligned}$$

EXAMPLE 6

Factor $x^3 + x^2 + x + 1$.

$$\begin{aligned} x^3 + x^2 + x + 1 &= x^2(x + 1) + (x + 1) \\ &= (x + 1)(x^2 + 1) \end{aligned}$$

If an algebraic expression can be expressed as the difference of two squares, it can be factored similarly to the form $x^2 - y^2 = (x + y)(x - y)$.

EXAMPLE 7

Factor $(a + b)^2 - 16$.

$$(a + b)^2 - 16 = (a + b + 4)(a + b - 4)$$

EXAMPLE 8

Factor $25a^2 - (a + 2b)^2$.

$$\begin{aligned} 25a^2 - (a + 2b)^2 &= (5a)^2 - (a + 2b)^2 \\ &= [5a + (a + 2b)][5a - (a + 2b)] \\ &= (6a + 2b)(4a - 2b) \\ &= 2(3a + b)(2)(2a - b) \\ &= 4(3a + b)(2a - b) \end{aligned}$$

EXAMPLE 9

Factor $y^2 - 4y + 4 - 25x^2$.

Group the first three terms which form a perfect square trinomial, factor into a perfect square, and then factor the resulting difference of two squares.

$$\begin{aligned} (y^2 - 4y + 4) - 25x^2 &= (y - 2)^2 - 25x^2 \\ &= (y - 2 + 5x)(y - 2 - 5x) \end{aligned}$$

The last type of group factoring involves algebraic expressions that can be grouped in the form $ax^2 + bx + c$.

EXAMPLE 10

Factor $(a + b)^2 + 8(a + b) + 15$.

Note that $(a + b)^2 + 8(a + b) + 15$ is in the form $x^2 + 8x + 15$ which factors $(x + 3)(x + 5)$. Therefore,

$$(a + b)^2 + 8(a + b) + 15 = (a + b + 3)(a + b + 5)$$

The last type of factoring we shall study here is factoring the sum and difference of two perfect cubes. The general forms and three examples of each follow.

SUM OF TWO PERFECT CUBES

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

EXAMPLE 11

Factor $a^3 + 8$.

$$a^3 + 8 = a^3 + 2^3 = (a + 2)(a^2 - 2a + 4)$$

EXAMPLE 12

Factor $y^3 + 27$.

$$y^3 + 27 = y^3 + 3^3 = (y + 3)(y^2 - 3y + 9)$$

EXAMPLE 13

Factor $27x^3 + 64y^6$.

$$27x^3 + 64y^6 = (3x)^3 + (4y^2)^3 = (3x + 4y^2)(9x^2 - 12xy^2 + 16y^4)$$

DIFFERENCE OF TWO PERFECT CUBES

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Note the similarities in these two forms.

EXAMPLE 14

Factor $a^3 - 1$.

$$a^3 - 1 = a^3 - 1^3 = (a - 1)(a^2 + a + 1)$$

EXAMPLE 15

Factor $125x^3 - 8y^3$.

$$125x^3 - 8y^3 = (5x)^3 - (2y)^3 = (5x - 2y)(25x^2 + 10xy + 4y^2)$$

EXAMPLE 16

Factor $x^6 - 64$.

$$\begin{aligned} x^6 - 64 &= (x^3 + 8)(x^3 - 8) \\ &= (x + 2)(x^2 - 2x + 4)(x - 2)(x^4 + 2x + 4) \end{aligned}$$

Exercises 5.3

Factor each expression completely.

1. $a(m + n) - b(m + n)$
2. $m(x^2 - 16) - 3(x^2 - 16)$
3. $mx + my + nx + ny$
4. $ab + 3a - bc - 3c$
5. $3x + y - 6x^2 - 2xy$
6. $x^2 + ax + xy + ay$
7. $6x^3 - 4x^2 + 3x - 2$
8. $x^4 - 8x + x^3y - 8y$
9. $(x + y)^2 - 4z^2$
10. $(x - y)^2 - 16a^4$
11. $100 - 49(x - y)^2$
12. $9(a - 4)^2 - 64b^4$
13. $x^2 - 6x + 9 - 4y^2$
14. $x^2 + 10x + 25 - 36y^2$
15. $4x^2 - 4y^2 + 4y - 1$
16. $4 - a^2 - 2ab - b^2$
17. $(x + y)^2 + 13(x + y) + 36$
18. $3(x - y)^2 - 14(x - y) + 8$
19. $24(a + b)^2 - 14(a + b) - 5$
20. $x^2 + 2x(y + z) + (y + z)^2$
21. $a^3 + b^3$
22. $27x^3 + 8y^3$
23. $x^3 - 64$
24. $27x^3 - 8y^3$
25. $a^3b^3 + c^3$
26. $x^3 - y^3z^3$
27. $a^3 - 27b^3$
28. $8a^3 + b^3$
29. $27a^3 + 64b^3$
30. $64x^6 - 125y^3$
31. $27a^6 - 8b^9$
32. $8a^3b^6 + 125c^{12}$
33. $x^6 - y^6$
34. $a^6b^6 - c^6$

5.4 EQUIVALENT FRACTIONS

In order to solve many problems that occur in the technical and applied sciences, it is necessary to use fractions. A **fraction** may be defined as the quotient of two numbers or algebraic expressions. Both the numerator and denominator of a fraction may be multiplied or divided by the same nonzero number without changing the value of the fraction.

Two fractions are **equivalent** when both the numerator and denominator of one fraction can be multiplied or divided by the same nonzero number in order to change one fraction to the other. For example, $\frac{2}{3}$ is equivalent to $\frac{6}{9}$, and $\frac{12}{16}$ is equivalent to $\frac{3}{4}$, because

$$\frac{2}{3} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{6}{9} \quad \text{and} \quad \frac{12}{16} = \frac{12 \div 4}{16 \div 4} = \frac{3}{4}$$

A fraction has three signs associated with it:

1. The sign of the fraction.
2. The sign of the numerator.
3. The sign of the denominator.

Any two of these three signs may be changed without changing the value of the fraction. For example,

$$-\frac{-3}{+4} = -\frac{+3}{-4} = +\frac{+3}{+4} = +\frac{-3}{-4}$$

When the numerator or the denominator is an algebraic expression, its sign is changed by placing a negative sign before the numerator or denominator in parentheses or by changing the signs of *all* the terms. For example,

$$-\frac{a+b-c}{d-3} = \frac{-(a+b-c)}{d-3} = \frac{-a-b+c}{d-3}$$

or

$$-\frac{a+b-c}{d-3} = \frac{a+b-c}{-(d-3)} = \frac{a+b-c}{-d+3} \quad \text{or} \quad \frac{a+b-c}{3-d}$$

A fraction is in **lowest terms** when its numerator and denominator have no common factors except 1. To reduce a fraction to lowest terms, divide the numerator and the denominator by their common factors.

EXAMPLE 1

Reduce $\frac{8xy^3}{12x^4y^2}$ to lowest terms.

The common factors of the numerator and the denominator are $4xy^2$. Dividing both numerator and denominator by $4xy^2$, we have

$$\frac{8xy^3}{12x^4y^2} = \frac{2y}{3x^3}$$

Some prefer to use "cancellation marks" to indicate this division; that is,

$$\frac{\cancel{8}^{\,2}\,\cancel{x}\,\cancel{y^3}^{\,y}}{\cancel{12}_{\,3}\,\cancel{x^4}\,\cancel{y^2}} = \frac{2y}{3x^3}$$

$3x^3$

Remember, *cancellation* is a nontechnical word for the mathematical operation of division. Think *division* whenever these marks are used.

EXAMPLE 2

Reduce $\frac{x^2+5x+6}{x^2+6x+9}$ to lowest terms.

Before finding common factors, we must first factor the numerator and the denominator.

$$\frac{x^2+5x+6}{x^2+6x+9} = \frac{(x+2)(x+3)}{(x+3)(x+3)} = \frac{x+2}{x+3}$$

EXAMPLE 3

Reduce $\frac{4-9x^2}{9x^2+3x-2}$ to lowest terms.

$$\frac{4-9x^2}{9x^2+3x-2} = \frac{(2-3x)(2+3x)}{(3x+2)(3x-1)} = \frac{2-3x}{3x-1}$$

(*Note:* $2+3x = 3x+2$.)

When factors *differ only in sign*, such as $x - y$ and $y - x$, one may be replaced by the opposite of the other; that is,

$$x - y = -(y - x) \quad \text{because} \quad -(y - x) = -y + x = x - y$$

or

$$y - x = -(x - y) \quad \text{because} \quad -(x - y) = -x + y = y - x$$

EXAMPLE 4

Reduce $\dfrac{x^2 - 2x - 8}{16 - x^2}$ to lowest terms.

$$\frac{x^2 - 2x - 8}{16 - x^2} = \frac{(x - 4)(x + 2)}{(4 + x)(4 - x)}$$

$$= \frac{-(4 - x)(x + 2)}{(4 + x)(4 - x)} \qquad [\textit{Note: } x - 4 = -(4 - x)]$$

$$= -\frac{x + 2}{4 + x} \quad \text{or} \quad \frac{-x - 2}{4 + x}$$

EXAMPLE 5

Reduce $\dfrac{30 - 19x - 5x^2}{5x^2 + 14x - 24}$ to lowest terms.

$$\frac{30 - 19x - 5x^2}{5x^2 + 14x - 24} = \frac{(5 + x)(6 - 5x)}{(5x - 6)(x + 4)}$$

$$= \frac{(5 + x)(6 - 5x)}{-(6 - 5x)(x + 4)} \qquad [\textit{Note: } 5x - 6 = -(6 - 5x)]$$

$$= -\frac{5 + x}{x + 4} \quad \text{or} \quad \frac{-x - 5}{x + 4}$$

Exercises 5.4

Reduce each fraction to lowest terms.

1. $\dfrac{3x}{18x}$
2. $\dfrac{18xy}{24y}$
3. $\dfrac{30x^2y^4}{54x^2y^5}$
4. $\dfrac{27x^2y^3z^4}{57x^5y^7}$
5. $\dfrac{8(x + 4)^3}{2(x + 4)^5}$
6. $\dfrac{6x^2(2x - 1)^6}{2x(2x - 1)^2}$
7. $\dfrac{18(x - 3)^3(1 - 5x)^2}{9(1 - 5x)^2(x - 3)}$
8. $\dfrac{12(3x - 4)^3(2x + 5)^2}{3(2x + 5)^3(3x - 4)^5}$
9. $\dfrac{5m + 15}{6m + 18}$
10. $\dfrac{6x^2 + 3x}{12x + 6}$
11. $\dfrac{6m + 6}{m^2 - 1}$
12. $\dfrac{2t - t^2}{4t - t^2}$

13. $\dfrac{x^2 + x}{x + 1}$

14. $\dfrac{3x^2 + 21x}{6x^2 - 12x}$

15. $\dfrac{x^2 + 5x + 4}{x^2 - 6x - 7}$

16. $\dfrac{x^2 - 2x - 24}{2x^2 + 7x - 4}$

17. $\dfrac{6x^2 + x - 12}{6x^2 + 19x - 36}$

18. $\dfrac{6x^2 + 19x + 10}{6x^2 - 5x - 6}$

19. $\dfrac{18 - 12x}{6x - 9}$

20. $\dfrac{15x - 25}{50 - 30x}$

21. $\dfrac{x^2 - 1}{1 - x}$

22. $\dfrac{6x^3 - 5x^2 - 4x}{16 - 9x^2}$

23. $\dfrac{a^2 - 4}{a^2 + 4a + 4}$

24. $\dfrac{4c^2 + 12c + 9}{4c^2 - 9}$

25. $\dfrac{m^2 - 16m}{16 - m}$

26. $\dfrac{1 - y}{y^2 - y}$

27. $\dfrac{t^2 - t^3}{t^2 - 1}$

28. $\dfrac{1 - 4x + 4x^2}{4x^2 - 1}$

29. $\dfrac{x^2 + xy}{x^2 + xy - 2x - 2y}$

30. $\dfrac{a^2 - ab + 3a - 3b}{a^2 - ab}$

31. $\dfrac{y^3 - 8}{y^2 + 2y + 4}$

32. $\dfrac{a^2b^2 - 16b^2}{a^2b + 9ab + 20b}$

33. $\dfrac{xy - 3x - 2y + 6}{y^3 - 27}$

34. $\dfrac{x^3 - y^3}{x^2 - y^2}$

35. $\dfrac{x^2 + 4x + 16}{x^3 - 64}$

36. $\dfrac{x^4 - 16}{x^4 - 2x^2 - 8}$

37. $\dfrac{3 - 4x - 4x^2}{4x^2 - 8x + 3}$

38. $\dfrac{y^2 - x^2}{x^2 - 2xy + y^2}$

39. $\dfrac{3a^3 + 3b^3}{3a^2 + 6ab + 3b^2}$

40. $\dfrac{x^2 - 2xy + y^2 - z^2}{z^2 + y^2 - x^2 + 2yz}$

5.5 MULTIPLICATION AND DIVISION OF ALGEBRAIC FRACTIONS

The product of two or more fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators; that is,

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}}$$

It is usually helpful to first factor each of the terms of all numerators and denominators; divide by those common factors, if any; and then multiply the numerators and multiply the denominators. Cancellation marks are useful for showing each completed division.

EXAMPLE 1

Multiply $\frac{3xy^2}{5ab} \cdot \frac{20a^2}{6x^2y}$

$$\frac{\cancel{3}\cancel{x}\cancel{y^2}}{\cancel{5}\cancel{a}\cancel{b}} \cdot \frac{\cancel{20}\,\cancel{a^2}}{\cancel{6}\cancel{x^2}\cancel{y}} = \frac{2ay}{bx}$$

EXAMPLE 2

Multiply $\frac{x^2 - 4x + 4}{x^2 + x - 6} \cdot \frac{x^2 + 3x}{x^2 - x}$.

Factor each numerator and denominator.

$$\frac{\cancel{(x-2)^2}}{\cancel{(x+3)}\cancel{(x-2)}} \cdot \frac{\cancel{x}(\cancel{x+3})}{\cancel{x}(x-1)} = \frac{x-2}{x-1}$$

To divide one fraction by another, "invert the divisor" and multiply, or multiply the first fraction by the reciprocal of the second.

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}}$$

EXAMPLE 3

Divide $\frac{xy}{a^2b} \div \frac{x^2y}{ab^3}$.

$$\frac{xy}{a^2b} \div \frac{x^2y}{ab^3} = \frac{\cancel{xy}}{\cancel{a^2b}} \cdot \frac{\cancel{ab^3}}{\cancel{x^2y}} = \frac{b^2}{ax}$$

EXAMPLE 4

Divide $\frac{x^2 - 9}{4x^2 - 9} \div \frac{2x^2 - x - 15}{2x^2 + x - 6}$.

$$\frac{x^2 - 9}{4x^2 - 9} \div \frac{2x^2 - x - 15}{2x^2 + x - 6} = \frac{x^2 - 9}{4x^2 - 9} \cdot \frac{2x^2 + x - 6}{2x^2 - x - 15}$$

$$= \frac{(x+3)(\cancel{x-3})}{(2x+3)(\cancel{2x-3})} \cdot \frac{(\cancel{2x-3})(x+2)}{(\cancel{x-3})(2x+5)}$$

$$= \frac{(x+3)(x+2)}{(2x+3)(2x+5)}$$

Note: Results left in factored form are usually preferred.

EXAMPLE 5

Divide $\dfrac{18+3t-t^2}{1+2t+t^2} \div \dfrac{2t^2-14t+12}{t^2-1}$

$$\frac{18+3t-t^2}{1+2t+t^2} \div \frac{2t^2-14t+12}{t^2-1} = \frac{18+3t-t^2}{1+2t+t^2} \cdot \frac{t^2-1}{2t^2-14t+12}$$

$$= \frac{(3+t)(6-t)}{(1+t)^2} \cdot \frac{(t+1)(t-1)}{2(t-6)(t-1)}$$

[Replace $6-t$ by $(-1)(t-6)$ in the first numerator.]

$$= \frac{(3+t)(-1)\cancel{(t-6)}}{\cancel{(1+t)^2}} \cdot \frac{\cancel{(t+1)}\cancel{(t-1)}}{2\cancel{(t-6)}\cancel{(t-1)}}$$

$$= \frac{-t-3}{2(t+1)}$$

EXAMPLE 6

Perform the indicated operations.

$$\frac{x^2+xy}{x^2+3xy+2y^2} \cdot \frac{x^2-xy+y^2}{x^2-2xy+y^2} \div \frac{x^4+xy^3}{(x^2-y^2)^2}$$

$$= \frac{x^2+xy}{x^2+3xy+2y^2} \cdot \frac{x^2-xy+y^2}{x^2-2xy+y^2} \cdot \frac{(x^2-y^2)^2}{x^4+xy^3}$$

$$= \frac{\cancel{x(x+y)}}{\cancel{(x+y)}(x+2y)} \cdot \frac{x^2-xy+y^2}{\cancel{(x-y)^2}} \cdot \frac{\cancel{(x+y)^2}\cancel{(x-y)^2}}{\cancel{x(x+y)}\cancel{(x^2-xy+y^2)}}$$

$$= \frac{x+y}{x+2y}$$

Exercises 5.5

Perform the indicated operations and simplify.

1. $\dfrac{9}{16} \cdot \dfrac{4}{3}$
2. $\dfrac{16}{25} \cdot \dfrac{5}{32}$
3. $\dfrac{15}{32} \div \dfrac{3}{16}$
4. $\dfrac{5}{9} \div \dfrac{5}{18}$
5. $\dfrac{x^4}{3} \cdot \dfrac{6x}{x^5}$
6. $\dfrac{4t^4}{6t} \cdot \dfrac{12t^2}{9t^3}$
7. $\dfrac{6a^4}{a^2} \div \dfrac{18a^2}{a^5}$
8. $\dfrac{8c^2}{4c^3} \div \dfrac{6c^4}{9c}$

Day: Date:

9. $\frac{3ab}{4x^2y} \cdot \frac{6x^2}{5ab^3}$

10. $\frac{15pq^2}{13m^5n^3} \cdot \frac{39mn^4}{5p^4q^3}$

11. $\frac{4ax^2}{9b^2y} \div \frac{5(ax)^2}{18(by^2)^2}$

12. $\frac{(5a)^3}{6x} \div \frac{5a^3}{-6x^2}$

13. $\frac{(2a^2b^3)^2}{(4ab^2)^3} \cdot \frac{(8a^2b)^2}{12a^3b}$

14. $\frac{(3x^2y)^3}{(6x^2y^2)^2} \div \frac{(12x^4y^4)^2}{(9x^2y)^3}$

15. $\frac{4x+8}{8xy} \cdot \frac{16x}{32x+64}$

16. $\frac{2x}{8x+4} \cdot \frac{16x+8}{6}$

17. $\frac{5k+5}{12} \div \frac{9k+9}{4}$

18. $(8c-16) \div \frac{3c-6}{12}$

19. $\frac{(y+2)^2}{y} \cdot \frac{y^2}{y^2-4}$

20. $\frac{9a}{(a-1)^2} \cdot \frac{a^2-1}{a^2}$

21. $(m^2-1) \cdot \frac{1+m}{1-m}$

22. $(s^2-9) \div \frac{3-s}{3s}$

23. $\frac{x^2-64}{x+2} \div \frac{8-x}{x}$

24. $\frac{y^2-25}{5y} \cdot \frac{10}{5-y}$

25. $\frac{x-2}{x-y} \cdot \frac{x^2-y^2}{x^2-4x+4}$

26. $\frac{x+1}{x+y} \cdot \frac{x^2-y^2}{x^2-1}$

27. $\frac{x}{(x-3)^2} \cdot \frac{9-x^2}{x^4}$

28. $\frac{x^2-y^2}{z^2-y^2} \cdot \frac{xy-xz}{y-x}$

29. $\frac{x^2+6x+9}{x^2-9} \cdot \frac{x^2-6x+9}{x-3}$

30. $\frac{t^2-10t+25}{t^2-25} \div \frac{t^2+10t+25}{t+5}$

31. $\frac{x^2-x-12}{x^2+2x+1} \cdot \frac{x^2-4x-5}{x^2-9x-20}$

32. $\frac{x^2-x-2}{x^2+7x+6} \cdot \frac{x^2+3x-18}{x^2-4x+4}$

33. $\frac{9x^2-4}{6x^2-5x-6} \div \frac{9x^2-12x+4}{6x^2-13x+6}$

34. $\frac{10x^2+11x+3}{3x^2-19x+28} \div \frac{4x^2-8x-5}{6x^2-29x+35}$

35. $\frac{15a^2+7ab-2b^2}{6a^2-11ab-10b^2} \cdot \frac{2a^2-13ab+20b^2}{25a^2-b^2}$

36. $\frac{18m^2+27mn+10n^2}{6m^2+19mn+10n^2} \div \frac{12m^2-8mn-15n^2}{16m^2+34mn-15n^2}$

37. $\frac{x^2-y^2}{x^2+2xy+y^2} \div \frac{x^2-2xy+y^2}{xy-y^2}$

38. $\frac{x^2y-16y^3}{xy^2-4y^3} \div \frac{x^2+3xy-4y^2}{y-x}$

39. $\frac{x^2-16}{x^2-9} \div \frac{3x^2+13x+4}{2x^2+x-21}$

40. $\frac{25a^2b^2-49}{4ab+1} \div \frac{5ab+7}{16a^2b^2+16ab+3}$

41. $\frac{x^3-8}{x^2-2x+4} \cdot \frac{x^3+8}{x^2+2x+4}$

42. $\frac{(x-y)^3}{(x+y)^2} \cdot \frac{x^2+2xy+y^2}{x^2-2xy+y^2}$

43. $\frac{xy+y-x^2-x}{12xy} \div \frac{x^2+x}{4x^2}$

44. $\frac{2x^2+10x+xy+5y}{x^2-25} \div \frac{x^2+5x+25}{x^3-125}$

45. $\frac{x^3+27}{x^2-9} \cdot \frac{5x^2+x-18}{5x^2-9x} \div \frac{x^2-3x+9}{5x^2-6x-27}$

46. $\frac{x^2-16}{x^2-x-12} \cdot \left(\frac{x+3}{x+4}\right)^2 \div \frac{x^3+27}{x^3+64}$

47. $\dfrac{18x^2 - 9x - 20}{6x^2 - 5x - 4} \div \left(\dfrac{2x^2 - 9x - 5}{9x^2 - 16} \cdot \dfrac{12x^2 - 10x}{4x^2 + 4x + 1}\right)$

48. $\dfrac{2x^2 - 9x + 7}{x^2 - 1} \div \left(\dfrac{2x^3 + 9x + 4}{x^2 + 4x + 3} \cdot \dfrac{x + 3}{x + 4}\right)^2$

5.6 ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS

Fractions may be added or subtracted if they have a common denominator. If they do not have a common denominator, the lowest common denominator is the most convenient to use because it involves the least amount of computation.

FINDING THE LOWEST COMMON DENOMINATOR (L.C.D.)

1. Factor each denominator into its prime factors; that is, factor each denominator completely.
2. Then the L.C.D. is the product formed by using each of the different factors the greatest number of times that it occurs in any *one* of the given denominators.

EXAMPLE 1

Find the L.C.D. for $\dfrac{3}{18}$, $\dfrac{15}{60}$, and $\dfrac{19}{27}$.

The prime factors of the denominators are

$$18 = 2 \cdot 3 \cdot 3$$
$$60 = 2 \cdot 2 \cdot 3 \cdot 5$$
$$27 = 3 \cdot 3 \cdot 3$$

Thus,

$$\text{L.C.D.} = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 = 540$$

because 2 occurs at most twice, 3 occurs at most three times, and 5 occurs at most one time in any *one* given denominator.

EXAMPLE 2

Find the L.C.D. for $\dfrac{1}{4x}$, $\dfrac{5}{6x^2y}$, and $\dfrac{-7}{15y^3}$.

The prime factors of the denominators are

$$4x = 2 \cdot 2 \cdot x$$
$$6x^2y = 2 \cdot 3 \cdot x \cdot x \cdot y$$
$$15y^3 = 3 \cdot 5 \cdot y \cdot y \cdot y$$

So

$$\text{L.C.D.} = 2 \cdot 2 \cdot 3 \cdot 5 \cdot x \cdot x \cdot y \cdot y \cdot y = 60x^2y^3$$

because 2 occurs at most twice, 3 occurs at most once, 5 occurs at most once, x occurs at most twice, and y occurs at most three times in any *one* given denominator.

EXAMPLE 3

Find the L.C.D. for

$$\frac{3x}{x^2 - 2xy + y^2} \qquad \frac{4y}{x^2 - y^2} \quad \text{and} \quad \frac{6x + 2y}{x^2 - xy - 2y^2}$$

First find the prime factors of the denominators.

$$x^2 - 2xy + y^2 = (x - y)(x - y)$$
$$x^2 - y^2 = (x + y)(x - y)$$
$$x^2 - xy - 2y^2 = (x + y)(x - 2y)$$
$$\text{L.C.D.} = (x - y)(x - y)(x + y)(x - 2y)$$

because in any *one* given denominator the factor $x - y$ occurs at most twice, the factor $x + y$ occurs at most once, and the factor $x - 2y$ occurs at most once.

ADDING OR SUBTRACTING FRACTIONS

1. If the fractions do not have a common denominator, change each fraction to an equivalent fraction having the L.C.D. (For each fraction, divide the L.C.D. by the denominator. Then multiply both numerator and denominator by this result in order to obtain an equivalent fraction having the L.C.D.)
2. Add or subtract the numerators in the order they occur, and place this result over the L.C.D.
3. Reduce the resulting fraction to lowest terms, if possible.

EXAMPLE 4

Perform the indicated operations.

$$\frac{5}{6a^2b} - \frac{3}{ab^2} + \frac{1}{4a^3}$$

The L.C.D. is $12a^3b^2$. Now divide $12a^3b^2$ by each denominator.

$$\frac{12a^3b^2}{6a^2b} = 2ab \qquad \frac{12a^3b^2}{ab^2} = 12a^2 \qquad \frac{12a^3b^2}{4a^3} = 3b^2$$

Multiply each numerator and denominator by the corresponding result.

$$\frac{5}{6a^2b} - \frac{3}{ab^2} + \frac{1}{4a^3} = \frac{5}{6a^2b} \cdot \frac{2ab}{2ab} - \frac{3}{ab^2} \cdot \frac{12a^2}{12a^2} + \frac{1}{4a^3} \cdot \frac{3b^2}{3b^2}$$
$$= \frac{10ab - 36a^2 + 3b^2}{12a^3b^2}$$

EXAMPLE 5

Perform the indicated operations and simplify.

$$\frac{x - 4}{2x} + \frac{x + 2}{x + 1} - \frac{3x + 2}{x^2 + x}$$

The L.C.D. is $2x(x + 1)$.

$$\frac{x-4}{2x}+\frac{x+2}{x+1}-\frac{3x+2}{\underset{x(x+1)}{x^2+x}}=\frac{x-4}{2x}\cdot\frac{x+1}{x+1}+\frac{x+2}{x+1}\cdot\frac{2x}{2x}-\frac{3x+2}{x(x+1)}\cdot\frac{2}{2}$$

$$=\frac{(x^2-3x-4)+(2x^2+4x)-(6x+4)}{2x(x+1)}$$

$$=\frac{3x^2-5x-8}{2x(x+1)}$$

$$=\frac{(3x-8)(x+1)}{2x(x+1)}$$

$$=\frac{3x-8}{2x}$$

EXAMPLE 6

Perform the indicated operations and simplify.

$$\frac{4x}{\underset{(2x+1)(2x-1)}{4x^2-1}}-\frac{3x+1}{\underset{-(2x-1)}{1-2x}}+\frac{2x}{1+2x}$$

Note: If we write $1-2x$ as $-(2x-1)$, the L.C.D. is $(2x+1)(2x-1)$.

$$\frac{4x}{(2x+1)(2x-1)}+\frac{3x+1}{2x-1}\cdot\frac{2x+1}{2x+1}+\frac{2x}{1+2x}\cdot\frac{2x-1}{2x-1}$$

$$=\frac{4x+(6x^2+5x+1)+(4x^2-2x)}{(2x+1)(2x-1)}$$

$$=\frac{10x^2+7x+1}{(2x+1)(2x-1)}$$

$$=\frac{(2x+1)(5x+1)}{(2x+1)(2x-1)}$$

$$=\frac{5x+1}{2x-1}$$

Exercises 5.6

Find the lowest common denominator for each set of denominators.

1. $6x, 8$

2. $15, 20m$

3. $2k, 4k^2$

4. $3t, 12t^3$

5. $6ab^2, 5a^2b$

6. $4ab^3c^4, 12a^2bc, 16ac^2$

7. $5x, x-1$

8. $x+3, 3x$

9. $3x+6, 6x+12$

10. $4a-12, 6a-18$

11. $x^2-25, (x-5)^2$

12. $b^2-16, (b+4)^2$

13. x^2+6x+8, x^2-x-6

14. $6x^2+3x-1, 6x^2-5x-4$

15. $3x^3-12x^2, 6x^2+12x, 2x^3-4x^2-16x$

16. $18a(a+5)^2, 6(a^2-25), 5a-25$

17. $6(c+2)(c-4)^2, 12c(c+2)(c-4), 3c(c-4)^2$

18. $t^2(t+3)^2, t^2(t-3)^2, t(t^2-9)$

Perform the indicated operations and simplify.

19. $\frac{3}{14} + \frac{5}{14}$

20. $\frac{9}{20} - \frac{3}{20}$

21. $\frac{16}{a} - \frac{9}{a}$

22. $\frac{x}{3a} + \frac{5x}{3a}$

23. $\frac{6}{x+1} + \frac{9}{x+1}$

24. $\frac{6a}{2x+3} - \frac{8a}{2x+3}$

25. $\frac{7}{36} + \frac{7}{45}$

26. $\frac{19}{24} - \frac{31}{60}$

27. $\frac{2x-1}{12} + \frac{3x+5}{20} - \frac{x}{4}$

28. $\frac{3a+b}{6} - \frac{2a-b}{8} + \frac{5a}{12}$

29. $\frac{5}{6y} - \frac{7}{2y}$

30. $\frac{2}{3y} + \frac{1}{y^2}$

31. $\frac{4}{3p} + \frac{2}{p^2}$

32. $\frac{3}{4t} - \frac{5}{3t}$

33. $\frac{5}{4x^2} + \frac{1}{3x^3} - \frac{3}{2x}$

34. $\frac{5}{3a^2b} - \frac{3}{2a} - \frac{1}{4ab^3}$

35. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

36. $\frac{a}{b} + \frac{c}{a} + 1$

37. $3x + \frac{4-x}{2x}$

38. $9 - \frac{2x-1}{3y}$

39. $\frac{2}{c-1} + \frac{1}{c}$

40. $\frac{4}{s-3} - \frac{1}{s}$

41. $\frac{6}{t+2} - \frac{3}{t-2}$

42. $\frac{5}{x-5} + \frac{2}{x+5}$

43. $\frac{2}{a} + \frac{1}{a+1}$

44. $\frac{2}{a+3} - \frac{4}{a+2}$

45. $\frac{d}{d+4} - \frac{2d}{d-5}$

46. $\frac{3t}{t-2} + \frac{4t}{t+4}$

47. $\frac{8}{x-4} + \frac{2}{4-x}$

48. $\frac{c}{c-6} + \frac{2c}{6-c}$

49. $\frac{4}{3a+9} - \frac{6}{4a+12}$

50. $\frac{12}{5r-25} - \frac{5}{3r-15}$

51. $\frac{5a+1}{a^2-1} - \frac{2a-3}{a+1}$

52. $\frac{2x+5}{x+6} + \frac{14x}{x^2-36}$

53. $\frac{6}{x-3} - \frac{5}{3-x}$

54. $\frac{2x}{x^2-1} + \frac{2-x}{1-x}$

55. $\frac{2}{r^2-4r+4} + \frac{1}{r^2+r-6}$

56. $\frac{4}{y^2-5y+6} - \frac{3}{y^2-y-2}$

57. $\frac{1}{x-2} + \frac{1}{x^2-5x+6}$

58. $\frac{1}{6x} + \frac{1}{3x-6} - \frac{1}{2x+4}$

59. $\frac{t+4}{2t-4} - \frac{2t+5}{t^2-t-2} + \frac{3}{4}$

60. $\frac{2}{x^2-y^2} - \frac{1}{x^2-3xy+2y^2} - \frac{3}{x^2-xy-2y^2}$

61. $\dfrac{3x-2}{x^2-3x-4}+\dfrac{2x-3}{x^2-x-12}$

62. $\dfrac{3x+7}{6x^2+x-35}-\dfrac{3x-7}{6x^2+29x+35}$

63. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a}{a+b}+\dfrac{b}{b-a}$

64. $\dfrac{2x}{x^2-4}-\dfrac{2+x}{2-x}-\dfrac{x-2}{x+2}$

65. $\dfrac{x+3}{x^2-7x+12}+\dfrac{x+3}{16-x^2}$

66. $\dfrac{3x^2-18x+5}{2x^2+x-1}-\dfrac{2x+1}{1-2x}-\dfrac{3x-7}{x+1}$

67. $\dfrac{5x}{8x+2}-\dfrac{3}{4x^2+x}+\dfrac{1}{2x}$

68. $\dfrac{1}{x^2+2x+1}+\dfrac{3}{2x+2}-\dfrac{5}{4x+4}$

69. $\dfrac{1}{t-2}-\dfrac{6t}{t^3-8}$

70. $\dfrac{c}{1-c^3}+\dfrac{1}{c^2-1}-\dfrac{2}{c^2+c+1}$

71. $\dfrac{x^2+x}{x^3+1}+1$

72. $\dfrac{x^2-1}{x^3-1}+\dfrac{x-1}{x^2+x+1}$

5.7 COMPLEX FRACTIONS

A **complex fraction** is a fraction that contains a fraction in the numerator, the denominator, or both. There are basically two ways to simplify a complex fraction.

Method 1: Multiply both the numerator and denominator of the complex fraction by the L.C.D. of all the fractions that appear in the numerator and denominator.

Method 2: Simplify the numerator and the denominator separately. Then divide the simplified numerator by the simplified denominator. Finally simplify that result, if possible.

EXAMPLE 1

Simplify.

$$\frac{1-\dfrac{1}{x}}{x-\dfrac{1}{x^2}}$$

Method 1: The L.C.D. of all the denominators of the fractions in the complex fraction is x^2.

$$\frac{\left(1-\dfrac{1}{x}\right)x^2}{\left(x-\dfrac{1}{x^2}\right)x^2}=\frac{x^2-x}{x^3-1}$$

$$=\frac{x(x-1)}{(x-1)(x^2+x+1)}=\frac{x}{x^2+x+1}$$

Method 2:

$$\frac{1-\dfrac{1}{x}}{x-\dfrac{1}{x^2}}=\frac{\dfrac{x-1}{x}}{\dfrac{x^3-1}{x^2}}=\frac{x-1}{x}\div\frac{x^3-1}{x^2}$$

$$=\frac{x-1}{x}\cdot\frac{x^2}{(x-1)(x^2+x+1)}=\frac{x}{x^2+x+1}$$

To simplify an expression containing a complex fraction, first simplify the complex fraction and then perform the remaining indicated operations.

EXAMPLE 2

Simplify.

$$x - \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = x - \frac{\frac{x}{1-x}}{\frac{1-x-x}{1-x}}$$

$$= x - \frac{\frac{x}{1-x}}{\frac{1-2x}{1-x}} \cdot \frac{(1-x)}{(1-x)}$$

$$= x - \frac{x(1-x)}{1-2x}$$

$$= \frac{x(1-2x)}{1-2x} - \frac{x(1-x)}{1-2x}$$

$$= \frac{(x-2x^2)-(x-x^2)}{1-2x} = \frac{-x^2}{1-2x}$$

Exercises 5.7

Perform the indicated operations and simplify.

1. $\dfrac{\frac{5}{x}}{\frac{15}{x+2}}$

2. $\dfrac{\frac{12}{r-3}}{\frac{3}{2r}}$

3. $\dfrac{\frac{t-1}{6t}}{\frac{t+1}{9t}}$

4. $\dfrac{\frac{3x^2y}{9z}}{\frac{12xy}{3z}}$

5. $\dfrac{\frac{s+t}{4k}}{\frac{2s+2t}{8k}}$

6. $\dfrac{\frac{x-4y}{6x}}{\frac{8x-32y}{12}}$

7. $\dfrac{2+\frac{7}{16}}{2-\frac{1}{4}}$

8. $\dfrac{\frac{2}{9}-6}{\frac{1}{3}+1}$

9. $\dfrac{a-4}{2-\frac{8}{a}}$

10. $\dfrac{a-12}{1-\frac{12}{a}}$

11. $\dfrac{a-\frac{25}{a}}{a+5}$

12. $\dfrac{a+\frac{27}{a^2}}{a+3}$

13. $\dfrac{\frac{3}{x}+\frac{5}{y}}{\frac{x}{3}+\frac{y}{5}}$

14. $\dfrac{\frac{x}{y}-\frac{y}{z}}{\frac{y}{x}-\frac{z}{y}}$

15. $\dfrac{1-y}{1-\frac{1}{y}}$

16. $\dfrac{\frac{1}{y}-1}{y-\frac{1}{y}}$

17. $\dfrac{1+\frac{1}{a}}{1-\frac{1}{a}}$

18. $\dfrac{\frac{2}{c}-1}{\frac{2}{c}+1}$

19. $\dfrac{\frac{1}{m}+\frac{1}{n}}{\frac{1}{m}-\frac{1}{n}}$

20. $\dfrac{\frac{x}{y}+2}{\frac{x}{y}-\frac{4y}{x}}$

21. $\dfrac{x-3-\dfrac{28}{x}}{x+10+\dfrac{24}{x}}$

22. $\dfrac{12x+16+\dfrac{5}{x}}{18x-27-\dfrac{35}{x}}$

23. $\dfrac{\dfrac{x^2-y^2}{x^3+y^3}}{\dfrac{x^2+2xy+y^2}{x^2-xy+y^2}}$

24. $\dfrac{\dfrac{8x^2-2x-21}{6x^2+11x+3}}{\dfrac{4x^2+20x+25}{12x^3+19x^2+5x}}$

25. $\dfrac{\dfrac{x}{x+y}+\dfrac{y}{x-y}}{\dfrac{x^2+y^2}{x^2-y^2}}$

26. $\dfrac{\dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}$

27. $\dfrac{\dfrac{6a^2-5ab-4b^2}{a-b}}{\dfrac{3a^2-7ab+4b^2}{2a+b}}$

28. $\dfrac{x+\dfrac{y^2}{x+y}}{y+\dfrac{x^2}{x+y}}$

29. $\dfrac{1+\dfrac{1}{x^2-1}}{1+\dfrac{1}{x-1}}$

30. $\dfrac{3+\dfrac{5x}{x^2-4}}{3+\dfrac{2}{x-2}}$

31. $1-\dfrac{1}{2-\dfrac{1}{x}}$

32. $3+\dfrac{1}{4-\dfrac{2}{y}}$

33. $1+\dfrac{1}{x+\dfrac{1}{x-1}}$

34. $1+\dfrac{x+1}{x+\dfrac{1}{x+2}}$

35. $\dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{x+6+\dfrac{5}{x}}-\dfrac{\dfrac{1}{x^2}-\dfrac{9}{x^4}}{1+\dfrac{2}{x}-\dfrac{15}{x^2}}$

36. $\dfrac{\dfrac{x^2+7x+10}{x+2}}{x-6}+\dfrac{x-2}{\dfrac{x^2-2x}{x-3}}$

37. $1-\dfrac{1}{1+\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{1-\dfrac{1}{1+1}}}}}$

38. $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+1}}}}}$

5.8 EQUATIONS WITH FRACTIONS

To solve an equation with fractions, multiply each side by the L.C.D. The resulting equation will not contain any fractions and thus may be solved as those equations in Chapter 2.

EXAMPLE 1

Solve.

$$\frac{x}{12} + \frac{1}{6} = \frac{x+1}{8}$$

The L.C.D. of 12, 6, and 8 is 24; therefore, multiply each side of the preceding equation by 24.

$$24\left(\frac{x}{12} + \frac{1}{6}\right) = 24\left(\frac{x+1}{8}\right)$$

$$2x + 4 = 3(x+1)$$

$$2x + 4 = 3x + 3$$

$$1 = x$$

Check: Substitute 1 for x each time it occurs in the *original equation.*

$$\frac{(1)}{12} + \frac{1}{6} = \frac{(1)+1}{8}$$

$$\frac{1}{12} + \frac{2}{12} = \frac{2}{8}$$

$$\frac{3}{12} = \frac{2}{8}$$

$$\frac{1}{4} = \frac{1}{4}$$

It is common practice in science and technology to solve a given formula for the unknown letter before the substitution of the data is made. To solve a formula involving fractions, follow the same procedure as above.

EXAMPLE 2

Solve for r.

$$\frac{E}{e} = \frac{R+r}{r}$$

The L.C.D. is er; therefore, multiply each side of the preceding equation by er.

$$er\left(\frac{E}{e}\right) = er\left(\frac{R+r}{r}\right)$$

$$Er = eR + er$$

$$Er - er = eR \qquad \text{(Subtract } er \text{ from each side.)}$$

$$(E-e)r = eR \qquad \text{(Factor.)}$$

$$r = \frac{eR}{E-e} \qquad \text{(Divide each side by } E-e.\text{)}$$

EXAMPLE 3

Solve.

$$\frac{x}{x-5} = \frac{5}{x-5} + 2$$

The L.C.D. is $x - 5$; therefore, multiply each side by $x - 5$.

$$(x-5)\left[\frac{x}{x-5}\right] = (x-5)\left[\frac{5}{x-5} + 2\right]$$

$$x = 5 + 2(x-5)$$

$$x = 5 + 2x - 10$$

$$x = 2x - 5$$

$$5 = x$$

Check:

$$\frac{5}{5-5} = \frac{5}{5-5} + 2$$

$$\frac{5}{0} \neq \frac{5}{0} + 2$$

because $\frac{5}{0}$ is meaningless.

The only apparent solution, $x = 5$, does not satisfy the *original* equation. Therefore, the original equation has no solution; that is, there is no real number that can replace x in the original equation and produce a true statement.

> The technique of multiplying each side of an equation by a **variable** quantity gives the possibility of obtaining *extraneous,* or false, solutions.

For example, the simple equation

$$x = 3$$

obviously has only one solution. If we multiply each side by x, we have

$$x^2 = 3x$$

But this equation has two solutions: 3 and 0. Here, 0 is an extraneous solution.

> In solving an equation using the technique of multiplying each side by a variable quantity, the only way to determine whether or not an apparent solution is extraneous is to check each apparent solution in the *original* equation.

EXAMPLE 4

Solve.

$$\frac{x}{x+2} + \frac{4}{x-2} = \frac{x^2}{\underset{(x+2)(x-2)}{x^2-4}}$$

The L.C.D. is $(x + 2)(x - 2)$.

$$x(x-2)+4(x+2)=x^2$$
$$x^2-2x+4x+8=x^2$$
$$2x+8=0$$
$$2x=-8$$
$$x=-4$$

Check: Substitute the apparent solution, $x=-4$, in the original equation.

$$\frac{(-4)}{(-4)+2}+\frac{4}{(-4)-2}=\frac{(-4)^2}{(-4)^2-4}$$
$$\frac{-4}{-2}+\frac{4}{-6}=\frac{16}{12}$$
$$2-\frac{2}{3}=\frac{4}{3}$$
$$\frac{4}{3}=\frac{4}{3}$$

Since $x=-4$ satisfies the original equation, $x=-4$ is the solution, or root.

As we have seen, multiplying each side of an equation by a variable quantity can lead to extraneous solutions. This operation requires checking the solutions. It is also important *not* to divide each side by a variable quantity, because this operation may result in the loss of solutions. For example, the equation

$$x^2=5x$$

has two solutions: 5 and 0. If we divide each side of this equation by x, we have

$$x=5$$

and we have lost a solution.

You should note that there is a great difference between a fractional expression involving addition, subtraction, multiplication, and division of algebraic fractions and an algebraic fractional equation. The only time that you multiply by the L.C.D. to eliminate fractions is when an equal sign is present, which indicates a fractional equation.

Exercises 5.8

Solve and check.

1. $\frac{x}{3}-\frac{x}{6}=2$

2. $\frac{y}{4}+\frac{y}{12}=5$

3. $\frac{3m}{4}+\frac{2m}{5}=\frac{23}{2}$

4. $\frac{5t}{4}-\frac{4t}{9}=\frac{29}{4}$

5. $\frac{x}{6}+2=\frac{x+3}{4}$

6. $\frac{t-6}{6}+\frac{1}{10}=\frac{t}{15}$

7. $\frac{s+4}{8}-1\frac{1}{8}=\frac{3s}{4}$

8. $\frac{4m}{5}+7\frac{2}{3}=\frac{m+5}{15}$

9. $\frac{x+5}{2}=\frac{x+4}{3}$

10. $\frac{x+3}{5}=\frac{x-7}{4}$

11. $\frac{x}{20}+\frac{1}{4}=\frac{x+2}{5}$

12. $\frac{x}{15}+\frac{1}{3}=\frac{x-3}{5}$

13. $\frac{2x+6}{4}+1=\frac{3x+1}{8}$

14. $\frac{4x-3}{9} - 2 = \frac{x-4}{3}$

15. $\frac{5-x}{7} - \frac{2x-1}{14} = 1$

16. $\frac{2-x}{24} - \frac{3x+4}{16} = 2$

In Exercises 17–28, solve for the indicated letter.

17. $F = \frac{mv^2}{r}$ for r

18. $R = \frac{\rho L}{A}$ for ρ

19. $I_L = \frac{V}{R + R_L}$ for R

20. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ for f

21. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R_1

22. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ for R_2

23. $y = \frac{x+y}{x}$ for y

24. $y = \frac{x-y}{x}$ for x

25. $\frac{x}{a+b} = \frac{x+a}{b}$ for x

26. $\frac{1}{x+a} - \frac{a}{x-a} = \frac{ax}{x^2-a^2}$ for x

27. $V = \frac{Q}{R_1} - \frac{Q}{R_2}$ for Q

28. $V = \frac{Q}{R_1} - \frac{Q}{R_2}$ for R_2

Solve and check.

29. $\frac{3}{2x} = \frac{5}{x} - 7$

30. $\frac{7}{2x} - \frac{5}{3} = \frac{1}{x}$

31. $\frac{x-3}{x} = \frac{1}{x} + \frac{2}{3}$

32. $\frac{2x+5}{x} = \frac{2}{x} + \frac{1}{4}$

33. $\frac{1}{x-2} = \frac{3}{x+4}$

34. $\frac{6}{3-2x} = \frac{2}{5-3x}$

35. $\frac{x+2}{x-3} = \frac{x+1}{x-1}$

36. $\frac{6t-3}{2t+1} = \frac{9t}{3t+4}$

37. $\frac{2x^2+8}{x^2-1} - \frac{x-4}{x+1} = \frac{x}{x-1}$

38. $\frac{x-3}{x+3} = \frac{2x+3}{x-3} - \frac{x^2}{x^2-9}$

39. $\frac{x+2}{x+7} = \frac{2}{x-2} + \frac{x^2+3x-28}{x^2+5x-14}$

40. $\frac{1}{x-4} + \frac{1}{x-5} + \frac{1}{x^2-9x+20} = 0$

41. $\frac{x}{x+4} + \frac{x+1}{x-5} = 2$

42. $\frac{-3x^2+5x}{x^2+6x+9} + 4 = \frac{x+5}{x+3}$

43. $\frac{x^2+3x+7}{x^2-x-12} + \frac{x+6}{x+3} = \frac{2x+1}{x-4}$

44. $\frac{7x^2+x-17}{2x^2-5x+3} - \frac{3x-4}{2x-3} = \frac{2x+1}{x-1}$

CHAPTER SUMMARY

1. Special products:
   (a) *Product of a monomial and a polynomial:*

$$a(x + y + z) = ax + ay + az$$

   (b) $(x + y)(x - y) = x^2 - y^2$
   (c) *Square of a binomial:*

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$

   (d) *Square of a trinomial:*

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

   (e) *Cube of a binomial:*

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

2. *Factoring:*
   (a) *Greatest common factor:*

$$ax + ay + az = a(x + y + z)$$

   (b) *Difference of two perfect squares:*

$$x^2 - y^2 = (x + y)(x - y)$$

   (c) *Trinomials by trial and error.*
   (d) *Perfect square trinomials:*

$$x^2 + 2xy + y^2 = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)^2$$

   (e) *Sum and difference of two perfect cubes:*

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

3. *Equivalent fractions:* Two fractions are equivalent when both numerator and denominator of one fraction can be multiplied or divided by the same nonzero number in order to change one fraction to the other.
4. *Signs of fractions:* A fraction has three signs—the sign of the fraction, the sign of the numerator, and the sign of the denominator. Any two of these three signs may be changed without changing the value of the fraction. For example,

$$-\frac{-a}{+b} = -\frac{+a}{-b} = +\frac{+a}{+b} = +\frac{-a}{-b}$$

5. *Lowest terms:* A fraction is in lowest terms when its numerator and denominator have no common factors except 1.

6. *Factors that differ in sign:* When factors differ in sign, one may be replaced by the opposite of the other; that is,

$$x - y = -(y - x) \quad \text{or} \quad y - x = -(x - y)$$

7. *To multiply fractions:*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

8. *To divide fractions:*

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

9. *To find the lowest common denominator (L.C.D.):*
   (a) Factor each denominator completely.
   (b) Then the L.C.D. is the product formed by using each of the different factors the greatest number of times that it occurs in any *one* of the given denominators.

10. *Adding or subtracting fractions:*
   (a) If the fractions do not have a common denominator, change each fraction to an equivalent fraction having the L.C.D. (For each fraction, divide the L.C.D. by the denominator. Then multiply both numerator and denominator by this result in order to obtain an equivalent fraction having the L.C.D.)
   (b) Add or subtract the numerators in the order they occur, and place this result over the L.C.D.
   (c) Reduce the resulting fraction to lowest terms, if possible.

11. *Complex fraction:* A fraction that contains a fraction in the numerator, the denominator, or both. There are basically two ways to simplify a complex fraction.

Method 1: Multiply both numerator and denominator of the complex fraction by the L.C.D. of all the fractions that appear in the numerator and denominator.

Method 2: Simplify the numerator and the denominator separately. Then divide the simplified numerator by the simplified denominator. Finally simplify that result, if possible.

12. *To solve an equation with fractions:* Multiply each side by the lowest common denominator. When you multiply each side by a variable quantity you *must check* each apparent solution in the original equation.

CHAPTER 5 REVIEW

Find each product mentally.

1. $-6a^2b^4(a^3 - 12ab^2 - 9b^5)$
2. $(3x + 13)(5x - 4)$
3. $(4x - 7)(4x + 7)$
4. $(5a^2 - 9b^2)^2$
5. $(4a - b^2)^3$
6. $(6x + 7yz^2)(5x - 2yz^2)$

Factor each expression ***completely.***

7. $18a^3b - 9a^2b^4 + 9a^3b^2$
8. $16x^2 - 9y^2$

9. $5x^2 + 28x + 32$

10. $14x^2 + 59x - 18$

11. $4a^2 - 20a + 25$

12. $m^3 - n^3$

13. $3x^2 - 18x - 21$

14. $x^4 - 81$

15. $15x^2 + 23x - 90$

16. $10x^2 + 25xy - 60y^2$

17. $x^2 + 6x + 9 - 25y^2$

18. $ax + 2bx - 4ay - 8by$

Reduce each fraction to lowest terms.

19. $\dfrac{90x^4y^9}{48x^8y^3}$

20. $\dfrac{16x^2 - 40x}{8x^2 - 20x}$

21. $\dfrac{x^2 - 4x - 32}{2x^2 + 11x + 12}$

22. $\dfrac{-2x^2 + 7x + 4}{x^2 - 16}$

Perform the indicated operations and simplify.

23. $\dfrac{2x^2 + 5x - 25}{4x^2 + 21x + 5} \cdot \dfrac{4x^2 + x}{4x^2 - 25}$

24. $\dfrac{6x^2 + 11x - 7}{4x^2 + 4x + 1} \div \dfrac{3x^2 + 4x - 7}{-2x^2 + x + 1}$

25. $\dfrac{y^6 + 125}{4y^2 + 20} \div \dfrac{y^4 - 5y^2 + 25}{12y}$

26. $\dfrac{2 - x}{x^2 + 10x + 21} \cdot \dfrac{3x^2 + 21x}{18x + 36} \div \dfrac{x^3 - 4x}{x + 3}$

27. $\dfrac{4}{3a} + \dfrac{9}{4a^2} - \dfrac{5}{2}$

28. $4 - \dfrac{3x + 4}{2x + 1}$

29. $\dfrac{x}{x^2 - 7x + 10} + \dfrac{x}{2x^2 - 5x + 2}$

30. $\dfrac{x - 1}{2x^2 + 3x + 1} + \dfrac{x - 3}{3x^2 - x - 4}$

31. $\dfrac{\dfrac{1}{x} + \dfrac{1}{x^2}}{x + \dfrac{1}{x^2}}$

32. $4 + \dfrac{2}{3 + \dfrac{1}{y}}$

Solve for x and check.

33. $\dfrac{x + 2}{5} + \dfrac{x + 4}{30} = \dfrac{x}{6}$

34. $\dfrac{ax + b}{b} = \dfrac{bx}{a} + \dfrac{a}{b}$

35. $\dfrac{x + 1}{x - 3} + 1 = \dfrac{4}{x - 3}$

36. $\dfrac{4}{x - 4} - \dfrac{x^2}{16 - x^2} = 1$

Many equations used by electronic technicians to solve problems involve fractions. In this chapter you will have an opportunity to expand your understanding of algebra by working with fractions, fractional equations, and literal equations containing fractions and by evaluating formulas.

9–1 SOLVING EQUATIONS CONTAINING FRACTIONS

Equations having numerical coefficients that are fractions are transformed and solved by the methods in earlier chapters. Equations with fractional coefficients are best solved by transforming them into equations without fractions. This is done by multiplying each member of the equation by the common denominator.

EXAMPLE 9–1 Solve $\frac{3}{8}y = \frac{3}{4}$.

Solution Multiply each member by the common denominator, 8; then simplify:

$$(\mathbf{8})\tfrac{3}{8}y = \tfrac{3}{4}(\mathbf{8})$$

$$3y = 6$$

$$y = 2$$

Check Does $\frac{3}{8}y = \frac{3}{4}$ when $y = 2$?

$$\tfrac{3}{8}(\mathbf{2}) = \tfrac{3}{4}$$

$$\tfrac{3}{4} = \tfrac{3}{4} \quad \text{Yes!}$$

$\therefore$ $y = 2$ is the solution.

EXAMPLE 9–2 Solve $\dfrac{2x}{5} - 3 = \dfrac{5 - 3x}{10}$.

Solution Multiply each member by the common denominator of 10:

$$\mathbf{10}\left(\frac{2x}{5} - 3\right) = \mathbf{10}\left(\frac{5 - 3x}{10}\right)$$

$$\mathbf{10}\left(\frac{2x}{5}\right) + \mathbf{10}(-3) = \mathbf{10}\left(\frac{5 - 3x}{10}\right)$$

$$2(2x) - 30 = 5 - 3x$$

$$4x + 3x = 5 + 30$$

$$7x = 35$$

$$x = 5$$

Check Does $\frac{2x}{5} - 3 = \frac{5 - 3x}{10}$ when $x = 5$?

$$\frac{2(5)}{5} - 3 = \frac{5 - 3(5)}{10}$$

$$2 - 3 = \frac{5 - 15}{10}$$

$$-1 = -1 \quad \text{Yes!}$$

$\therefore$ $x = 5$ is the solution.

EXAMPLE 9–3 Solve $\frac{3x + 3}{12} = \frac{x - 3}{18} + \frac{x}{6}$.

Solution Multiply each member by the common denominator of 36:

$$\mathbf{36}\left(\frac{3x + 3}{12}\right) = \mathbf{36}\left(\frac{x - 3}{18} + \frac{x}{6}\right)$$

$$36\left(\frac{3x + 3}{12}\right) = 36\left(\frac{x - 3}{18}\right) + 36\left(\frac{x}{6}\right)$$

$$3(3x + 3) = 2(x - 3) + 6x$$

$$9x + 9 = 2x - 6 + 6x$$

$$9x - 8x = -15$$

$$x = -15$$

Check Does $\frac{3x + 3}{12} = \frac{x - 3}{18} + \frac{x}{6}$ when $x = -15$?

$$\frac{3(\mathbf{-15}) + 3}{12} = \frac{\mathbf{-15} - 3}{18} + \frac{\mathbf{-15}}{6}$$

$$\frac{-45 + 3}{12} = -1 + \frac{-15}{6}$$

$$\frac{-42}{12} = \frac{-6 - 15}{6}$$

$$\frac{-7}{2} = \frac{-21}{6}$$

$$\frac{-7}{2} = \frac{-7}{2} \quad \text{Yes!}$$

$\therefore$ $x = -15$ is the solution.

EXERCISE 9–1

Solve the following equations and check your solutions.

1. $\frac{1}{8}x = \frac{1}{4}$ **2.** $\frac{-x}{2} = \frac{1}{6}$ **3.** $\frac{a}{15} = \frac{1}{10}$

4. $\frac{5}{4} = \frac{4}{5}y$ **5.** $\frac{1}{3}R = -\frac{15}{7}$ **6.** $\frac{x}{3} = \frac{4}{5}$

7. $\frac{m-7}{4} = 2m$ **8.** $\frac{2b-5}{3} = \frac{45}{3}$ **9.** $\frac{4}{7} = \frac{12V}{16}$

10. $\frac{a}{3} - \frac{a}{2} = 5$ **11.** $\frac{3I+4}{7} = I + 2$ **12.** $\frac{3}{4} + E = \frac{5}{4}E$

13. $5 - \frac{2}{3}x = \frac{-x}{2}$ **14.** $\frac{V}{2} = \frac{4V}{5} - 3$

15. $\frac{1}{6} = \frac{a-15}{3} + \frac{7-a}{2}$ **16.** $\frac{m-3}{3} = \frac{m+4}{5} - 3$

17. $\frac{3x}{10} - \frac{2}{10} = \frac{2x-7}{5} + \frac{1}{5}$ **18.** $\frac{2R}{5} = \frac{4R-1}{4} - \frac{3}{16}$

19. $\frac{2I-3}{3} - 3 = -\frac{3}{6} - \frac{I}{2}$ **20.** $\frac{8y-1}{3} = 3 - \frac{10y}{6} - \frac{7}{6}$

21. $\frac{3x-3}{4} = \frac{2x}{3} - \frac{x-5}{2}$ **22.** $\frac{4a-3}{2} = \frac{4a-3}{5}$

9–2 SOLVING FRACTIONAL EQUATIONS

A **fractional equation** is an equation that has a variable in the denominator of one or more terms of the equation. The solution of fractional equations is carried out in the same way as the solution of equations containing fractions: by multiplying both members of the equation by the common denominator.

EXAMPLE 9–4 Solve $\frac{2}{x} = \frac{5}{x} - \frac{3}{5}$.

Solution Multiply each member by the common denominator of $5x$:

$$5x\left(\frac{2}{x}\right) = 5x\left(\frac{5}{x}\right) + 5x\left(-\frac{3}{5}\right)$$

$$10 = 25 - 3x$$
$$3x = 15$$
$$x = 5$$

Check Does $\frac{2}{x} = \frac{5}{x} - \frac{3}{5}$ when $x = 5$?

$$\frac{2}{5} = \frac{5}{5} - \frac{3}{5}$$

$$\frac{2}{5} = \frac{2}{5} \quad \text{Yes!}$$

$\therefore$ $x = 5$ is the solution.

It is important to check the roots of fractional equations. When we multiply both members of an equation by an expression containing the unknown, we run the risk of introducing an **extraneous root.** An extraneous root will *check* in the derived equation, but **not** in the original equation. Extraneous roots result from multiplying by zero.

EXAMPLE 9–5 Solve $\frac{5x}{x-1} = \frac{5}{x-1}$.

Solution Multiply each member by $(x - 1)$, the common denominator:

$$(x-1)\frac{5x}{x-1} = (x-1)\frac{5}{x-1}$$

$$5x = 5$$

$$x = 1$$

Check Does $\frac{5x}{x-1} = \frac{5}{x-1}$ when $x = 1$?

$$\frac{5(1)}{1-1} = \frac{5}{1-1}$$

$$\frac{5}{0} = \frac{5}{0}$$

No! Both members of the equation are undefined.
Since division by zero is not permitted, this equation has no root.

$\therefore$ $x = 1$ does not satisfy the equation.

EXAMPLE 9–6 Solve $\frac{3x}{x+1} = 3 - \frac{4}{2x}$.

Day: Date:

Solution Multiply each member by the common denominator $2x(x + 1)$ and then factor:

$$\frac{2x(x+1)(3x)}{x+1} = 2x(x+1)(3) - \frac{2x(x+1)(4)}{2x}$$

$$6x^2 = 6x^2 + 6x - 4x - 4$$
$$6x^2 - 6x^2 = 6x - 4x - 4$$
$$0 = 2x - 4$$
$$2x = 4$$
$$x = 2$$

Check Does $\frac{3x}{x+1} = 3 - \frac{4}{2x}$ when $x = 2$?

$$\frac{3(2)}{2+1} = 3 - \frac{4}{2(2)}$$
$$\frac{6}{3} = 3 - 1$$
$$2 = 2 \quad \text{Yes!}$$

$\therefore$ $x = 2$ is the solution.

EXERCISE 9–2

Solve the following equations and check your solutions.

1. $\frac{3}{x} + \frac{5}{x} = 2$
2. $\frac{8}{x} = \frac{4}{x} - \frac{2}{3}$
3. $\frac{8}{y} - 1 = \frac{7}{y}$
4. $\frac{3}{x} - 4 = \frac{7}{2}$
5. $\frac{8}{x+1} = 4$
6. $\frac{x+3}{x-3} = 4$
7. $\frac{3y}{4y-5} = 2$
8. $\frac{2x+5}{x+4} = 1$
9. $\frac{4}{y-3} = \frac{2}{y}$
10. $\frac{3x-2}{5x-10} = \frac{4}{10}$
11. $\frac{4x-2}{4x} - \frac{1}{2} = \frac{3}{x}$
12. $\frac{3}{2} - \frac{5}{2y} = 2 - \frac{3}{y}$
13. $\frac{x-2}{5x} = \frac{4x-8}{30x}$
14. $\frac{1}{y-1} = \frac{3}{y-3}$
15. $\frac{12}{x-6} = \frac{2x}{x-6}$
16. $5 - \frac{5x}{x+1} = \frac{15}{4x}$

17. $\dfrac{2y}{y+1} = 2 - \dfrac{8}{2y}$

18. $\dfrac{3}{y} = \dfrac{3y}{y-4} - 3$

19. $\dfrac{5}{3+y} = \dfrac{3y}{y+3} - \dfrac{3y-4}{2y+6}$

20. $\dfrac{3}{x} + \dfrac{6}{x^2+5x} = \dfrac{6x}{x^2+5x}$

9–3 LITERAL EQUATIONS CONTAINING FRACTIONS

Equations that have one or more numbers represented by letters are **literal equations.** The formulas used to solve electrical problems are literal equations.

Variables and Constants

In literal equations, such as $\dfrac{x}{a} - 2 = b$, $\dfrac{c}{3} = y$, and $\dfrac{x}{m} - n = a$, the letters a, b, c, m, and n are *standing in* for numbers that have a fixed or constant value. The letters x and y are *standing in* for numbers that may change (or could be made to change) or vary.

In literal equations, it is conventional to represent constants by the letters a, b, c, g, h, k, l, m, and n. It is conventional to represent variables by the letters s, t, u, v, w, x, y, and z.

Constants are divided into two types. Those represented by letters (a, b, c, etc.) are called **arbitrary constants,** whereas numbers and symbols representing unvarying values are called **numerical constants.** Numerical constants include integers (1, 2, 3, etc.), fractions ($\frac{1}{2}$, $\frac{2}{3}$, etc.), and irrational numbers ($\pi = 3.14159\ldots$, $e = 2.71828\ldots$, $\sqrt{2} = 1.414\ldots$, etc.).

EXAMPLE 9–7 Solve the literal equation $\dfrac{x}{a} + 2 = b$ for x.

Solution

$$\frac{x}{a} + 2 = b$$

$$\frac{x}{a} = b - 2$$

$$\therefore \quad x = a(b-2)$$

EXAMPLE 9–8 Solve $\frac{7}{b} - \frac{y}{a} = 4$ for y.

Solution

$$\frac{7}{b} - \frac{y}{a} = 4$$

$$7a - yb = 4ab$$

$$-yb = 4ab - 7a$$

Change sign:

$$yb = 7a - 4ab$$

$$\therefore \quad y = \frac{7a - 4ab}{b}$$

EXAMPLE 9–9 Solve $\frac{xm}{2} - \frac{m}{3} = x$ for x.

Solution

$$\frac{xm}{2} - \frac{m}{3} = x$$

$$\frac{xm}{2} = x + \frac{m}{3}$$

$$\frac{xm}{2} - x = \frac{m}{3}$$

$$6\left(\frac{xm}{2} - x\right) = 6\left(\frac{m}{3}\right)$$

$$3xm - 6x = 2m$$

Factor the left member:

$$x(3m - 6) = 2m$$

$$\therefore \quad x = \frac{2m}{3m - 6}$$

EXERCISE 9–3

Solve each equation for the *conventional* variable s, t, u, v, w, x, y, or z.

1. $\frac{a}{x} = b$

2. $\frac{3}{\pi} = \frac{a}{W}$

Solve each equation for the *conventional* variable $s, t, u, v, w, x, y,$ or z.

3. $\frac{1}{z} - \frac{1}{a} = -\frac{2}{z}$

4. $\frac{hu}{m} + \frac{ku}{m} = 5$

5. $\frac{y}{2m} + \frac{y}{n} = 1$

6. $2g = \frac{hke + V}{aV}$

7. $c = \frac{b - W}{\pi^2 kW}$

8. $z - 1 - m = \frac{5z - 8}{m}$

9. $\frac{c}{s} = \frac{c}{3}(D - c)$

10. $\frac{\pi x + b}{\pi x - b} = 3$

11. $\frac{m - n}{ay} = b + 1$

12. $\frac{2u + 2}{au} - \frac{3}{u} = -\frac{7 - u}{u}$

13. $\frac{5x}{g - h} - 3 = \frac{3x}{k}$

14. $\frac{1}{4(a - b)} = \frac{1}{5(x - a)}$

15. $\frac{1}{u} - \frac{1}{m} = \frac{1}{n} - \frac{1}{u}$

16. $\frac{1}{w} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

17. $\frac{b}{a} = \frac{x - b}{x} + \frac{2b}{ax}$

18. $\frac{4k + 1}{4k} = \frac{1}{2z} + \frac{2k}{z}$

19. $\frac{4g}{3g} - \frac{6}{6V} = \frac{4g}{V} - \frac{1}{3g}$

20. $\frac{z}{ab} - \frac{1}{ac} = \frac{z}{bc}$

9–4 EVALUATING FORMULAS

Evaluating a formula is the process of finding the value of one variable in a formula when all the other variables are known. When evaluating a formula for only one set of values, it is simpler first to substitute the known values into the formula and then solve for the variable representing the desired quantity. This technique is demonstrated in Example 9–10.

EXAMPLE 9–10 Using the formula $F = \frac{9}{5}C + 32$, determine the temperature in Celsius (C) when the temperature in Fahrenheit (F) is 77°F.

Solution First substitute 77 for F and then solve for C:

$$F = \frac{9}{5}C + 32$$

$$77 = \frac{9}{5}C + 32$$

$$\frac{9}{5}C = 45$$

$$C = 45\left(\frac{5}{9}\right)$$

$$\therefore \quad C = 25°\text{C}$$

In Example 9–10, we first substituted into the formula, and then solved the problem. When we wish to find several values for a particular variable, it is best first to solve the formula for the desired variable and then substitute into the formula. This concept is demonstrated in Example 9–11.

EXAMPLE 9–11 Using $F = \frac{9}{5}C + 32$, determine the temperature in Celsius (C) for temperatures in Fahrenheit (F) of: (a) 105°F (b) 93°F (c) 85°F.

Solution First solve for C:

$$F = \frac{9}{5}C + 32$$

$$\frac{9}{5}C = F - 32$$

$$C = \frac{5(F - 32)}{9}\,°\text{C}$$

Now, substitute each of the given temperatures.

(a) $F = 105°\text{F}$

$$\therefore \quad C = \frac{5(\mathbf{105} - 32)}{9} = 40.6°\text{C}$$

(b) $F = 93°\text{F}$

$$\therefore \quad C = \frac{5(\mathbf{93} - 32)}{9} = 33.9°\text{C}$$

(c) $F = 85°\text{F}$

$$\therefore \quad C = \frac{5(\mathbf{85} - 32)}{9} = 29.4°\text{C}$$

In the study of electronics, formulas are routinely evaluated. Because of this, it is very important that you develop the skill of successfully working with formulas. The following exercise will give you some practice.

EXERCISE 9–4

In the following formulas, solve for the indicated variable.

	Formula:	***Solve for:***
1.	$C = \frac{5}{9}(\text{F} - 32)$	F
2.	$A = \frac{1}{2}h(b + c)$	c

Formula:	Solve for:
3. $X_C = \frac{1}{2\pi fC}$	f
4. $C = \frac{1}{C_1} + \frac{1}{C_2}$	C_2
5. $k = \frac{wv^2}{2g}$	w
6. $\beta = \frac{\alpha}{1 - \alpha}$	α
7. $\theta_{JA} = \frac{T_J - T_A}{P_D}$	T_A
8. $A_v' = \frac{A_v}{1 + \beta A_v}$	A_v
9. $C = \frac{E}{R_1 + R_2}$	R_2
10. $\frac{V_1}{V_2} = \frac{P_2}{P_1}$	P_1
11. $\theta = \frac{108(n - 2)}{n}$	n
12. $s = t\left(\frac{V_0 + V_1}{2}\right)$	V_1
13. $\eta = \frac{A_2 S_2}{F_1 S_1}$	S_2
14. $Q = \frac{tKA(T_0 - T_1)}{L}$	T_0
15. $F = \frac{q_1 q_2}{4\pi\epsilon_0 d^2}$	d^2
16. $V = \frac{R_2(V_{CC} - R_1 I_{\text{sat}} - 1)}{R_1 + R_2}$	R_1
17. $R_T = \frac{R_1 R_2}{R_1 + R_2}$	R_2
18. $I_1 = \frac{IR_2}{R_1 + R_2}$	R_2
19. $S = \frac{R_1 + R_2}{R_E + R_\beta(1 - \alpha)}$	α
20. $R_2 = \frac{Z_Z(k + 1)}{k - 1} - R_3$	k

Evaluate each of the following formulas. First assign the indicated values to the variables and then solve. Express the solution to an appropriate precision with a unit and unit prefix (as needed).

21. Determine the resistance (R) of a coil in ohms from

$$Q_0 = \frac{2\pi f_{ar} L}{R}$$

when $f_{ar} = 310$ kHz, $L = 5.00$ mH, $Q_0 = 200$, and $\pi \approx 3.14$.

22. Determine the collector current (I_c) in amperes from

$$I_{CBO} = \frac{I_c - \beta I_B}{\beta + 1}$$

when $I_{CBO} = 5.00\ \mu A$, $I_B = 60.0\ \mu A$, and $\beta = 100$.

23. Determine the total resistance of a parallel network (R_T) in ohms from

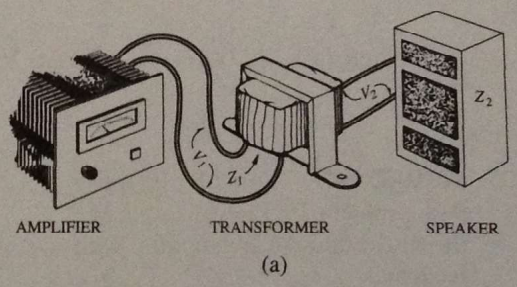
$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

when $R_1 = 2.20\ \text{k}\Omega$ and $R_2 = 1.50\ \text{k}\Omega$.

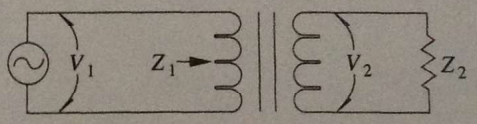
24. Determine the impedance of the load (Z_2) in ohms from

$$\frac{V^2_1}{V^2_2} = \frac{Z_1}{Z_2}$$

when $V_1 = 60$ V, $V_2 = 2$ V, and $Z_1 = 7.2\ \text{k}\Omega$. Refer to Figure 9–1.



(a)



(b)

FIGURE 9–1 Circuit for Exercise 9–4, problem 24: (a) pictorial diagram showing an output transformer being used to match the load impedance to the amplifier; (b) schematic diagram of the circuit.

25. Determine the frequency (f) in hertz from

$$X_C = \frac{1}{2\pi fC}$$

when $X_C = 410\ \Omega$, $C = 50$ nF, and $\pi \approx 3.14$.

26. Determine the output voltage (V_o') in volts from

$$\frac{V_o'}{V_s} = \frac{A_v}{1 - \beta A_v}$$

when $\beta = 0.050$, $A_v = 60$, and $V_s = 1.0$ V. Refer to Figure 9–2.

27. Determine the forward current transfer ratio of a transistor (β) from

$$\alpha = \frac{\beta}{\beta + 1}$$

when $\alpha = 0.985$.

28. Determine the load resistance (R_L) in ohms from

$$A_v = \frac{\mu R_L}{r_p + R_L}$$

when $\mu = 18$, $r_p = 0.20\ \text{k}\Omega$, and $A_v = 12$.

29. Determine the value of the bias resistance (R_1) in ohms from

$$I_B = \frac{V_{cc}}{R_1 + (\beta + 1)R_E}$$

when $I_B = 40.0\ \mu\text{A}$, $V_{cc} = 18.0$ V, $\beta = 100$, and $R_E = 470\ \Omega$.

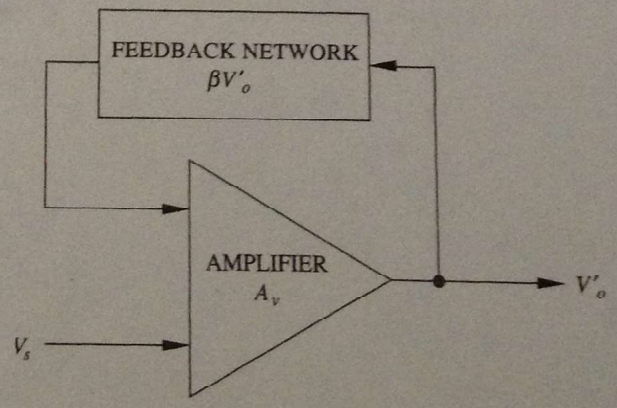


FIGURE 9–2 Circuit for Exercise 9–4, problem 26. Flow diagram of the signal through an amplifier with feedback.

30. Determine the temperature of the case of a power transistor (T_c) in degrees Celsius from

$$P_D = \frac{T_c - T_s}{\theta_{cs}}$$

when $P_D = 75$ W, $T_s = 55°C$, and $\theta_{cs} = 0.35°C/W$.

SELECTED TERMS

extraneous root A solution of a mathematical equation that is not a solution to the physical problem.

fractional equation Equation that has a variable in the denominator of one or more terms.

literal equation Equation with one or more numbers represented by letters.

INTRODUCTION

The properties that allow us to change the way algebraic expressions look without changing their values or their characteristics are very important. Expressions often need a different format so that we can simplify them or enter them into a calculator or a computer.

In this chapter we will study factoring, simplifying algebraic fractions, and solving equations with algebraic fractions.

Objectives

- In an algebraic expression, find the greatest common factor.
- Identify and factor the difference of two squares.
- Identify and factor the square of a binomial.
- Factor trinomials.
- Factor using the method of grouping.
- Identify and factor the sum or difference of two cubes.
- Add, subtract, multiply, and divide algebraic fractions.
- Solve equations containing algebraic fractions.

5.1 SPECIAL PRODUCTS

In Chapter 2 we introduced certain fundamental algebraic concepts and operations. Before proceeding, we must develop some additional algebraic techniques that are necessary in the development of later topics.

Certain types of algebraic products occur so often that we must be able to recognize them on sight. We must also be able to do the multiplications quickly and mentally.

To illustrate the first two types of special algebraic products, we give the general form of each followed by two examples.

$$\boxed{a(x + y + z) = ax + ay + az}$$

EXAMPLE 1

$$3x(x^2 + 2x - 4) = 3x^3 + 6x^2 - 12x$$

EXAMPLE 2

$$4a^2b^3(7a^4b^3 - 5a^2b^3 - ab + 3a^2) = 28a^6b^6 - 20a^4b^6 - 4a^3b^4 + 12a^4b^3$$

$$\boxed{(x + y)(x - y) = x^2 - y^2}$$

EXAMPLE 3

$$(3a + 4)(3a - 4) = 9a^2 - 16$$

EXAMPLE 4

$$(6a^2 + 5xy^3)(6a^2 - 5xy^3) = 36a^4 - 25x^2y^6$$

There are two general forms of the *square of a binomial.*

$$\boxed{(x + y)^2 = x^2 + 2xy + y^2}$$

EXAMPLE 5

$$(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$$

EXAMPLE 6

$$(5ab^2 + 4c)^2 = 25a^2b^4 + 40ab^2c + 16c^2$$

$$\boxed{(x - y)^2 = x^2 - 2xy + y^2}$$

EXAMPLE 7

$$(3x - 4)^2 = 9x^2 - 24x + 16$$

EXAMPLE 8

$$(2a^2 - 5y^3)^2 = 4a^4 - 20a^2y^3 + 25y^6$$

The *product of two binomials* may be found mentally using the FOIL method discussed in Section 2.3.

EXAMPLE 9

$$(x + 4)(x + 6) = x^2 + 10x + 24$$

EXAMPLE 10

$$(x + 3)(x - 5) = x^2 - 2x - 15$$

EXAMPLE 11

$$(2x + 3)(5x + 4) = 10x^2 + 23x + 12$$

EXAMPLE 12

$$(3x - 4)(-2x + 7) = -6x^2 + 29x - 28$$

The general form of the *square of a trinomial* is as follows.

$$\boxed{(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz}$$

EXAMPLE 13

$$(4x + y + 3z)^2 = 16x^2 + y^2 + 9z^2 + 8xy + 24xz + 6yz$$

The *square of a trinomial* may also be found using grouping and the square of a binomial as follows for Example 13:

$$\begin{aligned}(4x + y + 3z)^2 &= [(4x + y) + 3z]^2 \\ &= (4x + y)^2 + 2(4x + y)(3z) + (3z)^2 \\ &= 16x^2 + 8xy + y^2 + 24xz + 6yz + 9z^2\end{aligned}$$

EXAMPLE 14

$$(3a - b^2 - 2c^3)^2 = 9a^2 + b^4 + 4c^6 - 6ab^2 - 12ac^3 + 4b^2c^3$$

Or

$$\begin{aligned}(3a - b^2 - 2c^3)^2 &= [(3a - b^2) - 2c^3]^2 \\ &= (3a - b^2)^2 - 2(3a - b^2)(2c^3) + (2c^3)^2 \\ &= 9a^2 - 6ab^2 + b^4 - 12ac^3 + 4b^2c^3 + 4c^6\end{aligned}$$

There are two general forms of the *cube of a binomial.*

$$\boxed{(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3}$$

EXAMPLE 15

$$\begin{aligned}(2a + 5b)^3 &= (2a)^3 + 3(2a)^2(5b) + 3(2a)(5b)^2 + (5b)^3 \\ &= 8a^3 + 60a^2b + 150ab^2 + 125b^3\end{aligned}$$

EXAMPLE 16

$$\begin{aligned}(4x + y)^3 &= (4x)^3 + 3(4x)^2(y) + 3(4x)(y)^2 + (y)^3 \\ &= 64x^3 + 48x^2y + 12xy^2 + y^3\end{aligned}$$

$$\boxed{(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3}$$

EXAMPLE 17

$$(4a - b)^3 = (4a)^3 - 3(4a)^2(b) + 3(4a)(b)^2 - (b)^3$$
$$= 64a^3 - 48a^2b + 12ab^2 - b^3$$

EXAMPLE 18

$$(3x - 2y)^3 = (3x)^3 - 3(3x)^2(2y) + 3(3x)(2y)^2 - (2y)^3$$
$$= 27x^3 - 54x^2y + 36xy^2 - 8y^3$$

Exercises 5.1

Find each product mentally.

1. $-8x^2(5x^3 - 9x^2 + 10x)$
2. $5x^2(3x^5 - 4x^3 + 2x^2 - 6x + 7)$
3. $6x^3y^5(4xy - 7x^3y^2 + 9x^4)$
4. $6a^2b^4(-2a^2b^2 + 5ab^2 - 9b^3)$
5. $(2x + 7)(2x - 7)$
6. $(3x^2 + 8yz^2)(3x^2 - 8yz^2)$
7. $(7x^2 + 2y)^2$
8. $(8a^2b + 5c)^2$
9. $(2a^2 - 3b)^2$
10. $(9x - 4y^3)^2$
11. $(x + 4)(x + 12)$
12. $(x - 9)(x + 8)$
13. $(2x + 5)(-3x - 8)$
14. $(4x - 7)(5x - 6)$
15. $(2a - b + 3c)^2$
16. $(-3a + 2b - c)^2$
17. $(2a + b)^3$
18. $(4x + 5y)^3$
19. $(5 - 2x^2)^3$
20. $(2x - 4y^3)^3$
21. $(6a + 5b)(7a - 3b)$
22. $(5x - y)(4x - 8y)$
23. $(4x + 7y)^2$
24. $(2x - 10y)^2$
25. $(2a + 9b)(6a - 11b)$
26. $(2x^2y + 7z^3)^2$
27. $(3a^3 + 10b^2)^3$
28. $(8 - a^2b^3)^2$
29. $-4a^2b^3(6a^3b^4 - 9a^5b^7 + 20a^5)$
30. $(2 + a)(4 - 2a)$
31. $(9a^3 - 12b^2)^2$
32. $(12a - 5b)(15a + 2b)$
33. $(x - 3y - 5z)^2$
34. $(-3x + 7)(7x + 3)$
35. $(9a - 4b)(6a - 10b)$
36. $(80 - a^5)(80 + a^5)$
37. $(6x^3 + 4)(5x^3 - 4)$
38. $(4x^2 - 5)(3x^2 + 7)$
39. $(3xy + 9yz)(5xy - 8yz)$
40. $(-5y + z)^2$
41. $(5abc + 6)(5abc - 6)$
42. $(1 - a^2)^2$
43. $(3x^2 - \frac{1}{2}y)(x^2 + \frac{2}{3}y)$
44. $(2x^2 + \frac{1}{4}y^3)(2x^2 + \frac{1}{2}y^3)$
45. $(x - \frac{2}{3}y)^2$
46. $(4x^2 - \frac{1}{2}y)^2$
47. $x^2(1 + x^2)^2$
48. $x^2(1 - x^2)^3$

5.2 FACTORING ALGEBRAIC EXPRESSIONS

Factoring is the process of writing an algebraic expression as a product of two or more factors*—usually prime factors. The process is the opposite of multiplying algebraic expressions; that is, the reverse of the product forms in the previous section.

*Recall that in the product ab, a and b are called **factors.**

The first type of factoring involves finding the largest monomial factor that divides each factor "evenly," †dividing each term by this factor, and then writing the result as a product.

GREATEST COMMON FACTOR

$$ax + ay + az = a(x + y + z)$$

EXAMPLE 1

Factor $4x + 8y$.

$$4x + 8y = 4(x + 2y)$$

EXAMPLE 2

Factor $21x^3 - 14x^2 + 28x$.

$$21x^3 - 14x^2 + 28x = 7x(3x^2 - 2x + 4)$$

EXAMPLE 3

Factor $50xy^2z - 80xyz - 32xz$.

$$50xy^2z - 80xyz - 32xz = 2xz(25y^2 - 40y - 16)$$

Note: You should always try this type of factoring first, because many times it simplifies the remaining factor into something that factors more easily.

The difference of two perfect squares is the product of the square root of the first *plus* the square root of the second times the square root of the first *minus* the square root of the second.

DIFFERENCE OF TWO PERFECT SQUARES

$$x^2 - y^2 = (x + y)(x - y)$$

EXAMPLE 4

Factor $9a^2 - b^2$.

$$9a^2 - b^2 = (3a + b)(3a - b)$$

EXAMPLE 5

Factor $36x^2 - 49y^4$.

$$36x^2 - 49y^4 = (6x + 7y^2)(6x - 7y^2)$$

EXAMPLE 6

Factor $x^2 - 4a^2b^2$.

$$x^2 - 4a^2b^2 = (x + 2ab)(x - 2ab)$$

†The remainder is zero after dividing.

Note: The sum of two perfect squares, $x^2 + y^2$, cannot be factored.

EXAMPLE 7

Factor $81x^4 - y^8$.

$$81x^4 - y^8 = (9x^2 + y^4)(9x^2 - y^4)$$
$$= (9x^2 + y^4)(3x + y^2)(3x - y^2)$$

Note: To factor this expression completely, we must also factor $9x^2 - y^4$.

The factors of a general trinomial are often binomial factors. To find these binomial factors, you "undo" the FOIL multiplication.

EXAMPLE 8

Factor $x^2 + 7x + 10$.

This type of factoring involves some trial and error. Here we need two numbers whose sum is 7 and whose product is 10. They are $+2$ and $+5$.

$$x^2 + 7x + 10 = (x + 2)(x + 5)$$

EXAMPLE 9

Factor $x^2 - 9x + 18$.

We need two numbers whose sum is -9 and whose product is 18: -3 and -6.

$$x^2 - 9x + 18 = (x - 3)(x - 6)$$

EXAMPLE 10

Factor $x^2 - 3x - 28$.

We need two numbers whose sum is -3 and whose product is -28: $+4$ and -7.

$$x^2 - 3x - 28 = (x + 4)(x - 7)$$

A short summary about the signs in trinomials might be helpful. If the trinomial to be factored is one of the following forms, use the corresponding sign patterns.

1. $x^2 + px + q = (x + \quad)(x + \quad)$
2. $x^2 - px + q = (x - \quad)(x - \quad)$
3. $x^2 + px - q = (x + \quad)(x - \quad)$
4. $x^2 - px - q = (x + \quad)(x - \quad)$

EXAMPLE 11

Factor $x^2 + 10x + 24$.

We need two numbers whose sum is 10 and whose product is 24: $+4$ and $+6$.

$$x^2 + 10x + 24 = (x + 4)(x + 6)$$

EXAMPLE 12

Factor $x^2 - 15x + 56$.

We need two numbers whose sum is -15 and whose product is 56: -7 and -8.

$$x^2 - 15x + 56 = (x - 7)(x - 8)$$

EXAMPLE 13

Factor $x^2 + 3x - 40$.

We need two numbers whose sum is 3 and whose product is -40: $+8$ and -5.

$$x^2 + 3x - 40 = (x + 8)(x - 5)$$

EXAMPLE 14

Factor $x^2 - 7x - 18$.

We need two numbers whose sum is -7 and whose product is -18: $+2$ and -9.

$$x^2 - 7x - 18 = (x + 2)(x - 9)$$

EXAMPLE 15

Factor $8x^2 + 14x + 3$.

When the coefficient of the squared term is not 1, the factoring involves more trial and error. Here, we need the following:

1. Pairs of factors whose product is $8x^2$. There are two possible pairs:

$$(8x \quad)(x \quad) \text{ and } (2x \quad)(4x \quad)$$

2. Pairs of factors whose product is 3. There is only one pair: $3 \cdot 1$.
3. To arrange these factors so that the middle term is $14x$. If the coefficient of the squared term is positive, the signs in the factors take the same forms as discussed in the summary after Example 10. The possibilities are as follows.

Factors	*Middle term*	
$(8x + 1)(x + 3)$	$25x$	
$(8x + 3)(x + 1)$	$11x$	
$(2x + 1)(4x + 3)$	$10x$	
$(2x + 3)(4x + 1)$	$14x$	(the correct combination)

Therefore, $8x^2 + 14x + 3 = (2x + 3)(4x + 1)$.

EXAMPLE 16

Factor $12x^2 - 28x + 15$.

Step 1: The pairs of factors of $12x^2$ are

$$(12x \quad)(x \quad), \quad (2x \quad)(6x \quad), \text{ and } (3x \quad)(4x \quad)$$

Step 2: The pairs of factors of 15 are $1 \cdot 15$ and $3 \cdot 5$.

Step 3: Arrange these factors so that the middle term is $-28x$. The possibilities are as follows:

Factors	*Middle term*	
$(12x - 1)(x - 15)$	$-181x$	
$(12x - 15)(x - 1)$	$-27x$	
$(12x - 3)(x - 5)$	$-63x$	
$(12x - 5)(x - 3)$	$-41x$	
$(2x - 1)(6x - 15)$	$-36x$	
$(2x - 15)(6x - 1)$	$-92x$	
$(2x - 3)(6x - 5)$	$-28x$	(the correct combination)
$(2x - 5)(6x - 3)$	$-36x$	

Factors	*Middle term*
$(3x - 1)(4x - 15)$	$-49x$
$(3x - 15)(4x - 1)$	$-63x$
$(3x - 3)(4x - 5)$	$-27x$
$(3x - 5)(4x - 3)$	$-29x$

Note: Both signs inside the factors must be negative. Therefore,

$$12x^2 - 28x + 15 = (2x - 3)(6x - 5)$$

Of course, we do not go through this long listing process for every factoring. The work is often done mentally, and we stop when we have found the combination of factors that gives the desired middle term.

EXAMPLE 17

Factor $12x^2 - 7x - 12$.

$$12x^2 - 7x - 12 = (4x + 3)(3x - 4)$$

EXAMPLE 18

Factor $30x^2 - 59x + 9$.

$$30x^2 - 59x + 9 = (5x - 9)(6x - 1)$$

EXAMPLE 19

Factor $10x^2 + 27xy - 9y^2$.

$$10x^2 + 27xy - 9y^2 = (10x - 3y)(x + 3y)$$

EXAMPLE 20

Factor $4x^2 + 28xy^2 + 45y^4$.

$$4x^2 + 28xy^2 + 45y^4 = (2x + 5y^2)(2x + 9y^2)$$

EXAMPLE 21

Factor $-6x^2 - 11x + 10$.

$$-6x^2 - 11x + 10 = (-3x + 2)(2x + 5)$$

An alternate approach to factoring a trinomial whose first term is negative involves dividing each term first by -1 and then proceeding as usual. Following this approach in this example leads to the following:

$$\begin{aligned} -6x^2 - 11x + 10 &= -(6x^2 + 11x - 10) \\ &= -(3x - 2)(2x + 5) \end{aligned}$$

Do you see that this result is equivalent to our previous result?

EXAMPLE 22

Factor $60x^2 + 20x - 105$.

$$\begin{aligned} 60x^2 + 20x - 105 &= 5(12x^2 + 4x - 21) \\ &= 5(2x + 3)(6x - 7) \end{aligned}$$

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Note: Dividing each term by the common monomial factor, 5, simplified the remaining trinomial factor.

EXAMPLE 23

Factor $180x^2 + 279x + 108$.

$$180x^2 + 279x + 108 = 9(20x^2 + 31x + 12)$$
$$= 9(5x + 4)(4x + 3)$$

EXAMPLE 24

Factor $9x^2 + 24xy + 16y^2$.

$$9x^2 + 24xy + 16y^2 = (3x + 4y)(3x + 4y) = (3x + 4y)^2$$

EXAMPLE 25

Factor $16x^2 - 8xy + y^2$.

$$16x^2 - 8xy + y^2 = (4x - y)(4x - y) = (4x - y)^2$$

When the factors of a trinomial are the same two binomial factors, the trinomial is called a **perfect square trinomial.** The general forms are as follows.

PERFECT SQUARE TRINOMIALS

$$x^2 + 2xy + y^2 = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)^2$$

Exercises 5.2

Factor each expression completely.

1. $6x + 9y$
2. $15x - 20$
3. $10x + 25y - 45z$
4. $8x^2 - 12xy + 10xz$
5. $12x^2 - 30xy + 6xz$
6. $10x^2 + 25x$
7. $8x^3y^2 - 6xy^3 + 12xy^2z$
8. $15x^7y^3 - 6x^6y^2 + 12x^3yz$
9. $x^2 - 16$
10. $x^2 - 64$
11. $9x^2 - 25y^2$
12. $49x^2 - 100y^2$
13. $2e^2 - 72$
14. $20x^2 - 45$
15. $16d^2 - 100$
16. $64x^2 - 4y^2$
17. $4R^2 - 4r^2$
18. $49 - a^4$
19. $x^2 + 6x + 8$
20. $x^2 + 8x + 15$
21. $b^2 + 11b + 24$
22. $m^2 + 11m + 30$
23. $x^2 - 9x + 18$
24. $x^2 - 9x + 14$
25. $a^2 - 18a + 32$
26. $c^2 + 22c + 40$
27. $x^2 - 2x - 35$
28. $x^2 + 4x - 12$
29. $a^2 + 3a - 4$
30. $q^2 - 3q - 28$
31. $2x^2 - x - 6$
32. $5x^2 + 32x + 12$

33. $15x^2 - 31x + 14$
34. $8x^2 + 26x - 45$
35. $45y^2 + 59y + 6$
36. $8m^2 - 14m + 3$
37. $35a^2 + 2a - 1$
38. $16g^2 + 8g + 1$
39. $9c^2 - 24c + 16$
40. $6m^2 - 13m + 5$
41. $25b^2 + 60b + 20$
42. $15t^2 + 69t - 30$
43. $35t^2 - 4ts - 15s^2$
44. $12a^2 + 8ab - 15b^2$
45. $30k^2 - 95kt + 50t^2$
46. $24m^2 + 34m + 10$
47. $54x^2 + 27x - 42$
48. $8x^2 - 24xy - 1440y^2$
49. $4a^2 + 20a + 25$
50. $9x^2 + 42x + 49$
51. $9x^2 - 48xy^2 + 64y^4$
52. $4a^4 + 12a^2b + 9b^2$
53. $25x^2 + 10xy^3 + y^6$
54. $9c^4 - 12c^2y^2 + 4y^4$
55. $x^4 - 81$
56. $x^4 - 16$
57. $a^4 - 3a^2 - 40$
58. $b^4 + 21b^2 - 100$
59. $t^4 - 13t^2 + 36$
60. $x^4 - x^2 - 12$

5.3 OTHER FORMS OF FACTORING

Some algebraic expressions may be factored by grouping their terms so that they are of the types we have already studied. We shall consider those which can be grouped in one of the following ways:

1. With terms having a common binomial factor.
2. As the difference of two squares.
3. In the form of a trinomial.

Factoring out a common binomial factor is similar to factoring out a common monomial factor.

EXAMPLE 1

Factor $ax + bx$.

$$ax + bx = x(a + b)$$

EXAMPLE 2

Factor $a(x - y) + b(x - y)$.

$$a(x - y) + b(x - y) = (x - y)(a + b)$$

EXAMPLE 3

Factor $h(x - 1) - 2(x - 1)$.

$$h(x - 1) - 2(x - 1) = (x - 1)(h - 2)$$

In most cases, however, the algebraic expressions are not grouped so that the common binomial factors are so obvious.

EXAMPLE 4

Factor $3x - 3y + ax - ay$.

Group the first two terms and the last two terms, factor out common monomial factors, and then factor out the resulting binomial factors.

$$\begin{aligned} 3x - 3y + ax - ay &= 3(x - y) + a(x - y) \\ &= (x - y)(3 + a) \end{aligned}$$

EXAMPLE 5

Factor $a^2x - x - 3a^2 + 3$.

$$\begin{aligned} a^2x - x - 3a^2 + 3 &= x(a^2 - 1) - 3(a^2 - 1) \\ &= (a^2 - 1)(x - 3) \\ &= (a + 1)(a - 1)(x - 3) \end{aligned}$$

EXAMPLE 6

Factor $x^3 + x^2 + x + 1$.

$$\begin{aligned} x^3 + x^2 + x + 1 &= x^2(x + 1) + (x + 1) \\ &= (x + 1)(x^2 + 1) \end{aligned}$$

If an algebraic expression can be expressed as the difference of two squares, it can be factored similarly to the form $x^2 - y^2 = (x + y)(x - y)$.

EXAMPLE 7

Factor $(a + b)^2 - 16$.

$$(a + b)^2 - 16 = (a + b + 4)(a + b - 4)$$

EXAMPLE 8

Factor $25a^2 - (a + 2b)^2$.

$$\begin{aligned} 25a^2 - (a + 2b)^2 &= (5a)^2 - (a + 2b)^2 \\ &= [5a + (a + 2b)][5a - (a + 2b)] \\ &= (6a + 2b)(4a - 2b) \\ &= 2(3a + b)(2)(2a - b) \\ &= 4(3a + b)(2a - b) \end{aligned}$$

EXAMPLE 9

Factor $y^2 - 4y + 4 - 25x^2$.

Group the first three terms which form a perfect square trinomial, factor into a perfect square, and then factor the resulting difference of two squares.

$$\begin{aligned} (y^2 - 4y + 4) - 25x^2 &= (y - 2)^2 - 25x^2 \\ &= (y - 2 + 5x)(y - 2 - 5x) \end{aligned}$$

The last type of group factoring involves algebraic expressions that can be grouped in the form $ax^2 + bx + c$.

EXAMPLE 10

Factor $(a + b)^2 + 8(a + b) + 15$.

Note that $(a + b)^2 + 8(a + b) + 15$ is in the form $x^2 + 8x + 15$ which factors $(x + 3)(x + 5)$. Therefore,

$$(a + b)^2 + 8(a + b) + 15 = (a + b + 3)(a + b + 5)$$

The last type of factoring we shall study here is factoring the sum and difference of two perfect cubes. The general forms and three examples of each follow.

SUM OF TWO PERFECT CUBES

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

EXAMPLE 11

Factor $a^3 + 8$.

$$a^3 + 8 = a^3 + 2^3 = (a + 2)(a^2 - 2a + 4)$$

EXAMPLE 12

Factor $y^3 + 27$.

$$y^3 + 27 = y^3 + 3^3 = (y + 3)(y^2 - 3y + 9)$$

EXAMPLE 13

Factor $27x^3 + 64y^6$.

$$27x^3 + 64y^6 = (3x)^3 + (4y^2)^3 = (3x + 4y^2)(9x^2 - 12xy^2 + 16y^4)$$

DIFFERENCE OF TWO PERFECT CUBES

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Note the similarities in these two forms.

EXAMPLE 14

Factor $a^3 - 1$.

$$a^3 - 1 = a^3 - 1^3 = (a - 1)(a^2 + a + 1)$$

EXAMPLE 15

Factor $125x^3 - 8y^3$.

$$125x^3 - 8y^3 = (5x)^3 - (2y)^3 = (5x - 2y)(25x^2 + 10xy + 4y^2)$$

EXAMPLE 16

Factor $x^6 - 64$.

$$\begin{aligned} x^6 - 64 &= (x^3 + 8)(x^3 - 8) \\ &= (x + 2)(x^2 - 2x + 4)(x - 2)(x^4 + 2x + 4) \end{aligned}$$

Exercises 5.3

Factor each expression completely.

1. $a(m + n) - b(m + n)$
2. $m(x^2 - 16) - 3(x^2 - 16)$
3. $mx + my + nx + ny$
4. $ab + 3a - bc - 3c$
5. $3x + y - 6x^2 - 2xy$
6. $x^2 + ax + xy + ay$
7. $6x^3 - 4x^2 + 3x - 2$
8. $x^4 - 8x + x^3y - 8y$
9. $(x + y)^2 - 4z^2$
10. $(x - y)^2 - 16a^4$
11. $100 - 49(x - y)^2$
12. $9(a - 4)^2 - 64b^4$
13. $x^2 - 6x + 9 - 4y^2$
14. $x^2 + 10x + 25 - 36y^2$
15. $4x^2 - 4y^2 + 4y - 1$
16. $4 - a^2 - 2ab - b^2$
17. $(x + y)^2 + 13(x + y) + 36$
18. $3(x - y)^2 - 14(x - y) + 8$
19. $24(a + b)^2 - 14(a + b) - 5$
20. $x^2 + 2x(y + z) + (y + z)^2$
21. $a^3 + b^3$
22. $27x^3 + 8y^3$
23. $x^3 - 64$
24. $27x^3 - 8y^3$
25. $a^3b^3 + c^3$
26. $x^3 - y^3z^3$
27. $a^3 - 27b^3$
28. $8a^3 + b^3$
29. $27a^3 + 64b^3$
30. $64x^6 - 125y^3$
31. $27a^6 - 8b^9$
32. $8a^3b^6 + 125c^{12}$
33. $x^6 - y^6$
34. $a^6b^6 - c^6$

5.4 EQUIVALENT FRACTIONS

In order to solve many problems that occur in the technical and applied sciences, it is necessary to use fractions. A **fraction** may be defined as the quotient of two numbers or algebraic expressions. Both the numerator and denominator of a fraction may be multiplied or divided by the same nonzero number without changing the value of the fraction.

Two fractions are **equivalent** when both the numerator and denominator of one fraction can be multiplied or divided by the same nonzero number in order to change one fraction to the other. For example, $\frac{2}{3}$ is equivalent to $\frac{6}{9}$, and $\frac{12}{16}$ is equivalent to $\frac{3}{4}$, because

$$\frac{2}{3} = \frac{2 \cdot 3}{3 \cdot 3} = \frac{6}{9} \quad \text{and} \quad \frac{12}{16} = \frac{12 \div 4}{16 \div 4} = \frac{3}{4}$$

A fraction has three signs associated with it:

1. The sign of the fraction.
2. The sign of the numerator.
3. The sign of the denominator.

Any two of these three signs may be changed without changing the value of the fraction. For example,

$$-\frac{-3}{+4} = -\frac{+3}{-4} = +\frac{+3}{+4} = +\frac{-3}{-4}$$

When the numerator or the denominator is an algebraic expression, its sign is changed by placing a negative sign before the numerator or denominator in parentheses or by changing the signs of *all* the terms. For example,

$$-\frac{a+b-c}{d-3} = \frac{-(a+b-c)}{d-3} = \frac{-a-b+c}{d-3}$$

or

$$-\frac{a+b-c}{d-3} = \frac{a+b-c}{-(d-3)} = \frac{a+b-c}{-d+3} \quad \text{or} \quad \frac{a+b-c}{3-d}$$

A fraction is in **lowest terms** when its numerator and denominator have no common factors except 1. To reduce a fraction to lowest terms, divide the numerator and the denominator by their common factors.

EXAMPLE 1

Reduce $\frac{8xy^3}{12x^4y^2}$ to lowest terms.

The common factors of the numerator and the denominator are $4xy^2$. Dividing both numerator and denominator by $4xy^2$, we have

$$\frac{8xy^3}{12x^4y^2} = \frac{2y}{3x^3}$$

Some prefer to use "cancellation marks" to indicate this division; that is,

$$\frac{\cancel{8}^{\,2}\,\cancel{x}\,\cancel{y^3}^{\,y}}{\cancel{12}_{\,3}\,\cancel{x^4}\,\cancel{y^2}} = \frac{2y}{3x^3}$$

$3x^3$

Remember, *cancellation* is a nontechnical word for the mathematical operation of division. Think *division* whenever these marks are used.

EXAMPLE 2

Reduce $\frac{x^2+5x+6}{x^2+6x+9}$ to lowest terms.

Before finding common factors, we must first factor the numerator and the denominator.

$$\frac{x^2+5x+6}{x^2+6x+9} = \frac{(x+2)(x+3)}{(x+3)(x+3)} = \frac{x+2}{x+3}$$

EXAMPLE 3

Reduce $\frac{4-9x^2}{9x^2+3x-2}$ to lowest terms.

$$\frac{4-9x^2}{9x^2+3x-2} = \frac{(2-3x)(2+3x)}{(3x+2)(3x-1)} = \frac{2-3x}{3x-1}$$

(*Note:* $2+3x = 3x+2$.)

When factors *differ only in sign*, such as $x - y$ and $y - x$, one may be replaced by the opposite of the other; that is,

$$x - y = -(y - x) \quad \text{because} \quad -(y - x) = -y + x = x - y$$

or

$$y - x = -(x - y) \quad \text{because} \quad -(x - y) = -x + y = y - x$$

EXAMPLE 4

Reduce $\dfrac{x^2 - 2x - 8}{16 - x^2}$ to lowest terms.

$$\frac{x^2 - 2x - 8}{16 - x^2} = \frac{(x - 4)(x + 2)}{(4 + x)(4 - x)}$$

$$= \frac{-(4 - x)(x + 2)}{(4 + x)(4 - x)} \qquad [\textit{Note: } x - 4 = -(4 - x)]$$

$$= -\frac{x + 2}{4 + x} \quad \text{or} \quad \frac{-x - 2}{4 + x}$$

EXAMPLE 5

Reduce $\dfrac{30 - 19x - 5x^2}{5x^2 + 14x - 24}$ to lowest terms.

$$\frac{30 - 19x - 5x^2}{5x^2 + 14x - 24} = \frac{(5 + x)(6 - 5x)}{(5x - 6)(x + 4)}$$

$$= \frac{(5 + x)(6 - 5x)}{-(6 - 5x)(x + 4)} \qquad [\textit{Note: } 5x - 6 = -(6 - 5x)]$$

$$= -\frac{5 + x}{x + 4} \quad \text{or} \quad \frac{-x - 5}{x + 4}$$

Exercises 5.4

Reduce each fraction to lowest terms.

1. $\dfrac{3x}{18x}$
2. $\dfrac{18xy}{24y}$
3. $\dfrac{30x^2y^4}{54x^2y^5}$
4. $\dfrac{27x^2y^3z^4}{57x^5y^7}$
5. $\dfrac{8(x + 4)^3}{2(x + 4)^5}$
6. $\dfrac{6x^2(2x - 1)^6}{2x(2x - 1)^2}$
7. $\dfrac{18(x - 3)^3(1 - 5x)^2}{9(1 - 5x)^2(x - 3)}$
8. $\dfrac{12(3x - 4)^3(2x + 5)^2}{3(2x + 5)^3(3x - 4)^5}$
9. $\dfrac{5m + 15}{6m + 18}$
10. $\dfrac{6x^2 + 3x}{12x + 6}$
11. $\dfrac{6m + 6}{m^2 - 1}$
12. $\dfrac{2t - t^2}{4t - t^2}$

13. $\dfrac{x^2 + x}{x + 1}$

14. $\dfrac{3x^2 + 21x}{6x^2 - 12x}$

15. $\dfrac{x^2 + 5x + 4}{x^2 - 6x - 7}$

16. $\dfrac{x^2 - 2x - 24}{2x^2 + 7x - 4}$

17. $\dfrac{6x^2 + x - 12}{6x^2 + 19x - 36}$

18. $\dfrac{6x^2 + 19x + 10}{6x^2 - 5x - 6}$

19. $\dfrac{18 - 12x}{6x - 9}$

20. $\dfrac{15x - 25}{50 - 30x}$

21. $\dfrac{x^2 - 1}{1 - x}$

22. $\dfrac{6x^3 - 5x^2 - 4x}{16 - 9x^2}$

23. $\dfrac{a^2 - 4}{a^2 + 4a + 4}$

24. $\dfrac{4c^2 + 12c + 9}{4c^2 - 9}$

25. $\dfrac{m^2 - 16m}{16 - m}$

26. $\dfrac{1 - y}{y^2 - y}$

27. $\dfrac{t^2 - t^3}{t^2 - 1}$

28. $\dfrac{1 - 4x + 4x^2}{4x^2 - 1}$

29. $\dfrac{x^2 + xy}{x^2 + xy - 2x - 2y}$

30. $\dfrac{a^2 - ab + 3a - 3b}{a^2 - ab}$

31. $\dfrac{y^3 - 8}{y^2 + 2y + 4}$

32. $\dfrac{a^2b^2 - 16b^2}{a^2b + 9ab + 20b}$

33. $\dfrac{xy - 3x - 2y + 6}{y^3 - 27}$

34. $\dfrac{x^3 - y^3}{x^2 - y^2}$

35. $\dfrac{x^2 + 4x + 16}{x^3 - 64}$

36. $\dfrac{x^4 - 16}{x^4 - 2x^2 - 8}$

37. $\dfrac{3 - 4x - 4x^2}{4x^2 - 8x + 3}$

38. $\dfrac{y^2 - x^2}{x^2 - 2xy + y^2}$

39. $\dfrac{3a^3 + 3b^3}{3a^2 + 6ab + 3b^2}$

40. $\dfrac{x^2 - 2xy + y^2 - z^2}{z^2 + y^2 - x^2 + 2yz}$

5.5 MULTIPLICATION AND DIVISION OF ALGEBRAIC FRACTIONS

The product of two or more fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators; that is,

$$\boxed{\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}}$$

It is usually helpful to first factor each of the terms of all numerators and denominators; divide by those common factors, if any; and then multiply the numerators and multiply the denominators. Cancellation marks are useful for showing each completed division.

EXAMPLE 1

Multiply $\frac{3xy^2}{5ab} \cdot \frac{20a^2}{6x^2y}$

$$\frac{\cancel{3}\cancel{x}\cancel{y^2}}{\cancel{5}\cancel{a}\cancel{b}} \cdot \frac{\cancel{20}\,\cancel{a^2}}{\cancel{6}\cancel{x^2}\cancel{y}} = \frac{2ay}{bx}$$

EXAMPLE 2

Multiply $\frac{x^2 - 4x + 4}{x^2 + x - 6} \cdot \frac{x^2 + 3x}{x^2 - x}$.

Factor each numerator and denominator.

$$\frac{\cancel{(x-2)^2}}{\cancel{(x+3)}\cancel{(x-2)}} \cdot \frac{\cancel{x}(\cancel{x+3})}{\cancel{x}(x-1)} = \frac{x-2}{x-1}$$

To divide one fraction by another, "invert the divisor" and multiply, or multiply the first fraction by the reciprocal of the second.

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}}$$

EXAMPLE 3

Divide $\frac{xy}{a^2b} \div \frac{x^2y}{ab^3}$.

$$\frac{xy}{a^2b} \div \frac{x^2y}{ab^3} = \frac{\cancel{xy}}{\cancel{a^2b}} \cdot \frac{\cancel{ab^3}}{\cancel{x^2y}} = \frac{b^2}{ax}$$

EXAMPLE 4

Divide $\frac{x^2 - 9}{4x^2 - 9} \div \frac{2x^2 - x - 15}{2x^2 + x - 6}$.

$$\frac{x^2 - 9}{4x^2 - 9} \div \frac{2x^2 - x - 15}{2x^2 + x - 6} = \frac{x^2 - 9}{4x^2 - 9} \cdot \frac{2x^2 + x - 6}{2x^2 - x - 15}$$

$$= \frac{(x+3)(\cancel{x-3})}{(2x+3)(\cancel{2x-3})} \cdot \frac{(\cancel{2x-3})(x+2)}{(\cancel{x-3})(2x+5)}$$

$$= \frac{(x+3)(x+2)}{(2x+3)(2x+5)}$$

Note: Results left in factored form are usually preferred.

EXAMPLE 5

Divide $\dfrac{18+3t-t^2}{1+2t+t^2} \div \dfrac{2t^2-14t+12}{t^2-1}$

$$\frac{18+3t-t^2}{1+2t+t^2} \div \frac{2t^2-14t+12}{t^2-1} = \frac{18+3t-t^2}{1+2t+t^2} \cdot \frac{t^2-1}{2t^2-14t+12}$$

$$= \frac{(3+t)(6-t)}{(1+t)^2} \cdot \frac{(t+1)(t-1)}{2(t-6)(t-1)}$$

[Replace $6-t$ by $(-1)(t-6)$ in the first numerator.]

$$= \frac{(3+t)(-1)\cancel{(t-6)}}{\cancel{(1+t)^2}} \cdot \frac{\cancel{(t+1)}\cancel{(t-1)}}{2\cancel{(t-6)}\cancel{(t-1)}}$$

$$= \frac{-t-3}{2(t+1)}$$

EXAMPLE 6

Perform the indicated operations.

$$\frac{x^2+xy}{x^2+3xy+2y^2} \cdot \frac{x^2-xy+y^2}{x^2-2xy+y^2} \div \frac{x^4+xy^3}{(x^2-y^2)^2}$$

$$= \frac{x^2+xy}{x^2+3xy+2y^2} \cdot \frac{x^2-xy+y^2}{x^2-2xy+y^2} \cdot \frac{(x^2-y^2)^2}{x^4+xy^3}$$

$$= \frac{\cancel{x(x+y)}}{\cancel{(x+y)}(x+2y)} \cdot \frac{x^2-xy+y^2}{\cancel{(x-y)^2}} \cdot \frac{\cancel{(x+y)^2}\cancel{(x-y)^2}}{\cancel{x(x+y)}\cancel{(x^2-xy+y^2)}}$$

$$= \frac{x+y}{x+2y}$$

Exercises 5.5

Perform the indicated operations and simplify.

1. $\dfrac{9}{16} \cdot \dfrac{4}{3}$
2. $\dfrac{16}{25} \cdot \dfrac{5}{32}$
3. $\dfrac{15}{32} \div \dfrac{3}{16}$
4. $\dfrac{5}{9} \div \dfrac{5}{18}$
5. $\dfrac{x^4}{3} \cdot \dfrac{6x}{x^5}$
6. $\dfrac{4t^4}{6t} \cdot \dfrac{12t^2}{9t^3}$
7. $\dfrac{6a^4}{a^2} \div \dfrac{18a^2}{a^5}$
8. $\dfrac{8c^2}{4c^3} \div \dfrac{6c^4}{9c}$

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9. $\frac{3ab}{4x^2y} \cdot \frac{6x^2}{5ab^3}$

10. $\frac{15pq^2}{13m^5n^3} \cdot \frac{39mn^4}{5p^4q^3}$

11. $\frac{4ax^2}{9b^2y} \div \frac{5(ax)^2}{18(by^2)^2}$

12. $\frac{(5a)^3}{6x} \div \frac{5a^3}{-6x^2}$

13. $\frac{(2a^2b^3)^2}{(4ab^2)^3} \cdot \frac{(8a^2b)^2}{12a^3b}$

14. $\frac{(3x^2y)^3}{(6x^2y^2)^2} \div \frac{(12x^4y^4)^2}{(9x^2y)^3}$

15. $\frac{4x+8}{8xy} \cdot \frac{16x}{32x+64}$

16. $\frac{2x}{8x+4} \cdot \frac{16x+8}{6}$

17. $\frac{5k+5}{12} \div \frac{9k+9}{4}$

18. $(8c-16) \div \frac{3c-6}{12}$

19. $\frac{(y+2)^2}{y} \cdot \frac{y^2}{y^2-4}$

20. $\frac{9a}{(a-1)^2} \cdot \frac{a^2-1}{a^2}$

21. $(m^2-1) \cdot \frac{1+m}{1-m}$

22. $(s^2-9) \div \frac{3-s}{3s}$

23. $\frac{x^2-64}{x+2} \div \frac{8-x}{x}$

24. $\frac{y^2-25}{5y} \cdot \frac{10}{5-y}$

25. $\frac{x-2}{x-y} \cdot \frac{x^2-y^2}{x^2-4x+4}$

26. $\frac{x+1}{x+y} \cdot \frac{x^2-y^2}{x^2-1}$

27. $\frac{x}{(x-3)^2} \cdot \frac{9-x^2}{x^4}$

28. $\frac{x^2-y^2}{z^2-y^2} \cdot \frac{xy-xz}{y-x}$

29. $\frac{x^2+6x+9}{x^2-9} \cdot \frac{x^2-6x+9}{x-3}$

30. $\frac{t^2-10t+25}{t^2-25} \div \frac{t^2+10t+25}{t+5}$

31. $\frac{x^2-x-12}{x^2+2x+1} \cdot \frac{x^2-4x-5}{x^2-9x-20}$

32. $\frac{x^2-x-2}{x^2+7x+6} \cdot \frac{x^2+3x-18}{x^2-4x+4}$

33. $\frac{9x^2-4}{6x^2-5x-6} \div \frac{9x^2-12x+4}{6x^2-13x+6}$

34. $\frac{10x^2+11x+3}{3x^2-19x+28} \div \frac{4x^2-8x-5}{6x^2-29x+35}$

35. $\frac{15a^2+7ab-2b^2}{6a^2-11ab-10b^2} \cdot \frac{2a^2-13ab+20b^2}{25a^2-b^2}$

36. $\frac{18m^2+27mn+10n^2}{6m^2+19mn+10n^2} \div \frac{12m^2-8mn-15n^2}{16m^2+34mn-15n^2}$

37. $\frac{x^2-y^2}{x^2+2xy+y^2} \div \frac{x^2-2xy+y^2}{xy-y^2}$

38. $\frac{x^2y-16y^3}{xy^2-4y^3} \div \frac{x^2+3xy-4y^2}{y-x}$

39. $\frac{x^2-16}{x^2-9} \div \frac{3x^2+13x+4}{2x^2+x-21}$

40. $\frac{25a^2b^2-49}{4ab+1} \div \frac{5ab+7}{16a^2b^2+16ab+3}$

41. $\frac{x^3-8}{x^2-2x+4} \cdot \frac{x^3+8}{x^2+2x+4}$

42. $\frac{(x-y)^3}{(x+y)^2} \cdot \frac{x^2+2xy+y^2}{x^2-2xy+y^2}$

43. $\frac{xy+y-x^2-x}{12xy} \div \frac{x^2+x}{4x^2}$

44. $\frac{2x^2+10x+xy+5y}{x^2-25} \div \frac{x^2+5x+25}{x^3-125}$

45. $\frac{x^3+27}{x^2-9} \cdot \frac{5x^2+x-18}{5x^2-9x} \div \frac{x^2-3x+9}{5x^2-6x-27}$

46. $\frac{x^2-16}{x^2-x-12} \cdot \left(\frac{x+3}{x+4}\right)^2 \div \frac{x^3+27}{x^3+64}$

47. $\dfrac{18x^2 - 9x - 20}{6x^2 - 5x - 4} \div \left(\dfrac{2x^2 - 9x - 5}{9x^2 - 16} \cdot \dfrac{12x^2 - 10x}{4x^2 + 4x + 1}\right)$

48. $\dfrac{2x^2 - 9x + 7}{x^2 - 1} \div \left(\dfrac{2x^3 + 9x + 4}{x^2 + 4x + 3} \cdot \dfrac{x + 3}{x + 4}\right)^2$

5.6 ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS

Fractions may be added or subtracted if they have a common denominator. If they do not have a common denominator, the lowest common denominator is the most convenient to use because it involves the least amount of computation.

FINDING THE LOWEST COMMON DENOMINATOR (L.C.D.)

1. Factor each denominator into its prime factors; that is, factor each denominator completely.
2. Then the L.C.D. is the product formed by using each of the different factors the greatest number of times that it occurs in any *one* of the given denominators.

EXAMPLE 1

Find the L.C.D. for $\dfrac{3}{18}$, $\dfrac{15}{60}$, and $\dfrac{19}{27}$.

The prime factors of the denominators are

$$18 = 2 \cdot 3 \cdot 3$$
$$60 = 2 \cdot 2 \cdot 3 \cdot 5$$
$$27 = 3 \cdot 3 \cdot 3$$

Thus,

$$\text{L.C.D.} = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 = 540$$

because 2 occurs at most twice, 3 occurs at most three times, and 5 occurs at most one time in any *one* given denominator.

EXAMPLE 2

Find the L.C.D. for $\dfrac{1}{4x}$, $\dfrac{5}{6x^2y}$, and $\dfrac{-7}{15y^3}$.

The prime factors of the denominators are

$$4x = 2 \cdot 2 \cdot x$$
$$6x^2y = 2 \cdot 3 \cdot x \cdot x \cdot y$$
$$15y^3 = 3 \cdot 5 \cdot y \cdot y \cdot y$$

So

$$\text{L.C.D.} = 2 \cdot 2 \cdot 3 \cdot 5 \cdot x \cdot x \cdot y \cdot y \cdot y = 60x^2y^3$$

because 2 occurs at most twice, 3 occurs at most once, 5 occurs at most once, x occurs at most twice, and y occurs at most three times in any *one* given denominator.

EXAMPLE 3

Find the L.C.D. for

$$\frac{3x}{x^2 - 2xy + y^2} \qquad \frac{4y}{x^2 - y^2} \quad \text{and} \quad \frac{6x + 2y}{x^2 - xy - 2y^2}$$

First find the prime factors of the denominators.

$$x^2 - 2xy + y^2 = (x - y)(x - y)$$
$$x^2 - y^2 = (x + y)(x - y)$$
$$x^2 - xy - 2y^2 = (x + y)(x - 2y)$$
$$\text{L.C.D.} = (x - y)(x - y)(x + y)(x - 2y)$$

because in any *one* given denominator the factor $x - y$ occurs at most twice, the factor $x + y$ occurs at most once, and the factor $x - 2y$ occurs at most once.

ADDING OR SUBTRACTING FRACTIONS

1. If the fractions do not have a common denominator, change each fraction to an equivalent fraction having the L.C.D. (For each fraction, divide the L.C.D. by the denominator. Then multiply both numerator and denominator by this result in order to obtain an equivalent fraction having the L.C.D.)
2. Add or subtract the numerators in the order they occur, and place this result over the L.C.D.
3. Reduce the resulting fraction to lowest terms, if possible.

EXAMPLE 4

Perform the indicated operations.

$$\frac{5}{6a^2b} - \frac{3}{ab^2} + \frac{1}{4a^3}$$

The L.C.D. is $12a^3b^2$. Now divide $12a^3b^2$ by each denominator.

$$\frac{12a^3b^2}{6a^2b} = 2ab \qquad \frac{12a^3b^2}{ab^2} = 12a^2 \qquad \frac{12a^3b^2}{4a^3} = 3b^2$$

Multiply each numerator and denominator by the corresponding result.

$$\frac{5}{6a^2b} - \frac{3}{ab^2} + \frac{1}{4a^3} = \frac{5}{6a^2b} \cdot \frac{2ab}{2ab} - \frac{3}{ab^2} \cdot \frac{12a^2}{12a^2} + \frac{1}{4a^3} \cdot \frac{3b^2}{3b^2}$$
$$= \frac{10ab - 36a^2 + 3b^2}{12a^3b^2}$$

EXAMPLE 5

Perform the indicated operations and simplify.

$$\frac{x - 4}{2x} + \frac{x + 2}{x + 1} - \frac{3x + 2}{x^2 + x}$$

The L.C.D. is $2x(x + 1)$.

$$\frac{x-4}{2x}+\frac{x+2}{x+1}-\frac{3x+2}{\underset{x(x+1)}{x^2+x}}=\frac{x-4}{2x}\cdot\frac{x+1}{x+1}+\frac{x+2}{x+1}\cdot\frac{2x}{2x}-\frac{3x+2}{x(x+1)}\cdot\frac{2}{2}$$

$$=\frac{(x^2-3x-4)+(2x^2+4x)-(6x+4)}{2x(x+1)}$$

$$=\frac{3x^2-5x-8}{2x(x+1)}$$

$$=\frac{(3x-8)(x+1)}{2x(x+1)}$$

$$=\frac{3x-8}{2x}$$

EXAMPLE 6

Perform the indicated operations and simplify.

$$\frac{4x}{\underset{(2x+1)(2x-1)}{4x^2-1}}-\frac{3x+1}{\underset{-(2x-1)}{1-2x}}+\frac{2x}{1+2x}$$

Note: If we write $1-2x$ as $-(2x-1)$, the L.C.D. is $(2x+1)(2x-1)$.

$$\frac{4x}{(2x+1)(2x-1)}+\frac{3x+1}{2x-1}\cdot\frac{2x+1}{2x+1}+\frac{2x}{1+2x}\cdot\frac{2x-1}{2x-1}$$

$$=\frac{4x+(6x^2+5x+1)+(4x^2-2x)}{(2x+1)(2x-1)}$$

$$=\frac{10x^2+7x+1}{(2x+1)(2x-1)}$$

$$=\frac{(2x+1)(5x+1)}{(2x+1)(2x-1)}$$

$$=\frac{5x+1}{2x-1}$$

Exercises 5.6

Find the lowest common denominator for each set of denominators.

1. $6x, 8$

2. $15, 20m$

3. $2k, 4k^2$

4. $3t, 12t^3$

5. $6ab^2, 5a^2b$

6. $4ab^3c^4, 12a^2bc, 16ac^2$

7. $5x, x-1$

8. $x+3, 3x$

9. $3x+6, 6x+12$

10. $4a-12, 6a-18$

11. $x^2-25, (x-5)^2$

12. $b^2-16, (b+4)^2$

13. x^2+6x+8, x^2-x-6

14. $6x^2+3x-1, 6x^2-5x-4$

15. $3x^3-12x^2, 6x^2+12x, 2x^3-4x^2-16x$

16. $18a(a+5)^2, 6(a^2-25), 5a-25$

17. $6(c+2)(c-4)^2, 12c(c+2)(c-4), 3c(c-4)^2$

18. $t^2(t+3)^2, t^2(t-3)^2, t(t^2-9)$

Perform the indicated operations and simplify.

19. $\frac{3}{14} + \frac{5}{14}$

20. $\frac{9}{20} - \frac{3}{20}$

21. $\frac{16}{a} - \frac{9}{a}$

22. $\frac{x}{3a} + \frac{5x}{3a}$

23. $\frac{6}{x+1} + \frac{9}{x+1}$

24. $\frac{6a}{2x+3} - \frac{8a}{2x+3}$

25. $\frac{7}{36} + \frac{7}{45}$

26. $\frac{19}{24} - \frac{31}{60}$

27. $\frac{2x-1}{12} + \frac{3x+5}{20} - \frac{x}{4}$

28. $\frac{3a+b}{6} - \frac{2a-b}{8} + \frac{5a}{12}$

29. $\frac{5}{6y} - \frac{7}{2y}$

30. $\frac{2}{3y} + \frac{1}{y^2}$

31. $\frac{4}{3p} + \frac{2}{p^2}$

32. $\frac{3}{4t} - \frac{5}{3t}$

33. $\frac{5}{4x^2} + \frac{1}{3x^3} - \frac{3}{2x}$

34. $\frac{5}{3a^2b} - \frac{3}{2a} - \frac{1}{4ab^3}$

35. $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

36. $\frac{a}{b} + \frac{c}{a} + 1$

37. $3x + \frac{4-x}{2x}$

38. $9 - \frac{2x-1}{3y}$

39. $\frac{2}{c-1} + \frac{1}{c}$

40. $\frac{4}{s-3} - \frac{1}{s}$

41. $\frac{6}{t+2} - \frac{3}{t-2}$

42. $\frac{5}{x-5} + \frac{2}{x+5}$

43. $\frac{2}{a} + \frac{1}{a+1}$

44. $\frac{2}{a+3} - \frac{4}{a+2}$

45. $\frac{d}{d+4} - \frac{2d}{d-5}$

46. $\frac{3t}{t-2} + \frac{4t}{t+4}$

47. $\frac{8}{x-4} + \frac{2}{4-x}$

48. $\frac{c}{c-6} + \frac{2c}{6-c}$

49. $\frac{4}{3a+9} - \frac{6}{4a+12}$

50. $\frac{12}{5r-25} - \frac{5}{3r-15}$

51. $\frac{5a+1}{a^2-1} - \frac{2a-3}{a+1}$

52. $\frac{2x+5}{x+6} + \frac{14x}{x^2-36}$

53. $\frac{6}{x-3} - \frac{5}{3-x}$

54. $\frac{2x}{x^2-1} + \frac{2-x}{1-x}$

55. $\frac{2}{r^2-4r+4} + \frac{1}{r^2+r-6}$

56. $\frac{4}{y^2-5y+6} - \frac{3}{y^2-y-2}$

57. $\frac{1}{x-2} + \frac{1}{x^2-5x+6}$

58. $\frac{1}{6x} + \frac{1}{3x-6} - \frac{1}{2x+4}$

59. $\frac{t+4}{2t-4} - \frac{2t+5}{t^2-t-2} + \frac{3}{4}$

60. $\frac{2}{x^2-y^2} - \frac{1}{x^2-3xy+2y^2} - \frac{3}{x^2-xy-2y^2}$

61. $\dfrac{3x-2}{x^2-3x-4}+\dfrac{2x-3}{x^2-x-12}$

62. $\dfrac{3x+7}{6x^2+x-35}-\dfrac{3x-7}{6x^2+29x+35}$

63. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a}{a+b}+\dfrac{b}{b-a}$

64. $\dfrac{2x}{x^2-4}-\dfrac{2+x}{2-x}-\dfrac{x-2}{x+2}$

65. $\dfrac{x+3}{x^2-7x+12}+\dfrac{x+3}{16-x^2}$

66. $\dfrac{3x^2-18x+5}{2x^2+x-1}-\dfrac{2x+1}{1-2x}-\dfrac{3x-7}{x+1}$

67. $\dfrac{5x}{8x+2}-\dfrac{3}{4x^2+x}+\dfrac{1}{2x}$

68. $\dfrac{1}{x^2+2x+1}+\dfrac{3}{2x+2}-\dfrac{5}{4x+4}$

69. $\dfrac{1}{t-2}-\dfrac{6t}{t^3-8}$

70. $\dfrac{c}{1-c^3}+\dfrac{1}{c^2-1}-\dfrac{2}{c^2+c+1}$

71. $\dfrac{x^2+x}{x^3+1}+1$

72. $\dfrac{x^2-1}{x^3-1}+\dfrac{x-1}{x^2+x+1}$

5.7 COMPLEX FRACTIONS

A **complex fraction** is a fraction that contains a fraction in the numerator, the denominator, or both. There are basically two ways to simplify a complex fraction.

Method 1: Multiply both the numerator and denominator of the complex fraction by the L.C.D. of all the fractions that appear in the numerator and denominator.

Method 2: Simplify the numerator and the denominator separately. Then divide the simplified numerator by the simplified denominator. Finally simplify that result, if possible.

EXAMPLE 1

Simplify.

$$\frac{1-\dfrac{1}{x}}{x-\dfrac{1}{x^2}}$$

Method 1: The L.C.D. of all the denominators of the fractions in the complex fraction is x^2.

$$\frac{\left(1-\dfrac{1}{x}\right)x^2}{\left(x-\dfrac{1}{x^2}\right)x^2}=\frac{x^2-x}{x^3-1}$$

$$=\frac{x(x-1)}{(x-1)(x^2+x+1)}=\frac{x}{x^2+x+1}$$

Method 2:

$$\frac{1-\dfrac{1}{x}}{x-\dfrac{1}{x^2}}=\frac{\dfrac{x-1}{x}}{\dfrac{x^3-1}{x^2}}=\frac{x-1}{x}\div\frac{x^3-1}{x^2}$$

$$=\frac{x-1}{x}\cdot\frac{x^2}{(x-1)(x^2+x+1)}=\frac{x}{x^2+x+1}$$

To simplify an expression containing a complex fraction, first simplify the complex fraction and then perform the remaining indicated operations.

EXAMPLE 2

Simplify.

$$x - \frac{\frac{x}{1-x}}{1-\frac{x}{1-x}} = x - \frac{\frac{x}{1-x}}{\frac{1-x-x}{1-x}}$$

$$= x - \frac{\frac{x}{1-x}}{\frac{1-2x}{1-x}} \cdot \frac{(1-x)}{(1-x)}$$

$$= x - \frac{x(1-x)}{1-2x}$$

$$= \frac{x(1-2x)}{1-2x} - \frac{x(1-x)}{1-2x}$$

$$= \frac{(x-2x^2)-(x-x^2)}{1-2x} = \frac{-x^2}{1-2x}$$

Exercises 5.7

Perform the indicated operations and simplify.

1. $\dfrac{\frac{5}{x}}{\frac{15}{x+2}}$

2. $\dfrac{\frac{12}{r-3}}{\frac{3}{2r}}$

3. $\dfrac{\frac{t-1}{6t}}{\frac{t+1}{9t}}$

4. $\dfrac{\frac{3x^2y}{9z}}{\frac{12xy}{3z}}$

5. $\dfrac{\frac{s+t}{4k}}{\frac{2s+2t}{8k}}$

6. $\dfrac{\frac{x-4y}{6x}}{\frac{8x-32y}{12}}$

7. $\dfrac{2+\frac{7}{16}}{2-\frac{1}{4}}$

8. $\dfrac{\frac{2}{9}-6}{\frac{1}{3}+1}$

9. $\dfrac{a-4}{2-\frac{8}{a}}$

10. $\dfrac{a-12}{1-\frac{12}{a}}$

11. $\dfrac{a-\frac{25}{a}}{a+5}$

12. $\dfrac{a+\frac{27}{a^2}}{a+3}$

13. $\dfrac{\frac{3}{x}+\frac{5}{y}}{\frac{x}{3}+\frac{y}{5}}$

14. $\dfrac{\frac{x}{y}-\frac{y}{z}}{\frac{y}{x}-\frac{z}{y}}$

15. $\dfrac{1-y}{1-\frac{1}{y}}$

16. $\dfrac{\frac{1}{y}-1}{y-\frac{1}{y}}$

17. $\dfrac{1+\frac{1}{a}}{1-\frac{1}{a}}$

18. $\dfrac{\frac{2}{c}-1}{\frac{2}{c}+1}$

19. $\dfrac{\frac{1}{m}+\frac{1}{n}}{\frac{1}{m}-\frac{1}{n}}$

20. $\dfrac{\frac{x}{y}+2}{\frac{x}{y}-\frac{4y}{x}}$

21. $\dfrac{x-3-\dfrac{28}{x}}{x+10+\dfrac{24}{x}}$

22. $\dfrac{12x+16+\dfrac{5}{x}}{18x-27-\dfrac{35}{x}}$

23. $\dfrac{\dfrac{x^2-y^2}{x^3+y^3}}{\dfrac{x^2+2xy+y^2}{x^2-xy+y^2}}$

24. $\dfrac{\dfrac{8x^2-2x-21}{6x^2+11x+3}}{\dfrac{4x^2+20x+25}{12x^3+19x^2+5x}}$

25. $\dfrac{\dfrac{x}{x+y}+\dfrac{y}{x-y}}{\dfrac{x^2+y^2}{x^2-y^2}}$

26. $\dfrac{\dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}$

27. $\dfrac{\dfrac{6a^2-5ab-4b^2}{a-b}}{\dfrac{3a^2-7ab+4b^2}{2a+b}}$

28. $\dfrac{x+\dfrac{y^2}{x+y}}{y+\dfrac{x^2}{x+y}}$

29. $\dfrac{1+\dfrac{1}{x^2-1}}{1+\dfrac{1}{x-1}}$

30. $\dfrac{3+\dfrac{5x}{x^2-4}}{3+\dfrac{2}{x-2}}$

31. $1-\dfrac{1}{2-\dfrac{1}{x}}$

32. $3+\dfrac{1}{4-\dfrac{2}{y}}$

33. $1+\dfrac{1}{x+\dfrac{1}{x-1}}$

34. $1+\dfrac{x+1}{x+\dfrac{1}{x+2}}$

35. $\dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{x+6+\dfrac{5}{x}}-\dfrac{\dfrac{1}{x^2}-\dfrac{9}{x^4}}{1+\dfrac{2}{x}-\dfrac{15}{x^2}}$

36. $\dfrac{\dfrac{x^2+7x+10}{x+2}}{x-6}+\dfrac{x-2}{\dfrac{x^2-2x}{x-3}}$

37. $1-\dfrac{1}{1+\dfrac{1}{1-\dfrac{1}{1+\dfrac{1}{1-\dfrac{1}{1+1}}}}}$

38. $1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+\dfrac{1}{1+1}}}}}$

5.8 EQUATIONS WITH FRACTIONS

To solve an equation with fractions, multiply each side by the L.C.D. The resulting equation will not contain any fractions and thus may be solved as those equations in Chapter 2.

EXAMPLE 1

Solve.

$$\frac{x}{12} + \frac{1}{6} = \frac{x+1}{8}$$

The L.C.D. of 12, 6, and 8 is 24; therefore, multiply each side of the preceding equation by 24.

$$24\left(\frac{x}{12} + \frac{1}{6}\right) = 24\left(\frac{x+1}{8}\right)$$

$$2x + 4 = 3(x+1)$$

$$2x + 4 = 3x + 3$$

$$1 = x$$

Check: Substitute 1 for x each time it occurs in the *original equation.*

$$\frac{(1)}{12} + \frac{1}{6} = \frac{(1)+1}{8}$$

$$\frac{1}{12} + \frac{2}{12} = \frac{2}{8}$$

$$\frac{3}{12} = \frac{2}{8}$$

$$\frac{1}{4} = \frac{1}{4}$$

It is common practice in science and technology to solve a given formula for the unknown letter before the substitution of the data is made. To solve a formula involving fractions, follow the same procedure as above.

EXAMPLE 2

Solve for r.

$$\frac{E}{e} = \frac{R+r}{r}$$

The L.C.D. is er; therefore, multiply each side of the preceding equation by er.

$$er\left(\frac{E}{e}\right) = er\left(\frac{R+r}{r}\right)$$

$$Er = eR + er$$

$$Er - er = eR \qquad \text{(Subtract } er \text{ from each side.)}$$

$$(E-e)r = eR \qquad \text{(Factor.)}$$

$$r = \frac{eR}{E-e} \qquad \text{(Divide each side by } E-e.\text{)}$$

EXAMPLE 3

Solve.

$$\frac{x}{x-5} = \frac{5}{x-5} + 2$$

The L.C.D. is $x - 5$; therefore, multiply each side by $x - 5$.

$$(x-5)\left[\frac{x}{x-5}\right] = (x-5)\left[\frac{5}{x-5} + 2\right]$$

$$x = 5 + 2(x-5)$$

$$x = 5 + 2x - 10$$

$$x = 2x - 5$$

$$5 = x$$

Check:

$$\frac{5}{5-5} = \frac{5}{5-5} + 2$$

$$\frac{5}{0} \neq \frac{5}{0} + 2$$

because $\frac{5}{0}$ is meaningless.

The only apparent solution, $x = 5$, does not satisfy the *original* equation. Therefore, the original equation has no solution; that is, there is no real number that can replace x in the original equation and produce a true statement.

> The technique of multiplying each side of an equation by a **variable** quantity gives the possibility of obtaining *extraneous,* or false, solutions.

For example, the simple equation

$$x = 3$$

obviously has only one solution. If we multiply each side by x, we have

$$x^2 = 3x$$

But this equation has two solutions: 3 and 0. Here, 0 is an extraneous solution.

> In solving an equation using the technique of multiplying each side by a variable quantity, the only way to determine whether or not an apparent solution is extraneous is to check each apparent solution in the *original* equation.

EXAMPLE 4

Solve.

$$\frac{x}{x+2} + \frac{4}{x-2} = \frac{x^2}{\underset{(x+2)(x-2)}{x^2-4}}$$

The L.C.D. is $(x + 2)(x - 2)$.

$$x(x-2)+4(x+2)=x^2$$
$$x^2-2x+4x+8=x^2$$
$$2x+8=0$$
$$2x=-8$$
$$x=-4$$

Check: Substitute the apparent solution, $x=-4$, in the original equation.

$$\frac{(-4)}{(-4)+2}+\frac{4}{(-4)-2}=\frac{(-4)^2}{(-4)^2-4}$$
$$\frac{-4}{-2}+\frac{4}{-6}=\frac{16}{12}$$
$$2-\frac{2}{3}=\frac{4}{3}$$
$$\frac{4}{3}=\frac{4}{3}$$

Since $x=-4$ satisfies the original equation, $x=-4$ is the solution, or root.

As we have seen, multiplying each side of an equation by a variable quantity can lead to extraneous solutions. This operation requires checking the solutions. It is also important *not* to divide each side by a variable quantity, because this operation may result in the loss of solutions. For example, the equation

$$x^2=5x$$

has two solutions: 5 and 0. If we divide each side of this equation by x, we have

$$x=5$$

and we have lost a solution.

You should note that there is a great difference between a fractional expression involving addition, subtraction, multiplication, and division of algebraic fractions and an algebraic fractional equation. The only time that you multiply by the L.C.D. to eliminate fractions is when an equal sign is present, which indicates a fractional equation.

Exercises 5.8

Solve and check.

1. $\frac{x}{3}-\frac{x}{6}=2$

2. $\frac{y}{4}+\frac{y}{12}=5$

3. $\frac{3m}{4}+\frac{2m}{5}=\frac{23}{2}$

4. $\frac{5t}{4}-\frac{4t}{9}=\frac{29}{4}$

5. $\frac{x}{6}+2=\frac{x+3}{4}$

6. $\frac{t-6}{6}+\frac{1}{10}=\frac{t}{15}$

7. $\frac{s+4}{8}-1\frac{1}{8}=\frac{3s}{4}$

8. $\frac{4m}{5}+7\frac{2}{3}=\frac{m+5}{15}$

9. $\frac{x+5}{2}=\frac{x+4}{3}$

10. $\frac{x+3}{5}=\frac{x-7}{4}$

11. $\frac{x}{20}+\frac{1}{4}=\frac{x+2}{5}$

12. $\frac{x}{15}+\frac{1}{3}=\frac{x-3}{5}$

13. $\frac{2x+6}{4}+1=\frac{3x+1}{8}$

14. $\frac{4x-3}{9} - 2 = \frac{x-4}{3}$

15. $\frac{5-x}{7} - \frac{2x-1}{14} = 1$

16. $\frac{2-x}{24} - \frac{3x+4}{16} = 2$

In Exercises 17–28, solve for the indicated letter.

17. $F = \frac{mv^2}{r}$ for r

18. $R = \frac{\rho L}{A}$ for ρ

19. $I_L = \frac{V}{R + R_L}$ for R

20. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$ for f

21. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R_1

22. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ for R_2

23. $y = \frac{x+y}{x}$ for y

24. $y = \frac{x-y}{x}$ for x

25. $\frac{x}{a+b} = \frac{x+a}{b}$ for x

26. $\frac{1}{x+a} - \frac{a}{x-a} = \frac{ax}{x^2-a^2}$ for x

27. $V = \frac{Q}{R_1} - \frac{Q}{R_2}$ for Q

28. $V = \frac{Q}{R_1} - \frac{Q}{R_2}$ for R_2

Solve and check.

29. $\frac{3}{2x} = \frac{5}{x} - 7$

30. $\frac{7}{2x} - \frac{5}{3} = \frac{1}{x}$

31. $\frac{x-3}{x} = \frac{1}{x} + \frac{2}{3}$

32. $\frac{2x+5}{x} = \frac{2}{x} + \frac{1}{4}$

33. $\frac{1}{x-2} = \frac{3}{x+4}$

34. $\frac{6}{3-2x} = \frac{2}{5-3x}$

35. $\frac{x+2}{x-3} = \frac{x+1}{x-1}$

36. $\frac{6t-3}{2t+1} = \frac{9t}{3t+4}$

37. $\frac{2x^2+8}{x^2-1} - \frac{x-4}{x+1} = \frac{x}{x-1}$

38. $\frac{x-3}{x+3} = \frac{2x+3}{x-3} - \frac{x^2}{x^2-9}$

39. $\frac{x+2}{x+7} = \frac{2}{x-2} + \frac{x^2+3x-28}{x^2+5x-14}$

40. $\frac{1}{x-4} + \frac{1}{x-5} + \frac{1}{x^2-9x+20} = 0$

41. $\frac{x}{x+4} + \frac{x+1}{x-5} = 2$

42. $\frac{-3x^2+5x}{x^2+6x+9} + 4 = \frac{x+5}{x+3}$

43. $\frac{x^2+3x+7}{x^2-x-12} + \frac{x+6}{x+3} = \frac{2x+1}{x-4}$

44. $\frac{7x^2+x-17}{2x^2-5x+3} - \frac{3x-4}{2x-3} = \frac{2x+1}{x-1}$

CHAPTER SUMMARY

1. Special products:
   (a) *Product of a monomial and a polynomial:*

$$a(x + y + z) = ax + ay + az$$

   (b) $(x + y)(x - y) = x^2 - y^2$
   (c) *Square of a binomial:*

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$

   (d) *Square of a trinomial:*

$$(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

   (e) *Cube of a binomial:*

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

2. *Factoring:*
   (a) *Greatest common factor:*

$$ax + ay + az = a(x + y + z)$$

   (b) *Difference of two perfect squares:*

$$x^2 - y^2 = (x + y)(x - y)$$

   (c) *Trinomials by trial and error.*
   (d) *Perfect square trinomials:*

$$x^2 + 2xy + y^2 = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)^2$$

   (e) *Sum and difference of two perfect cubes:*

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

3. *Equivalent fractions:* Two fractions are equivalent when both numerator and denominator of one fraction can be multiplied or divided by the same nonzero number in order to change one fraction to the other.
4. *Signs of fractions:* A fraction has three signs—the sign of the fraction, the sign of the numerator, and the sign of the denominator. Any two of these three signs may be changed without changing the value of the fraction. For example,

$$-\frac{-a}{+b} = -\frac{+a}{-b} = +\frac{+a}{+b} = +\frac{-a}{-b}$$

5. *Lowest terms:* A fraction is in lowest terms when its numerator and denominator have no common factors except 1.

6. *Factors that differ in sign:* When factors differ in sign, one may be replaced by the opposite of the other; that is,

$$x - y = -(y - x) \quad \text{or} \quad y - x = -(x - y)$$

7. *To multiply fractions:*

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

8. *To divide fractions:*

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

9. *To find the lowest common denominator (L.C.D.):*
   (a) Factor each denominator completely.
   (b) Then the L.C.D. is the product formed by using each of the different factors the greatest number of times that it occurs in any *one* of the given denominators.

10. *Adding or subtracting fractions:*
   (a) If the fractions do not have a common denominator, change each fraction to an equivalent fraction having the L.C.D. (For each fraction, divide the L.C.D. by the denominator. Then multiply both numerator and denominator by this result in order to obtain an equivalent fraction having the L.C.D.)
   (b) Add or subtract the numerators in the order they occur, and place this result over the L.C.D.
   (c) Reduce the resulting fraction to lowest terms, if possible.

11. *Complex fraction:* A fraction that contains a fraction in the numerator, the denominator, or both. There are basically two ways to simplify a complex fraction.

Method 1: Multiply both numerator and denominator of the complex fraction by the L.C.D. of all the fractions that appear in the numerator and denominator.

Method 2: Simplify the numerator and the denominator separately. Then divide the simplified numerator by the simplified denominator. Finally simplify that result, if possible.

12. *To solve an equation with fractions:* Multiply each side by the lowest common denominator. When you multiply each side by a variable quantity you *must check* each apparent solution in the original equation.

CHAPTER 5 REVIEW

Find each product mentally.

1. $-6a^2b^4(a^3 - 12ab^2 - 9b^5)$
2. $(3x + 13)(5x - 4)$
3. $(4x - 7)(4x + 7)$
4. $(5a^2 - 9b^2)^2$
5. $(4a - b^2)^3$
6. $(6x + 7yz^2)(5x - 2yz^2)$

Factor each expression ***completely.***

7. $18a^3b - 9a^2b^4 + 9a^3b^2$
8. $16x^2 - 9y^2$

9. $5x^2 + 28x + 32$

10. $14x^2 + 59x - 18$

11. $4a^2 - 20a + 25$

12. $m^3 - n^3$

13. $3x^2 - 18x - 21$

14. $x^4 - 81$

15. $15x^2 + 23x - 90$

16. $10x^2 + 25xy - 60y^2$

17. $x^2 + 6x + 9 - 25y^2$

18. $ax + 2bx - 4ay - 8by$

Reduce each fraction to lowest terms.

19. $\dfrac{90x^4y^9}{48x^8y^3}$

20. $\dfrac{16x^2 - 40x}{8x^2 - 20x}$

21. $\dfrac{x^2 - 4x - 32}{2x^2 + 11x + 12}$

22. $\dfrac{-2x^2 + 7x + 4}{x^2 - 16}$

Perform the indicated operations and simplify.

23. $\dfrac{2x^2 + 5x - 25}{4x^2 + 21x + 5} \cdot \dfrac{4x^2 + x}{4x^2 - 25}$

24. $\dfrac{6x^2 + 11x - 7}{4x^2 + 4x + 1} \div \dfrac{3x^2 + 4x - 7}{-2x^2 + x + 1}$

25. $\dfrac{y^6 + 125}{4y^2 + 20} \div \dfrac{y^4 - 5y^2 + 25}{12y}$

26. $\dfrac{2 - x}{x^2 + 10x + 21} \cdot \dfrac{3x^2 + 21x}{18x + 36} \div \dfrac{x^3 - 4x}{x + 3}$

27. $\dfrac{4}{3a} + \dfrac{9}{4a^2} - \dfrac{5}{2}$

28. $4 - \dfrac{3x + 4}{2x + 1}$

29. $\dfrac{x}{x^2 - 7x + 10} + \dfrac{x}{2x^2 - 5x + 2}$

30. $\dfrac{x - 1}{2x^2 + 3x + 1} + \dfrac{x - 3}{3x^2 - x - 4}$

31. $\dfrac{\dfrac{1}{x} + \dfrac{1}{x^2}}{x + \dfrac{1}{x^2}}$

32. $4 + \dfrac{2}{3 + \dfrac{1}{y}}$

Solve for x and check.

33. $\dfrac{x + 2}{5} + \dfrac{x + 4}{30} = \dfrac{x}{6}$

34. $\dfrac{ax + b}{b} = \dfrac{bx}{a} + \dfrac{a}{b}$

35. $\dfrac{x + 1}{x - 3} + 1 = \dfrac{4}{x - 3}$

36. $\dfrac{4}{x - 4} - \dfrac{x^2}{16 - x^2} = 1$

Many equations used by electronic technicians to solve problems involve fractions. In this chapter you will have an opportunity to expand your understanding of algebra by working with fractions, fractional equations, and literal equations containing fractions and by evaluating formulas.

9–1 SOLVING EQUATIONS CONTAINING FRACTIONS

Equations having numerical coefficients that are fractions are transformed and solved by the methods in earlier chapters. Equations with fractional coefficients are best solved by transforming them into equations without fractions. This is done by multiplying each member of the equation by the common denominator.

EXAMPLE 9–1 Solve $\frac{3}{8}y = \frac{3}{4}$.

Solution Multiply each member by the common denominator, 8; then simplify:

$$(\mathbf{8})\tfrac{3}{8}y = \tfrac{3}{4}(\mathbf{8})$$

$$3y = 6$$

$$y = 2$$

Check Does $\frac{3}{8}y = \frac{3}{4}$ when $y = 2$?

$$\tfrac{3}{8}(\mathbf{2}) = \tfrac{3}{4}$$

$$\tfrac{3}{4} = \tfrac{3}{4} \quad \text{Yes!}$$

$\therefore$ $y = 2$ is the solution.

EXAMPLE 9–2 Solve $\dfrac{2x}{5} - 3 = \dfrac{5 - 3x}{10}$.

Solution Multiply each member by the common denominator of 10:

$$\mathbf{10}\left(\frac{2x}{5} - 3\right) = \mathbf{10}\left(\frac{5 - 3x}{10}\right)$$

$$\mathbf{10}\left(\frac{2x}{5}\right) + \mathbf{10}(-3) = \mathbf{10}\left(\frac{5 - 3x}{10}\right)$$

$$2(2x) - 30 = 5 - 3x$$

$$4x + 3x = 5 + 30$$

$$7x = 35$$

$$x = 5$$

Check Does $\frac{2x}{5} - 3 = \frac{5 - 3x}{10}$ when $x = 5$?

$$\frac{2(5)}{5} - 3 = \frac{5 - 3(5)}{10}$$

$$2 - 3 = \frac{5 - 15}{10}$$

$$-1 = -1 \quad \text{Yes!}$$

$\therefore$ $x = 5$ is the solution.

EXAMPLE 9–3 Solve $\frac{3x + 3}{12} = \frac{x - 3}{18} + \frac{x}{6}$.

Solution Multiply each member by the common denominator of 36:

$$\mathbf{36}\left(\frac{3x + 3}{12}\right) = \mathbf{36}\left(\frac{x - 3}{18} + \frac{x}{6}\right)$$

$$36\left(\frac{3x + 3}{12}\right) = 36\left(\frac{x - 3}{18}\right) + 36\left(\frac{x}{6}\right)$$

$$3(3x + 3) = 2(x - 3) + 6x$$

$$9x + 9 = 2x - 6 + 6x$$

$$9x - 8x = -15$$

$$x = -15$$

Check Does $\frac{3x + 3}{12} = \frac{x - 3}{18} + \frac{x}{6}$ when $x = -15$?

$$\frac{3(\mathbf{-15}) + 3}{12} = \frac{\mathbf{-15} - 3}{18} + \frac{\mathbf{-15}}{6}$$

$$\frac{-45 + 3}{12} = -1 + \frac{-15}{6}$$

$$\frac{-42}{12} = \frac{-6 - 15}{6}$$

$$\frac{-7}{2} = \frac{-21}{6}$$

$$\frac{-7}{2} = \frac{-7}{2} \quad \text{Yes!}$$

$\therefore$ $x = -15$ is the solution.

EXERCISE 9–1

Solve the following equations and check your solutions.

1. $\frac{1}{8}x = \frac{1}{4}$ **2.** $\frac{-x}{2} = \frac{1}{6}$ **3.** $\frac{a}{15} = \frac{1}{10}$

4. $\frac{5}{4} = \frac{4}{5}y$ **5.** $\frac{1}{3}R = -\frac{15}{7}$ **6.** $\frac{x}{3} = \frac{4}{5}$

7. $\frac{m-7}{4} = 2m$ **8.** $\frac{2b-5}{3} = \frac{45}{3}$ **9.** $\frac{4}{7} = \frac{12V}{16}$

10. $\frac{a}{3} - \frac{a}{2} = 5$ **11.** $\frac{3I+4}{7} = I + 2$ **12.** $\frac{3}{4} + E = \frac{5}{4}E$

13. $5 - \frac{2}{3}x = \frac{-x}{2}$ **14.** $\frac{V}{2} = \frac{4V}{5} - 3$

15. $\frac{1}{6} = \frac{a-15}{3} + \frac{7-a}{2}$ **16.** $\frac{m-3}{3} = \frac{m+4}{5} - 3$

17. $\frac{3x}{10} - \frac{2}{10} = \frac{2x-7}{5} + \frac{1}{5}$ **18.** $\frac{2R}{5} = \frac{4R-1}{4} - \frac{3}{16}$

19. $\frac{2I-3}{3} - 3 = -\frac{3}{6} - \frac{I}{2}$ **20.** $\frac{8y-1}{3} = 3 - \frac{10y}{6} - \frac{7}{6}$

21. $\frac{3x-3}{4} = \frac{2x}{3} - \frac{x-5}{2}$ **22.** $\frac{4a-3}{2} = \frac{4a-3}{5}$

9–2 SOLVING FRACTIONAL EQUATIONS

A **fractional equation** is an equation that has a variable in the denominator of one or more terms of the equation. The solution of fractional equations is carried out in the same way as the solution of equations containing fractions: by multiplying both members of the equation by the common denominator.

EXAMPLE 9–4 Solve $\frac{2}{x} = \frac{5}{x} - \frac{3}{5}$.

Solution Multiply each member by the common denominator of $5x$:

$$5x\left(\frac{2}{x}\right) = 5x\left(\frac{5}{x}\right) + 5x\left(-\frac{3}{5}\right)$$

$$10 = 25 - 3x$$
$$3x = 15$$
$$x = 5$$

Check Does $\frac{2}{x} = \frac{5}{x} - \frac{3}{5}$ when $x = 5$?

$$\frac{2}{5} = \frac{5}{5} - \frac{3}{5}$$

$$\frac{2}{5} = \frac{2}{5} \quad \text{Yes!}$$

$\therefore$ $x = 5$ is the solution.

It is important to check the roots of fractional equations. When we multiply both members of an equation by an expression containing the unknown, we run the risk of introducing an **extraneous root.** An extraneous root will *check* in the derived equation, but **not** in the original equation. Extraneous roots result from multiplying by zero.

EXAMPLE 9–5 Solve $\frac{5x}{x-1} = \frac{5}{x-1}$.

Solution Multiply each member by $(x - 1)$, the common denominator:

$$(x-1)\frac{5x}{x-1} = (x-1)\frac{5}{x-1}$$

$$5x = 5$$

$$x = 1$$

Check Does $\frac{5x}{x-1} = \frac{5}{x-1}$ when $x = 1$?

$$\frac{5(1)}{1-1} = \frac{5}{1-1}$$

$$\frac{5}{0} = \frac{5}{0}$$

No! Both members of the equation are undefined.
Since division by zero is not permitted, this equation has no root.

$\therefore$ $x = 1$ does not satisfy the equation.

EXAMPLE 9–6 Solve $\frac{3x}{x+1} = 3 - \frac{4}{2x}$.

Day: Date:

Solution Multiply each member by the common denominator $2x(x + 1)$ and then factor:

$$\frac{2x(x+1)(3x)}{x+1} = 2x(x+1)(3) - \frac{2x(x+1)(4)}{2x}$$

$$6x^2 = 6x^2 + 6x - 4x - 4$$
$$6x^2 - 6x^2 = 6x - 4x - 4$$
$$0 = 2x - 4$$
$$2x = 4$$
$$x = 2$$

Check Does $\frac{3x}{x+1} = 3 - \frac{4}{2x}$ when $x = 2$?

$$\frac{3(2)}{2+1} = 3 - \frac{4}{2(2)}$$
$$\frac{6}{3} = 3 - 1$$
$$2 = 2 \quad \text{Yes!}$$

$\therefore$ $x = 2$ is the solution.

EXERCISE 9–2

Solve the following equations and check your solutions.

1. $\frac{3}{x} + \frac{5}{x} = 2$
2. $\frac{8}{x} = \frac{4}{x} - \frac{2}{3}$
3. $\frac{8}{y} - 1 = \frac{7}{y}$
4. $\frac{3}{x} - 4 = \frac{7}{2}$
5. $\frac{8}{x+1} = 4$
6. $\frac{x+3}{x-3} = 4$
7. $\frac{3y}{4y-5} = 2$
8. $\frac{2x+5}{x+4} = 1$
9. $\frac{4}{y-3} = \frac{2}{y}$
10. $\frac{3x-2}{5x-10} = \frac{4}{10}$
11. $\frac{4x-2}{4x} - \frac{1}{2} = \frac{3}{x}$
12. $\frac{3}{2} - \frac{5}{2y} = 2 - \frac{3}{y}$
13. $\frac{x-2}{5x} = \frac{4x-8}{30x}$
14. $\frac{1}{y-1} = \frac{3}{y-3}$
15. $\frac{12}{x-6} = \frac{2x}{x-6}$
16. $5 - \frac{5x}{x+1} = \frac{15}{4x}$

17. $\dfrac{2y}{y+1} = 2 - \dfrac{8}{2y}$

18. $\dfrac{3}{y} = \dfrac{3y}{y-4} - 3$

19. $\dfrac{5}{3+y} = \dfrac{3y}{y+3} - \dfrac{3y-4}{2y+6}$

20. $\dfrac{3}{x} + \dfrac{6}{x^2+5x} = \dfrac{6x}{x^2+5x}$

9–3 LITERAL EQUATIONS CONTAINING FRACTIONS

Equations that have one or more numbers represented by letters are **literal equations.** The formulas used to solve electrical problems are literal equations.

Variables and Constants

In literal equations, such as $\dfrac{x}{a} - 2 = b$, $\dfrac{c}{3} = y$, and $\dfrac{x}{m} - n = a$, the letters a, b, c, m, and n are *standing in* for numbers that have a fixed or constant value. The letters x and y are *standing in* for numbers that may change (or could be made to change) or vary.

In literal equations, it is conventional to represent constants by the letters a, b, c, g, h, k, l, m, and n. It is conventional to represent variables by the letters s, t, u, v, w, x, y, and z.

Constants are divided into two types. Those represented by letters (a, b, c, etc.) are called **arbitrary constants,** whereas numbers and symbols representing unvarying values are called **numerical constants.** Numerical constants include integers (1, 2, 3, etc.), fractions ($\frac{1}{2}$, $\frac{2}{3}$, etc.), and irrational numbers ($\pi = 3.14159\ldots$, $e = 2.71828\ldots$, $\sqrt{2} = 1.414\ldots$, etc.).

EXAMPLE 9–7 Solve the literal equation $\dfrac{x}{a} + 2 = b$ for x.

Solution

$$\frac{x}{a} + 2 = b$$

$$\frac{x}{a} = b - 2$$

$$\therefore \quad x = a(b-2)$$

EXAMPLE 9–8 Solve $\frac{7}{b} - \frac{y}{a} = 4$ for y.

Solution

$$\frac{7}{b} - \frac{y}{a} = 4$$

$$7a - yb = 4ab$$

$$-yb = 4ab - 7a$$

Change sign:

$$yb = 7a - 4ab$$

$$\therefore \quad y = \frac{7a - 4ab}{b}$$

EXAMPLE 9–9 Solve $\frac{xm}{2} - \frac{m}{3} = x$ for x.

Solution

$$\frac{xm}{2} - \frac{m}{3} = x$$

$$\frac{xm}{2} = x + \frac{m}{3}$$

$$\frac{xm}{2} - x = \frac{m}{3}$$

$$6\left(\frac{xm}{2} - x\right) = 6\left(\frac{m}{3}\right)$$

$$3xm - 6x = 2m$$

Factor the left member:

$$x(3m - 6) = 2m$$

$$\therefore \quad x = \frac{2m}{3m - 6}$$

EXERCISE 9–3

Solve each equation for the *conventional* variable s, t, u, v, w, x, y, or z.

1. $\frac{a}{x} = b$

2. $\frac{3}{\pi} = \frac{a}{W}$

Solve each equation for the *conventional* variable $s, t, u, v, w, x, y,$ or z.

3. $\frac{1}{z} - \frac{1}{a} = -\frac{2}{z}$

4. $\frac{hu}{m} + \frac{ku}{m} = 5$

5. $\frac{y}{2m} + \frac{y}{n} = 1$

6. $2g = \frac{hke + V}{aV}$

7. $c = \frac{b - W}{\pi^2 kW}$

8. $z - 1 - m = \frac{5z - 8}{m}$

9. $\frac{c}{s} = \frac{c}{3}(D - c)$

10. $\frac{\pi x + b}{\pi x - b} = 3$

11. $\frac{m - n}{ay} = b + 1$

12. $\frac{2u + 2}{au} - \frac{3}{u} = -\frac{7 - u}{u}$

13. $\frac{5x}{g - h} - 3 = \frac{3x}{k}$

14. $\frac{1}{4(a - b)} = \frac{1}{5(x - a)}$

15. $\frac{1}{u} - \frac{1}{m} = \frac{1}{n} - \frac{1}{u}$

16. $\frac{1}{w} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$

17. $\frac{b}{a} = \frac{x - b}{x} + \frac{2b}{ax}$

18. $\frac{4k + 1}{4k} = \frac{1}{2z} + \frac{2k}{z}$

19. $\frac{4g}{3g} - \frac{6}{6V} = \frac{4g}{V} - \frac{1}{3g}$

20. $\frac{z}{ab} - \frac{1}{ac} = \frac{z}{bc}$

9–4 EVALUATING FORMULAS

Evaluating a formula is the process of finding the value of one variable in a formula when all the other variables are known. When evaluating a formula for only one set of values, it is simpler first to substitute the known values into the formula and then solve for the variable representing the desired quantity. This technique is demonstrated in Example 9–10.

EXAMPLE 9–10 Using the formula $F = \frac{9}{5}C + 32$, determine the temperature in Celsius (C) when the temperature in Fahrenheit (F) is 77°F.

Solution First substitute 77 for F and then solve for C:

$$F = \frac{9}{5}C + 32$$

$$77 = \frac{9}{5}C + 32$$

$$\frac{9}{5}C = 45$$

$$C = 45\left(\frac{5}{9}\right)$$

$$\therefore \quad C = 25°\text{C}$$

In Example 9–10, we first substituted into the formula, and then solved the problem. When we wish to find several values for a particular variable, it is best first to solve the formula for the desired variable and then substitute into the formula. This concept is demonstrated in Example 9–11.

EXAMPLE 9–11 Using $F = \frac{9}{5}C + 32$, determine the temperature in Celsius (C) for temperatures in Fahrenheit (F) of: (a) 105°F (b) 93°F (c) 85°F.

Solution First solve for C:

$$F = \frac{9}{5}C + 32$$

$$\frac{9}{5}C = F - 32$$

$$C = \frac{5(F - 32)}{9}\,°\text{C}$$

Now, substitute each of the given temperatures.

(a) $F = 105°\text{F}$

$$\therefore \quad C = \frac{5(\mathbf{105} - 32)}{9} = 40.6°\text{C}$$

(b) $F = 93°\text{F}$

$$\therefore \quad C = \frac{5(\mathbf{93} - 32)}{9} = 33.9°\text{C}$$

(c) $F = 85°\text{F}$

$$\therefore \quad C = \frac{5(\mathbf{85} - 32)}{9} = 29.4°\text{C}$$

In the study of electronics, formulas are routinely evaluated. Because of this, it is very important that you develop the skill of successfully working with formulas. The following exercise will give you some practice.

EXERCISE 9–4

In the following formulas, solve for the indicated variable.

	Formula:	***Solve for:***
1.	$C = \frac{5}{9}(\text{F} - 32)$	F
2.	$A = \frac{1}{2}h(b + c)$	c

Formula:	Solve for:
3. $X_C = \frac{1}{2\pi fC}$	f
4. $C = \frac{1}{C_1} + \frac{1}{C_2}$	C_2
5. $k = \frac{wv^2}{2g}$	w
6. $\beta = \frac{\alpha}{1 - \alpha}$	α
7. $\theta_{JA} = \frac{T_J - T_A}{P_D}$	T_A
8. $A_v' = \frac{A_v}{1 + \beta A_v}$	A_v
9. $C = \frac{E}{R_1 + R_2}$	R_2
10. $\frac{V_1}{V_2} = \frac{P_2}{P_1}$	P_1
11. $\theta = \frac{108(n - 2)}{n}$	n
12. $s = t\left(\frac{V_0 + V_1}{2}\right)$	V_1
13. $\eta = \frac{A_2 S_2}{F_1 S_1}$	S_2
14. $Q = \frac{tKA(T_0 - T_1)}{L}$	T_0
15. $F = \frac{q_1 q_2}{4\pi\epsilon_0 d^2}$	d^2
16. $V = \frac{R_2(V_{CC} - R_1 I_{\text{sat}} - 1)}{R_1 + R_2}$	R_1
17. $R_T = \frac{R_1 R_2}{R_1 + R_2}$	R_2
18. $I_1 = \frac{IR_2}{R_1 + R_2}$	R_2
19. $S = \frac{R_1 + R_2}{R_E + R_\beta(1 - \alpha)}$	α
20. $R_2 = \frac{Z_Z(k + 1)}{k - 1} - R_3$	k

Evaluate each of the following formulas. First assign the indicated values to the variables and then solve. Express the solution to an appropriate precision with a unit and unit prefix (as needed).

21. Determine the resistance (R) of a coil in ohms from

$$Q_0 = \frac{2\pi f_{ar} L}{R}$$

when $f_{ar} = 310$ kHz, $L = 5.00$ mH, $Q_0 = 200$, and $\pi \approx 3.14$.

22. Determine the collector current (I_c) in amperes from

$$I_{CBO} = \frac{I_c - \beta I_B}{\beta + 1}$$

when $I_{CBO} = 5.00\ \mu A$, $I_B = 60.0\ \mu A$, and $\beta = 100$.

23. Determine the total resistance of a parallel network (R_T) in ohms from

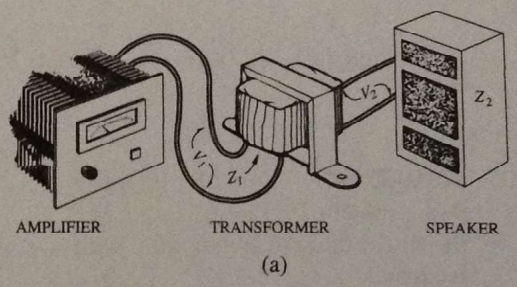
$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$$

when $R_1 = 2.20\ \text{k}\Omega$ and $R_2 = 1.50\ \text{k}\Omega$.

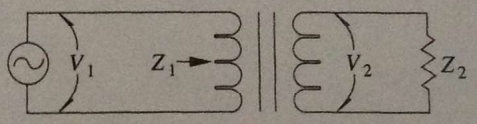
24. Determine the impedance of the load (Z_2) in ohms from

$$\frac{V^2_1}{V^2_2} = \frac{Z_1}{Z_2}$$

when $V_1 = 60$ V, $V_2 = 2$ V, and $Z_1 = 7.2\ \text{k}\Omega$. Refer to Figure 9–1.



(a)



(b)

FIGURE 9–1 Circuit for Exercise 9–4, problem 24: (a) pictorial diagram showing an output transformer being used to match the load impedance to the amplifier; (b) schematic diagram of the circuit.

25. Determine the frequency (f) in hertz from

$$X_C = \frac{1}{2\pi fC}$$

when $X_C = 410\ \Omega$, $C = 50$ nF, and $\pi \approx 3.14$.

26. Determine the output voltage (V_o') in volts from

$$\frac{V_o'}{V_s} = \frac{A_v}{1 - \beta A_v}$$

when $\beta = 0.050$, $A_v = 60$, and $V_s = 1.0$ V. Refer to Figure 9–2.

27. Determine the forward current transfer ratio of a transistor (β) from

$$\alpha = \frac{\beta}{\beta + 1}$$

when $\alpha = 0.985$.

28. Determine the load resistance (R_L) in ohms from

$$A_v = \frac{\mu R_L}{r_p + R_L}$$

when $\mu = 18$, $r_p = 0.20\ \text{k}\Omega$, and $A_v = 12$.

29. Determine the value of the bias resistance (R_1) in ohms from

$$I_B = \frac{V_{cc}}{R_1 + (\beta + 1)R_E}$$

when $I_B = 40.0\ \mu\text{A}$, $V_{cc} = 18.0$ V, $\beta = 100$, and $R_E = 470\ \Omega$.

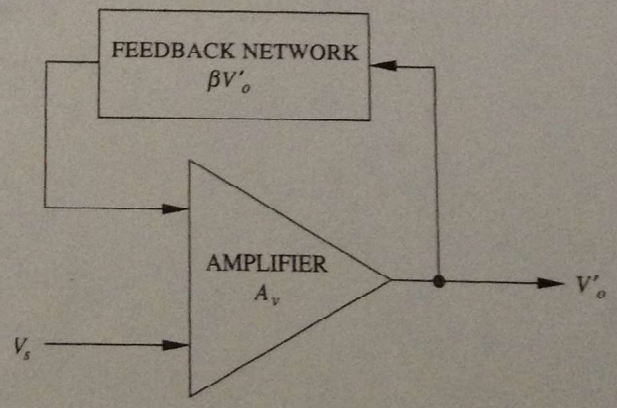


FIGURE 9–2 Circuit for Exercise 9–4, problem 26. Flow diagram of the signal through an amplifier with feedback.

30. Determine the temperature of the case of a power transistor (T_c) in degrees Celsius from

$$P_D = \frac{T_c - T_s}{\theta_{cs}}$$

when $P_D = 75$ W, $T_s = 55°C$, and $\theta_{cs} = 0.35°C/W$.

SELECTED TERMS

extraneous root A solution of a mathematical equation that is not a solution to the physical problem.

fractional equation Equation that has a variable in the denominator of one or more terms.

literal equation Equation with one or more numbers represented by letters.

Day: Date:

Chapter 8

Systems of Linear Equations

INTRODUCTION

Joyce owns a field with an irrigation system consisting of two networks of pipes. During the month of July she contracted to have the water turned on the first and third Fridays. On the first Friday the larger network operated 9 hours and the other 8 hours, delivering 730 gallons of water. On the third Friday, the larger network operated 5 hours and the other 7 hours, delivering 495 gallons. What is the rate of flow (gallons per hour) in each network?

To solve this problem, we need to use two equations with two unknowns to represent all the information. This very powerful problem-solving method is called a *system of equations*. It allows separate conditions or limitations to be expressed in a relatively simple manner while tying all the conditions together in a system.

In this chapter we describe methods of solving systems of linear equations. In the preceding problem the larger network delivers 50 gal/h, while the smaller delivers 35 gal/h.

Objectives

- Solve systems of equations by substitution.
- Solve systems of equations by the addition-subtraction method.
- Evaluate determinants using determinant properties.
- Use Cramer's rule.
- Use the method of partial fractions to rewrite rational expressions as the sum or difference of simpler expressions.

6.1 SOLVING A SYSTEM OF TWO LINEAR EQUATIONS

Systems of linear equations in the form

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

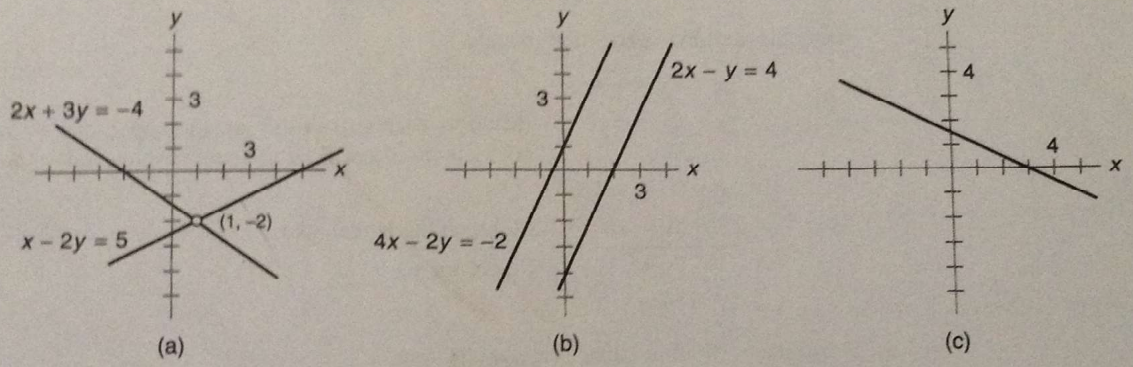
may be solved in a number of ways. In this section we shall study solutions by graphing, by the addition-subtraction method, and by the method of substitution. Any ordered pair (x, y) that satisfies both equations is called a **solution,** or root, of the system.

We saw in Chapter 4 that the graph of a linear equation with two variables is a straight line. The solution of a system of two linear equations with two variables can be discussed in general terms of their graphs as follows.

GRAPHS OF LINEAR SYSTEMS OF EQUATIONS WITH TWO VARIABLES

1. The two lines may intersect at a common, single point. This point, in ordered pair form (x, y), is the solution of the system.
2. The two lines may be parallel with no points in common; hence, the system has no solution.
3. The two lines may coincide; the solution of the system is the set of all points on the common line.

These results are illustrated in Fig. 6.1.



$2x + 3y = -4$
$x - 2y = 5$

The lines intersect at the point $(1, -2)$, which is the common solution.

$2x - y = 4$
$4x - 2y = -2$

The lines are parallel, and hence there is no common solution.

$x + 2y = 3$
$2x + 4y = 6$

The lines coincide, and as a result any solution of one equation is a solution of the other.

Figure 6.1

When the two lines intersect, the system of equations is called **independent and consistent.** When the two lines are parallel, the system of equations is called **inconsistent.** When the two lines coincide, the system of equations is called **dependent.**

Graphical methods of solving systems of linear equations usually give only approximate solutions. Algebraic methods give exact solutions.* The first algebraic method we shall study is called the **addition-subtraction method.**

*Do not use the rules for calculating with measurements in this chapter.

SOLVING A PAIR OF LINEAR EQUATIONS BY THE ADDITION-SUBTRACTION METHOD

1. If necessary, multiply each side of one or both equations by some number so that the numerical coefficients of one of the variables are of equal absolute value.
2. If these coefficients of equal absolute value have like signs, subtract one equation from the other. If they have unlike signs, add the equations. That is, do whatever is necessary to eliminate that variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute the solution for the variable found in Step 3 in either of the original equations, and solve this resulting equation for the second variable.
5. Check by substituting the solution in both original equations.

EXAMPLE 1

Solve

$$2x + 3y = -4$$
$$x - 2y = 5$$

using the addition-subtraction method.

$$\begin{aligned} 2x + 3y &= -4 \\ x - 2y &= 5 \end{aligned}$$

(Multiply each side of this equation by 2, and the numerical coefficients of x will be of equal absolute value.)

$$\begin{aligned} 2x + 3y &= -4 \\ 2x - 4y &= 10 \end{aligned}$$

(Subtract the equations.)

$$7y = -14$$

(Solve for y.)

$$y = -2$$

Substitute $y = -2$ in either original equation.

$$x - 2y = 5$$
$$x - 2(-2) = 5$$
$$x = 1$$

The solution is $(1, -2)$.

Check: Substitute the solution in both original equations.

$$2x + 3y = -4 \qquad x - 2y = 5$$
$$2(1) + 3(-2) = -4 \qquad (1) - 2(-2) = 5$$
$$-4 = -4 \qquad 5 = 5$$

EXAMPLE 2

Solve

$$3x - 4y = -36$$
$$10x + 3y = -22$$

using the addition-subtraction method.

$$\begin{aligned} 3x - 4y &= -36 \\ 10x + 3y &= -22 \end{aligned}$$ (Multiply each side of the first equation by 3 and each side of the second equation by 4; the numerical coefficients of y will be of equal absolute value.)

$$\begin{aligned} 9x - 12y &= -108 \\ 40x + 12y &= -88 \end{aligned}$$ (Add the equations.)

$$49x = -196$$ (Solve for x.)

$$x = -4$$

Substitute $x = -4$ in either original equation.

$$\begin{aligned} 3x - 4y &= -36 \\ 3(-4) - 4y &= -36 \\ -4y &= -24 \\ y &= 6 \end{aligned}$$

The solution is $(-4, 6)$.

Check: Substitute the solution in both original equations as before.

The second algebraic method of solving systems of linear equations is called the **method of substitution.**

SOLVING A PAIR OF LINEAR EQUATIONS BY THE METHOD OF SUBSTITUTION

1. From either of the two given equations, solve for one variable in terms of the other.
2. Substitute this result from Step 1 in the *other* equation. Note that this step eliminates one variable.
3. Solve the equation obtained from Step 2 for the remaining variable.
4. From the equation obtained in Step 1, substitute the solution for the variable found in Step 3, and solve this resulting equation for the second variable.
5. Check by substituting the solution in both original equations.

EXAMPLE 3

Solve

$$\begin{aligned} 3x + 4y &= -15 \\ y &= -2x \end{aligned}$$

using the method of substitution.

Since the second equation is already solved for y, substitute $y = -2x$ in the first equation.

$$\begin{aligned} 3x + 4y &= -15 \\ 3x + 4(-2x) &= -15 \quad \text{(Substitute for } y\text{.)} \\ 3x - 8x &= -15 \quad \text{(Solve for } x\text{.)} \\ -5x &= -15 \\ x &= 3 \end{aligned}$$

Substitute $x = 3$ in the second equation.

$$y = -2x$$
$$y = -2(3)$$
$$y = -6$$

The solution is $(3, -6)$.

Check: Substitute in both original equations.

EXAMPLE 4

Solve

$$3x + y = 3$$
$$2x - 4y = 16$$

by the method of substitution.

Solve the first equation for y.

$$3x + y = 3$$
$$y = -3x + 3$$

Substitute $y = -3x + 3$ in the second equation.

$$2x - 4y = 16$$
$$2x - 4(-3x + 3) = 16 \quad \text{(Substitute for } y\text{.)}$$
$$2x + 12x - 12 = 16 \quad \text{(Solve for } x\text{.)}$$
$$14x = 28$$
$$x = 2$$

Substitute $x = 2$ in the equation $y = -3x + 3$.

$$y = -3x + 3$$
$$y = -3(2) + 3$$
$$y = -3$$

The solution is $(2, -3)$.

Check: Substitute the solution in both original equations.

EXAMPLE 5

Solve

$$8x + 5y = -16$$
$$2x - 3y = 13$$

by the method of substitution.

Solve the second equation for x.

$$2x - 3y = 13$$
$$2x = 3y + 13$$
$$x = \frac{3y + 13}{2}$$

Substitute $x = \dfrac{3y + 13}{2}$ in the first equation.

$$8x + 5y = -16$$

$$8\left(\frac{3y + 13}{2}\right) + 5y = -16 \qquad \text{(Substitute for } x\text{.)}$$

$$4(3y + 13) + 5y = -16 \qquad \text{(Solve for } y\text{.)}$$

$$12y + 52 + 5y = -16$$

$$17y = -68$$

$$y = -4$$

Substitute $y = -4$ in the equation $x = \dfrac{3y + 13}{2}$.

$$x = \frac{3y + 13}{2}$$

$$x = \frac{3(-4) + 13}{2}$$

$$x = \frac{1}{2}$$

The solution is $\left(\frac{1}{2}, -4\right)$.

Check: Substitute the solution in both original equations.

A special case of the substitution method is the **comparison method.** That is,

if $a = c$ and $b = c$

then $a = b$

EXAMPLE 6

Solve

$$3x - 4 = 5y$$

$$6 - 2x = 5y$$

Since the left side of each equation equals the same quantity, we have

$$3x - 4 = 6 - 2x$$

which eliminates a variable. Then

$$5x = 10$$

$$x = 2$$

Substitute $x = 2$ in the first equation.

$$3x - 4 = 5y$$

$$3(2) - 4 = 5y$$

$$2 = 5y$$

$$\frac{2}{5} = y$$

The solution is $\left(2, \frac{2}{5}\right)$.

Check: Substitute the solution in both original equations.

Note: The basic idea of solving any system of equations is to choose the method that eliminates a variable most easily. This is determined by the form of the given equations.

EXAMPLE 7

Solve

$$2x - y = 4$$
$$4x - 2y = -2$$

Using the addition-subtraction method, multiply each side of the first equation by 2.

$$\begin{array}{rl} 4x - 2y = 8 & \\ \underline{4x - 2y = -2} & \text{(Subtract the equations.)} \\ 0 = 10 & \end{array}$$

When we obtain the result $0 = a$ $(a \neq 0)$, this means that there is no common solution, the lines when graphed are parallel, and the system of equations is inconsistent.

EXAMPLE 8

Solve

$$x + 2y = 3$$
$$2x + 4y = 6$$

Using the addition-subtraction method, multiply each side of the first equation by 2.

$$\begin{array}{rl} 2x + 4y = 6 & \\ \underline{2x + 4y = 6} & \text{(Subtract the equations.)} \\ 0 = 0 & \end{array}$$

When we obtain the result $0 = 0$, this means that the system is dependent and the lines coincide when graphed. Since one equation is a multiple of the other, the equations are equivalent and therefore form the same straight line when graphed.

Many technical applications can be expressed mathematically as a system of linear equations. Examples will be given in this section and in the following sections to demonstrate how to set up an appropriate system of equations for a given problem. Except for Steps 4, 5, and 6, the steps in approaching the solution of a problem stated verbally are the same as for problems involving only one linear equation. You may wish to review Section 2.8 at this time.

STEPS FOR PROBLEM SOLVING

1. Read the problem carefully at least two times.
2. If possible, draw a picture or a diagram.
3. Write what facts are given and what unknown quantities are to be found.
4. Choose a symbol to represent each quantity to be found (there will be more than one). Be sure to label each symbol to indicate what it represents.

5. Write appropriate equations relating these variables from the information given in the problem.* Watch for information that is not stated but which should be assumed. For example, rate × time = distance.
6. Solve for the unknown variables by the methods presented in this chapter.
7. Check your solution in the original equations.
8. Check your solution in the original verbal problem.

EXAMPLE 9

The sum of two currents is 200 mA. The larger current is three times the smaller. What are the two currents?

Let

x = the smaller current

y = the larger current

The first sentence gives the first equation that follows; the second sentence gives the second equation.

$$x + y = 200$$
$$y = 3x$$

Substitute the second equation into the first.

$$x + 3x = 200$$
$$4x = 200$$
$$x = 50$$

Substitute $x = 50$ into $y = 3x$.

$$y = 3(50) = 150$$

The solution is (50, 150). That is, the smaller current is 50 mA and the larger current is 150 mA.

The solution checks in both original equations and in the problem as it is originally stated.

EXAMPLE 10

A boat can travel 24 mi upstream in 6 h. It takes 4 h to make the return trip downstream. Find the rate of the current and the speed of the boat (in still water).

Let

x = the speed of the boat

y = the rate of the current

Then

$x + y$ = the speed of the boat going downstream (the boat is traveling with the current)

$x - y$ = the speed of the boat going upstream (the boat is traveling against the current)

*There should be as many equations as there are unknown variables.

Let's make a chart of the information.

	d	r	t
Traveling downstream	24	$x + y$	4
Traveling upstream	24	$x - y$	6

We now have

$$4(x + y) = 24 \qquad \text{(time} \times \text{rate} = \text{distance)}$$
$$6(x - y) = 24$$

This gives us the system

$$4x + 4y = 24$$
$$6x - 6y = 24$$

Use the addition-subtraction method.

$$\begin{array}{r} 12x + 12y = 72 \\ 12x - 12y = 48 \\ \hline 24x \qquad = 120 \\ x \qquad = 5 \end{array}$$

Substitute $x = 5$ in one of the original equations.

$$\begin{aligned} 6x - 6y &= 24 \\ 6(5) - 6y &= 24 \\ 30 - 6y &= 24 \\ -6y &= -6 \\ y &= 1 \end{aligned}$$

The solution is (5, 1). The rate of the current is 1 mi/h and the speed of the boat is 5 mi/h. The solution should be checked in both original equations and in the original problem.

Exercises 6.1

Solve each system of equations graphically by graphing each equation and finding the point of intersection. Check by substituting the solution ordered pair in both original equations.

1. $2x + 3y = 5$
$x - 2y = 6$

2. $3x - 2y = 0$
$3x + y = -9$

3. $5x + 2y = -5$
$2x + 3y = 9$

4. $-2x + 4y = -6$
$3x - 2y = 9$

5. $9x - 6y = 15$
$-3x + 2y = -5$

6. $-3x + y = 4$
$y = 3x$

Solve each system of equations by the addition-subtraction method. Check by substituting the solution in both original equations.

7. $4x + y = 15$
$3x + y = 13$

8. $-2x + 5y = -30$
$2x - 3y = 22$

9. $-4x + 3y = -22$
$4x - 5y = 34$

10. $7x + 3y = 1$
$2x + 3y = 11$

11. $4x - 5y = -34$
$2x + 3y = 16$

12. $3x - 2y = -3$
$7x - 10y = 1$

13. $3x + 2y = -15$
$x + 5y = -5$

14. $x + 3y = 6$
$5x - 2y = -4$

15. $4x - 5y = 7$
$-2x + 3y = -3$

16. $6x + 7y = -8$
$18x - 4y = 26$

17. $4x + 3y = -3$
$12x + 9y = -12$

18. $2x - y = 6$
$10x - 5y = 30$

19. $12x + 5y = -18$
$8x - 7y = -74$

20. $7x - 10y = -6$
$24x - 16y = 16$

21. $-12x + 15y = -43$
$9x - 12y = 34$

22. $24x + 84y = 185$
$42x + 12y = 155$

Solve each system of equations by the method of substitution and check.

23. $2x - 5y = -36$
$y = 4x$

24. $6x - 11y = -3$
$x = 2y$

25. $8x + 9y = -5$
$x = -3y$

26. $12x + 3y = -6$
$y = -5x$

27. $3x - 5y = 64$
$y = 6x - 2$

28. $6x + 8y = 154$
$x = 3y + 4$

29. $3x + 4y = -6$
$2x - y = -15$

30. $2x + 3y = -29$
$x + 4y = -32$

31. $5x - 7y = 23$
$3x + 2y = -11$

32. $6x + 7y = 16$
$-5x - 6y = -13$

Solve each system of equations by any method and check.

33. $\frac{2}{3}x + \frac{1}{2}y = \frac{4}{9}$
$\frac{3}{4}x - \frac{2}{3}y = -\frac{23}{72}$

34. $\frac{5}{4}x - \frac{2}{5}y = -1$
$\frac{1}{4}x + \frac{2}{3}y = 11$

35. $\frac{3}{5}x + 2y = \frac{5}{2}$
$\frac{2}{5}x - 4y = 3$

36. $\frac{7}{3}x + \frac{2}{5}y = -17$
$\frac{2}{5}x + \frac{4}{3}y = 54$

37. $1.4x - 2.7y = 5.66$
$0.5x + 2y = 3.8$

38. $x + y = 40.3$
$0.02x + 0.05y = 1.46$

39. $0.002x + 0.008y = 2.28$
$0.04x + 0.09y = 28.8$

40. $1.57x + 2.04y = 20.262$
$2.16x - 8.42y = -47.342$

41. The sum of two capacitors is 55 microfarads (μF). The difference between them is 25 μF. What is the size of each capacitor?

42. Find the two acute angles in a right triangle if the difference of their measures is 20°.

43. In one concrete mix there is four times as much gravel as cement. If the total volume of the mix is 11.5 m^3, how much of each ingredient is in the mix?

44. The sum of two inductors is 90 millihenries (mH). The larger is 3.5 times the smaller. What is the size of each inductor?

45. A farmer has a 3% solution and an 8% solution of a pesticide. How much of each must he mix to get 2000 L of 4% solution for his sprayer?

46. In testing gasohol mixtures, two mixtures are on hand: mixture A (90% gasoline, 10% alcohol) and mixture B (80% gasoline, 20% alcohol). How much of each mixture must be combined to get 100 gal of an 84% gasoline, 16% alcohol mixture?

47. A boat can travel 20 km upstream in 5 h. It takes 4 h to make the return trip downstream. Find the rate of current and the speed of the boat (in still water).

48. Two planes are 60 km apart. Flying toward each other, they meet in 10 min. When flying in the same direction (at the same speeds as before), the faster plane overtakes the slower plane in 30 min. What is the speed of each plane in kilometres per hour?

49. A rectangular yard has been fenced using 96 m of fence. Find the length and the width of the yard if the length of the yard is 12 m longer than the width.

50. The perimeter of an isosceles triangle is 87 cm. The shortest side is 12 cm less than the longer sides. Find the length of the three sides of the triangle.

51. In a parallel electrical circuit, the products of the current and the resistance are equal in all branches. If the total current is 900 mA through any two branches, find the current flowing through branches having resistances of 50 Ω and 300 Ω.

52. The total current in a parallel circuit equals the sum of the currents in the branches. If the total current in six parallel branches is 1.90 A, and if some of the branches have currents of 0.25 A and others have 0.45 A, how many of each type of branch are in the circuit?

53. The circuit in Fig. 6.2 gives the following system of linear equations:

$$4.3I_1 + 2.3(I_1 + I_2) = 27$$
$$2.8I_2 + 2.3(I_1 + I_2) = 35.2$$

Find I_1 and I_2 in mA.

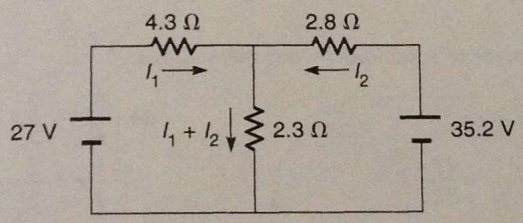


Figure 6.2

54. Harold needs a loan for $40,000. He gets part at 11% at a credit union. He gets the rest at 14.5% at a bank. If the total annual interest he must pay is $5520, how much did he receive from each lending institution?

55. Find the distance between the parallel lines $3x + 4y = 25$ and $6x + 8y = 15$.

56. Find the distance between the parallel lines $y = 3x - 4$ and $y = 3x + 5$.

6.2 OTHER SYSTEMS OF EQUATIONS

A **literal equation** is one in which letter coefficients are used in place of numerical coefficients. For example, in the following equations, a and b represent known quantities or coefficients, and x and y are the variables or unknown quantities. To solve such systems of literal equations, use either of the algebraic methods discussed in Section 6.1.

EXAMPLE 1

Solve

$$ax + by = ab$$
$$bx - ay = b^2$$

using the addition-subtraction method.

$$a^2x + aby = a^2b \qquad \text{(Multiply each side of the first equation by } a\text{.)}$$
$$b^2x - aby = b^3 \qquad \text{(Multiply each side of the second equation by } b\text{.)}$$
$$a^2x + b^2x = a^2b + b^3 \qquad \text{(Add the two equations.)}$$
$$(a^2 + b^2)x = b(a^2 + b^2) \qquad \text{(Factor.)}$$
$$x = \frac{b(a^2 + b^2)}{a^2 + b^2} \qquad \text{(Solve for } x\text{.)}$$
$$x = b$$

Substitute $x = b$ in the first equation.

$$ax + by = ab$$
$$a(b) + by = ab$$
$$by = 0$$
$$y = 0$$

The solution is $(b, 0)$.

The equations in the system

$$\frac{6}{x} + \frac{4}{y} = -2$$
$$\frac{9}{x} - \frac{7}{y} = 10$$

are not linear, or first-degree, equations. When each is multiplied by the L.C.D., xy, they become

$$6y + 4x = -2xy$$
$$9y - 7x = 10xy$$

each of which is second degree. However, we must assume that $x \neq 0$ and $y \neq 0$. Such a pair of equations can be solved using the algebraic methods from Section 6.1.

EXAMPLE 2

Solve

$$\frac{6}{x} + \frac{4}{y} = -2$$
$$\frac{9}{x} - \frac{7}{y} = 10$$

using the addition-subtraction method.

$$\frac{18}{x} + \frac{12}{y} = -6 \qquad \text{(Multiply each side of the first equation by 3.)}$$
$$\frac{18}{x} - \frac{14}{y} = 20 \qquad \text{(Multiply each side of the second equation by 2.)}$$
$$\frac{26}{y} = -26 \qquad \text{(Subtract.)}$$
$$26 = -26y \qquad \text{(Solve for } y\text{.)}$$
$$-1 = y$$

Substitute $y = -1$ in the first equation.

$$\frac{6}{x} + \frac{4}{y} = -2$$

$$\frac{6}{x} + \frac{4}{(-1)} = -2$$

$$\frac{6}{x} = 2$$

$$6 = 2x$$

$$3 = x$$

The solution is $(3, -1)$.

Check: Substitute the solution in both original equations.

Note: A system of equations as the preceding one can also be solved by first letting $A = \frac{1}{x}$ and $B = \frac{1}{y}$. Then solve for A and B, and finally solve for x and y.

Exercises 6.2

Solve each system of equations.

1. $ax + y = b$
$bx - y = a$

2. $x + y = a - b$
$x - y = a + b$

3. $x + y = a^2 + ab$
$ax = by$

4. $2ax + b^2y = a^2b^4 + 2ab^2$
$y = a^2x$

5. $ax - by = 0$
$bx + ay = 1$

6. $ax + by = 1$
$bx + ay = 1$

7. $ax + by = c$
$bx + ay = c$

8. $ax + by = b - a$
$bx - ay = b - a$

9. $(a + 3b)x - by = a$
$(a - b)x - ay = b$

10. $(a + b)x - (a - b)y = 4ab$
$(a - b)x - (a + b)y = 0$

11. $\frac{3}{x} + \frac{4}{y} = 2$
$\frac{6}{x} + \frac{12}{y} = 5$

12. $\frac{12}{x} - \frac{15}{y} = 7$
$\frac{9}{x} + \frac{30}{y} = -3$

13. $\frac{3}{x} + \frac{2}{y} = 0$
$\frac{2}{x} - \frac{5}{y} = 19$

14. $\frac{6}{x} - \frac{4}{y} = 12$
$\frac{5}{x} - \frac{3}{y} = 11$

15. $\frac{1}{x} + \frac{1}{y} = \frac{13}{6}$
$\frac{1}{x} - \frac{1}{y} = \frac{5}{6}$

16. $\frac{3}{x} + \frac{12}{y} = 32$
$\frac{4}{x} + \frac{9}{y} = 21\frac{2}{3}$

17. $\frac{6}{s} + \frac{4}{t} = 16$
$\frac{9}{s} - \frac{5}{t} = 2$

18. $\frac{24}{m} + \frac{18}{n} = 72$
$\frac{36}{m} + \frac{45}{n} = 116$

19. $\frac{1}{x} + \frac{1}{y} = a$
$\frac{1}{x} - \frac{1}{y} = b$

20. $\frac{a}{x} + \frac{b}{y} = 2$
$\frac{b}{x} + \frac{a}{y} = 1$

6.3 SOLVING A SYSTEM OF THREE LINEAR EQUATIONS

The graph of a linear equation with three variables in the form

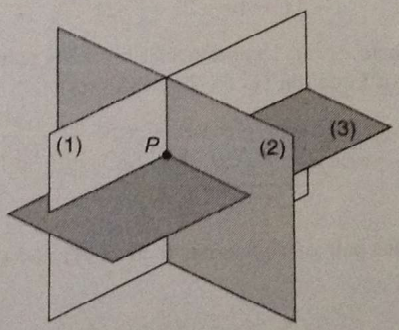
$$ax + by + cz = d$$

is a **plane.** The solution of a system of three linear equations with three variables can be discussed in general terms of their graphs as follows:

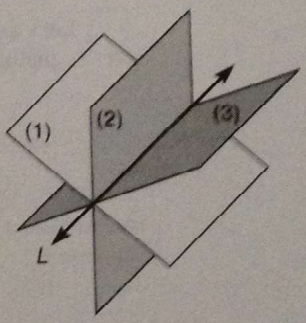
GRAPHS OF LINEAR SYSTEMS OF EQUATIONS WITH THREE VARIABLES

1. The three planes may intersect at a common, single point. This point, in ordered triple form (x, y, z), is then the solution of the system.
2. The three planes may intersect along a common line. The infinite set of points that satisfy the equation of the line is the solution of the system.
3. The three planes may not have any points in common; the system has no solution. For example, the planes may be parallel, or they may intersect triangularly with no points common to all three planes.
4. The three planes may coincide; the solution of the system is the set of all points in the common plane.

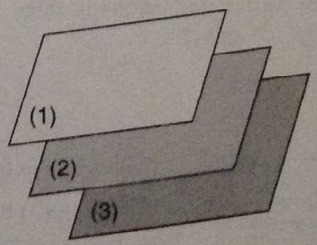
Graphical solutions of three linear equations with three unknowns are not used because three-dimensional graphing is required and is not practical by hand. However, three-dimensional graphing is useful to illustrate the general cases (see Fig. 6.3).



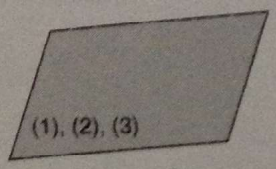
(a) One common solution at point P



(b) Common solutions at line L



(c) No points in common



(d) All points in common

Figure 6.3

A linear equation with four or more variables cannot be represented in three-dimensional space. However, systems of linear equations with four or more variables have many applications, and the methods presented in this chapter will be extended to solve such systems.

The first method presented here for solving a system of linear equations with three variables is the **addition-subtraction method.**

THE ADDITION-SUBTRACTION METHOD FOR A SYSTEM OF THREE LINEAR EQUATIONS WITH THREE VARIABLES

1. Choose a variable to be eliminated. Eliminate it from any pair of equations by using the techniques of the addition-subtraction method from Section 6.1.
2. Eliminate this same variable from *any other pair* of equations.
3. The result of Steps 1 and 2 is a pair of linear equations in two unknowns. Solve this pair for the two variables.
4. Solve for the third variable by substituting the results from Step 3 in any one of the original equations.
5. Check by substituting the solution in all three original equations.

EXAMPLE 1

Solve

$$x + 2y - 6z = -17 \quad \textbf{(1)}$$
$$2x - 5y + z = 28 \quad \textbf{(2)}$$
$$-3x + 4y + 2z = -21 \quad \textbf{(3)}$$

Let's choose to eliminate x first. To eliminate x from any pair of equations, such as (1) and (2), multiply each side of Equation (1) by 2 and subtract.

$$\begin{array}{r} 2x + 4y - 12z = -34 \\ 2x - 5y + z = 28 \\ \hline 9y - 13z = -62 \end{array} \quad \textbf{(4)}$$

To eliminate x from any other pair of equations, such as (1) and (3), multiply each side of Equation (1) by 3 and add.

$$\begin{array}{r} 3x + 6y - 18z = -51 \\ -3x + 4y + 2z = -21 \\ \hline 10y - 16z = -72 \end{array} \quad \textbf{(5)}$$

We have now reduced the system of three equations in three variables to a system of two equations in two variables, namely, (4) and (5).

$$9y - 13z = -62 \quad \textbf{(4)}$$
$$10y - 16z = -72 \quad \textbf{(5)}$$
$$90y - 130z = -620 \quad \text{[Multiply (4) by 10.]}$$
$$\underline{90y - 144z = -648} \quad \text{[Multiply (5) by 9.]}$$
$$14z = 28 \quad \text{(Subtract.)}$$
$$z = 2$$

Substitute $z = 2$ in (5).

$$\begin{aligned} 10y - 16z &= -72 \\ 10y - 16(2) &= -72 \\ 10y &= -40 \\ y &= -4 \end{aligned}$$

Substitute $z = 2$ and $y = -4$ in any original equation; we shall use Equation (1).

$$\begin{aligned} x + 2y - 6z &= -17 \\ x + 2(-4) - 6(2) &= -17 \\ x &= 3 \end{aligned}$$

The solution is $(3, -4, 2)$.

Check: Substitute the solution in all three original equations.

EXAMPLE 2

Solve

$$\begin{aligned} 4x - 2y + z &= 12 && (1) \\ -y + 3z &= -11 && (2) \\ 5x + 3y &= 45 && (3) \end{aligned}$$

Let's choose to eliminate z first. To eliminate z from Equations (1) and (2), multiply each side of the first equation by 3 and subtract.

$$\begin{array}{rl} 12x - 6y + 3z = 36 & \\ -y + 3z = -11 & \\ \hline 12x - 5y = 47 & (4) \end{array}$$

We now have reduced the system to Equations (3) and (4).

$$\begin{array}{rll} 5x + 3y = 45 & & (3) \\ 12x - 5y = 47 & & (4) \\ 25x + 15y = 225 & [\text{Multiply (3) by 5.}] & \\ \underline{36x - 15y = 141} & [\text{Multiply (4) by 3.}] & \\ 61x = 366 & (\text{Add.}) & \\ x = 6 & & \end{array}$$

Substitute $x = 6$ in (3).

$$\begin{aligned} 5x + 3y &= 45 \\ 5(6) + 3y &= 45 \\ 3y &= 15 \\ y &= 5 \end{aligned}$$

Substitute $x = 6$ and $y = 5$ in any original equation; we shall use (1).

$$\begin{aligned} 4x - 2y + z &= 12 \\ 4(6) - 2(5) + z &= 12 \\ z &= -2 \end{aligned}$$

The solution is $(6, 5, -2)$.

Check: Substitute the solution in all three original equations.

EXAMPLE 3

A person deposits \$6000 into three different savings accounts. One account earns interest at 5%, another at $5\frac{1}{2}$%, and another at 6%. The total interest earned at the end of one year was \$340. How much was deposited into each account if the 6% account earned \$135 more than the 5% account?

Let

x = the amount deposited at 6%
y = the amount deposited at $5\frac{1}{2}$%
z = the amount deposited at 5%

Then

$0.06x + 0.055y + 0.05z = 340$	(the sum of the interest earned from each account)
$0.06x - 0.05z = 135$	(the difference in earnings between the 6% and 5% accounts)
$x + y + z = 6000$	(the total sum deposited)

This gives us a system of three linear equations in three unknowns. Multiply each equation by some quantity which will result in removing the decimal points.

$$60x + 55y + 50z = 340{,}000 \quad \textbf{(1)}$$
$$6x - 5z = 13{,}500 \quad \textbf{(2)}$$
$$x + y + z = 6{,}000 \quad \textbf{(3)}$$

Let us choose to eliminate y first. To eliminate y from Equations (1) and (3), multiply each side of (3) by 55 and subtract.

$$\begin{array}{r} 60x + 55y + 50z = 340{,}000 \\ \underline{55x + 55y + 55z = 330{,}000} \\ 5x - 5z = 10{,}000 \end{array} \quad \textbf{(4)}$$

We have now reduced the system to Equations (2) and (4).

$$6x - 5z = 13{,}500 \quad \textbf{(2)}$$
$$\underline{5x - 5z = 10{,}000} \quad \textbf{(4)}$$
$$x = 3500 \quad \text{(Subtract.)}$$

Substitute $x = 3500$ in (2).

$$\begin{aligned} 6x - 5z &= 13{,}500 \\ 6(3500) - 5z &= 13{,}500 \\ 21{,}000 - 5z &= 13{,}500 \\ -5z &= -7500 \\ z &= 1500 \end{aligned}$$

Substitute $x = 3500$ and $z = 1500$ in (3).

$$\begin{aligned} x + y + z &= 6000 \\ (3500) + y + (1500) &= 6000 \\ y &= 1000 \end{aligned}$$

The solution is (3500, 1000, 1500), which means that

$3500 was deposited at 6%
$1000 was deposited at $5\frac{1}{2}$%
$1500 was deposited at 5%

Check: Substitute the solution in all three equations and in the original verbal problem.

Exercises 6.3

Solve each system of equations and check.

1. $x + y + z = 4$
$x - y + z = 0$
$x - y - z = 2$

2. $x + y + z = 9$
$x - 2y + 3z = 36$
$2x - y + z = 27$

3. $3x - 5y - 6z = -19$
$3y + 6z = 15$
$4x - 2y - 5z = -2$

4. $x + 2z = -8$
$y - 3z = 2$
$x - y + z = 2$

5. $2x + 3y - 5z = 56$
$6x - 4y + 7z = -42$
$x - 2y + 3z = -26$

6. $5x - y + 3z = 28$
$8x + 2y - 4z = 24$
$9x - 4y + 3z = 32$

7. $x + y + z = 16$
$y + z = 3$
$x - z = 11$

8. $9x - 7y + 3z = -48$
$2x + 3y + 4z = -4$
$-3x + 2y - 5z = 8$

9. $2x - 4y + z = 17$
$4x + 5y - z = -8$
$x - 3y + 5z = 16$

10. $x + 3y - 6z = -1$
$2x - y + z = 10$
$5x - 2y + 3z = 27$

11. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3\frac{5}{6}$
$\frac{3}{x} - \frac{5}{y} - \frac{2}{z} = \frac{10}{3}$
$\frac{2}{x} + \frac{3}{y} + \frac{6}{z} = 7$

12. $\frac{3}{x} + \frac{2}{y} - \frac{3}{z} = -\frac{7}{2}$
$\frac{5}{x} + \frac{3}{y} - \frac{4}{z} = -\frac{53}{12}$
$-\frac{2}{y} + \frac{8}{z} = 11\frac{1}{6}$

13. $2x + 3y + 4z - w = 7$
$-x - 2y + 3z + 2w = -3$
$3x + y - 3w = -5$
$4x + 2z - 5w = -19$

14. $5x - 2y - z + 2w = -44$
$3x + 3y - 4z + w = -27$
$-2x + 4y + 5z - 2w = 48$
$4x - 5y + 6z + 4w = -23$

15. The sum of three resistors, R_1, R_2, and R_3, connected in series is 1950 ohms (Ω). The sum of R_1 and R_3 is 1800 Ω. If R_1 is eight times R_2, what is the size of each resistor?

16. The following equations are derived from a circuit diagram. Solve for the currents I_1, I_2, and I_3 in amperes (A).

$$5I_1 + 9I_2 = 37$$
$$15I_2 + 5I_3 = 70$$
$$I_1 + I_2 - I_3 = 0$$

17. The perimeter of a triangle is 65 cm. The longest side is 5 cm longer than the medium side. The medium side is twice the length of the shortest side. Find the length of each side of the triangle.

18. Seventy-five acres of land were purchased for $142,500. The land facing the highway cost $2700 per acre. The land facing the railroad cost $2200 per acre, and the remainder cost $1450 per acre. There were 5 acres more facing the railroad than the highway. How much land was sold at each price?

19. Three cylindrical rods are welded together as shown in Fig. 6.4. Find the diameter of each rod.

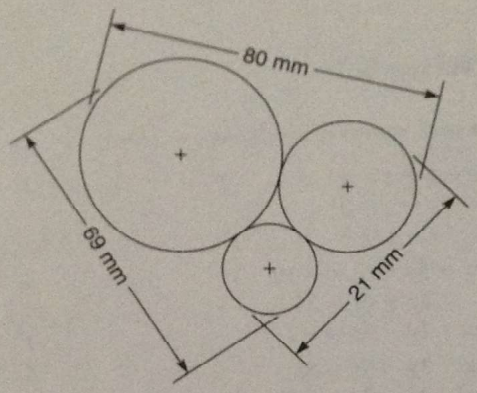


Figure 6.4

20. A chemist has three acid solutions. The first contains 20% acid, the second contains 30% acid, and the third contains 60% acid. She uses all three solutions to obtain a mixture of 80 L containing 40% acid. She uses twice as much of the 60% solution as the 20% solution. How many litres of each solution should she use?

21. If $f(x) = ax^2 + bx + c$, find a, b, and c such that the graph passes through the points $P_1(1, 4)$, $P_2(3, 10)$, and $P_3(-1, 14)$.

6.4 DETERMINANTS

A **determinant** is a square array of numbers such as

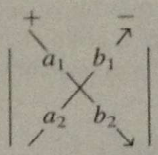
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

where the array has as many rows as columns. The number of rows or columns determines the order of the determinant. The preceding one is a *determinant of second order.* The numbers a_1, b_1, a_2, and b_2 are called **elements.** The elements a_1 and b_1 form the first row; the elements a_2 and b_2 form the second row; the elements a_1 and a_2 form the first column; and the elements b_1 and b_2 form the second column. The value of a second-order determinant is given by the following.

VALUE OF SECOND-ORDER DETERMINANT

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

The elements a_1 and b_2 form what is called the **principal diagonal.** The value of a second-order determinant is the product of the elements of the principal diagonal minus the product of the elements of the other diagonal. This can also be shown using the following diagram.



EXAMPLE 1

Evaluate $\begin{vmatrix} 2 & 4 \\ 6 & 3 \end{vmatrix}$.

$$\begin{vmatrix} 2 & 4 \\ 6 & 3 \end{vmatrix} = (2)(3) - (6)(4) = 6 - 24 = -18$$

EXAMPLE 2

Evaluate $\begin{vmatrix} -3 & 9 \\ -2 & 4 \end{vmatrix}$.

$$\begin{vmatrix} -3 & 9 \\ -2 & 4 \end{vmatrix} = (-3)(4) - (-2)(9) = -12 + 18 = 6$$

EXAMPLE 3

Evaluate $\begin{vmatrix} 0 & -4 \\ -2 & 7 \end{vmatrix}$.

$$\begin{vmatrix} 0 & -4 \\ -2 & 7 \end{vmatrix} = (0)(7) - (-2)(-4) = 0 - 8 = -8$$

A determinant of third order has three rows and three columns. A third-order determinant is in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

The elements, rows, and columns of a determinant of third order are defined in the same way as a determinant of second order. The value of a third-order determinant is defined as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

Let's rearrange the terms on the right side of the equation as

$$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

These terms can then be expressed as the following 2×2 determinants:

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Each 2×2 determinant is called a **minor** of an element in the 3×3 determinant. In general, the minor of a given element of a determinant is the resulting determinant after the row and the column that contain the element have been deleted, as shown in Table 6.1.

TABLE 6.1

Element	Minor	Original Determinant
a_1	$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$
a_2	$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$
a_3	$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

The value of any determinant of any order may be found by finding the sums and differences of *any* row or column of the products of the elements and the corresponding minors. The following diagram should help you remember how to determine the signs (sum or difference) of the various products of elements and minors:

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix}$$

Note: Begin with a + sign in the upper-left position. Then alternate the signs across each row and down each column.

EXAMPLE 4

Evaluate the determinant

$$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix}$$

using expansion by minors (a) down the first column and (b) across the second row.

(a) The elements and the corresponding minors down the first column are determined as shown in Table 6.2.

TABLE 6.2

Element	Minor	Original Determinant
1	$\begin{vmatrix} -7 & -2 \\ 8 & 2 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix}$
6	$\begin{vmatrix} 3 & -5 \\ 8 & 2 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix}$
-4	$\begin{vmatrix} 3 & -5 \\ -7 & -2 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix}$

The signs down the first column are $+ - +$. Therefore,

$$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix} = +1\begin{vmatrix} -7 & -2 \\ 8 & 2 \end{vmatrix} - 6\begin{vmatrix} 3 & -5 \\ 8 & 2 \end{vmatrix} + (-4)\begin{vmatrix} 3 & -5 \\ -7 & -2 \end{vmatrix}$$

$$= 1[(-7)(2) - (8)(-2)] - 6[(3)(2) - (8)(-5)] + (-4)[(3)(-2) - (-7)(-5)]$$

$$= 1[-14 + 16] - 6[6 + 40] + (-4)[-6 - 35]$$

$$= 2 - 276 + 164$$

$$= -110$$

(b) The elements and the corresponding minors across the second row are determined as shown in Table 6.3.

TABLE 6.3

Element	Minor	Original Determinant
6	$\begin{vmatrix} 3 & -5 \\ 8 & 2 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix}$
-7	$\begin{vmatrix} 1 & -5 \\ -4 & 2 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix}$
-2	$\begin{vmatrix} 1 & 3 \\ -4 & 8 \end{vmatrix}$	$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix}$

The signs across the second row are $- + -$. Therefore,

$$\begin{vmatrix} 1 & 3 & -5 \\ 6 & -7 & -2 \\ -4 & 8 & 2 \end{vmatrix} = -6\begin{vmatrix} 3 & -5 \\ 8 & 2 \end{vmatrix} + (-7)\begin{vmatrix} 1 & -5 \\ -4 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 3 \\ -4 & 8 \end{vmatrix}$$

$$= -6[(3)(2) - (8)(-5)] + (-7)[(1)(2) - (-4)(-5)] - (-2)[(1)(8) - (-4)(3)]$$

$$= -6[6 + 40] - 7[2 - 20] + 2[8 + 12]$$

$$= -276 + 126 + 40$$

$$= -110$$

EXAMPLE 5

Evaluate the determinant

$$\begin{vmatrix} 3 & 0 & 5 & 4 \\ 2 & 1 & 3 & 4 \\ 0 & -2 & -1 & -3 \\ 1 & 0 & 2 & 1 \end{vmatrix}$$

Since the value of a determinant may be found by expansion by minors across any row or down any column, let's expand down the second column; it has the most zero elements and, therefore, involves the least work.

The signs down the second column are $- + - +$. Therefore,

$$\begin{vmatrix} 3 & 0 & 5 & 4 \\ 2 & 1 & 3 & 4 \\ 0 & -2 & -1 & -3 \\ 1 & 0 & 2 & 1 \end{vmatrix} = -0\begin{vmatrix} 2 & 3 & 4 \\ 0 & -1 & -3 \\ 1 & 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 3 & 5 & 4 \\ 0 & -1 & -3 \\ 1 & 2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 5 & 4 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

$$+ 0\begin{vmatrix} 3 & 5 & 4 \\ 2 & 3 & 4 \\ 0 & -1 & -3 \end{vmatrix}$$

$$= 0 + 1(4) + 2(-1) + 0$$

$$= 2$$

Note: The second and third 3×3 determinants may be evaluated by expansion of three 2×2 minors.

The method for evaluating determinants by expansion by minors works for *any* order determinant. In Appendix C, an optional method for evaluating *only* third-order determinants is shown.

Exercises 6.4

Evaluate each determinant.

1. $\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}$ **2.** $\begin{vmatrix} 6 & -1 \\ 4 & 5 \end{vmatrix}$ **3.** $\begin{vmatrix} 2 & -3 \\ -4 & 5 \end{vmatrix}$

4. $\begin{vmatrix} 5 & 1 \\ 3 & -4 \end{vmatrix}$

5. $\begin{vmatrix} 3 & -7 \\ 6 & -1 \end{vmatrix}$

6. $\begin{vmatrix} -5 & -2 \\ 7 & 3 \end{vmatrix}$

7. $\begin{vmatrix} -5 & 8 \\ 9 & -6 \end{vmatrix}$

8. $\begin{vmatrix} 4 & 12 \\ 5 & 8 \end{vmatrix}$

9. $\begin{vmatrix} 4 & 0 \\ 6 & -3 \end{vmatrix}$

10. $\begin{vmatrix} 2 & -5 \\ -7 & 0 \end{vmatrix}$

11. $\begin{vmatrix} -4 & -2 \\ -6 & -7 \end{vmatrix}$

12. $\begin{vmatrix} -1 & 5 \\ -4 & 2 \end{vmatrix}$

13. $\begin{vmatrix} -7 & -2 \\ 4 & 1 \end{vmatrix}$

14. $\begin{vmatrix} 5 & -7 \\ -2 & -6 \end{vmatrix}$

15. $\begin{vmatrix} 8 & -9 \\ 7 & 4 \end{vmatrix}$

16. $\begin{vmatrix} 6 & -9 \\ -8 & 7 \end{vmatrix}$

17. $\begin{vmatrix} m & -n \\ n^2 & n \end{vmatrix}$

18. $\begin{vmatrix} -mn & -n \\ m & -mn \end{vmatrix}$

19. $\begin{vmatrix} 1 & 1 & -4 \\ -3 & 7 & 11 \\ 2 & 1 & -5 \end{vmatrix}$

20. $\begin{vmatrix} 2 & 0 & -6 \\ 4 & -1 & -7 \\ -3 & 3 & 2 \end{vmatrix}$

21. $\begin{vmatrix} -1 & 3 & 8 \\ 0 & 0 & -6 \\ -5 & 2 & -3 \end{vmatrix}$

22. $\begin{vmatrix} 2 & -5 & 7 \\ 0 & -8 & 1 \\ 3 & -2 & -1 \end{vmatrix}$

23. $\begin{vmatrix} 1 & 5 & -3 \\ 2 & -2 & 2 \\ -5 & -6 & 1 \end{vmatrix}$

24. $\begin{vmatrix} 2 & 3 & -6 \\ -5 & 1 & -2 \\ 0 & 3 & 0 \end{vmatrix}$

25. $\begin{vmatrix} 1 & -3 & 4 \\ 4 & 6 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

26. $\begin{vmatrix} -2 & -5 & -2 \\ 1 & -7 & 1 \\ 3 & 2 & 3 \end{vmatrix}$

27. $\begin{vmatrix} 3 & 6 & -9 \\ 1 & -5 & 2 \\ 0 & 0 & 0 \end{vmatrix}$

28. $\begin{vmatrix} -6 & 0 & 5 \\ 3 & 0 & -2 \\ 7 & 0 & 4 \end{vmatrix}$

29. $\begin{vmatrix} 1 & 3 & -7 & 2 \\ -5 & 3 & 0 & -2 \\ 1 & -2 & 5 & 1 \\ 3 & 1 & 0 & 2 \end{vmatrix}$

30. $\begin{vmatrix} 2 & -1 & 7 & -5 \\ -3 & -2 & 7 & 4 \\ -1 & 0 & 3 & 2 \\ -2 & 0 & 5 & 1 \end{vmatrix}$

31. $\begin{vmatrix} 3 & -1 & 6 & 2 \\ -5 & 3 & -8 & 7 \\ 1 & 0 & -5 & 0 \\ 2 & -6 & 3 & 1 \end{vmatrix}$

32. $\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix}$

33. $\begin{vmatrix} 1 & 3 & 6 & -2 \\ 0 & 2 & -5 & 7 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{vmatrix}$

34. $\begin{vmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

35. $\begin{vmatrix} 3 & -2 & 0 & 2 & -1 \\ 1 & 2 & -3 & 1 & 2 \\ 0 & -1 & 0 & 6 & 1 \\ -3 & 2 & 0 & 6 & -7 \\ 0 & 0 & 0 & 5 & 2 \end{vmatrix}$

36. $\begin{vmatrix} -1 & 1 & 5 & 6 & 2 \\ 3 & 2 & 0 & 0 & -1 \\ -2 & 0 & -3 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 6 & -1 & 0 & 1 & 2 \end{vmatrix}$

6.5 PROPERTIES OF DETERMINANTS

Evaluation of determinants using expansion by minors allows us to evaluate a determinant of any order. As you saw in Section 6.4, the amount of work is considerable. However, a few basic properties of determinants significantly lessen the effort and time needed to evaluate a determinant.

PROPERTY 1

If every element in a row (or a column) of a determinant is zero, the value of the determinant is zero.

EXAMPLE 1

(a) $\begin{vmatrix} 3 & -2 & 6 \\ 0 & 0 & 0 \\ 1 & 5 & -8 \end{vmatrix} = 0$

(b) $\begin{vmatrix} 5 & 3 & 0 \\ -7 & 2 & 0 \\ 8 & -6 & 0 \end{vmatrix} = 0$

PROPERTY 2

If two rows (or two columns) of a determinant are identical, the value of the determinant is zero.

EXAMPLE 2

(a) $\begin{vmatrix} 1 & -3 & 5 \\ -2 & -6 & 3 \\ 1 & -3 & 5 \end{vmatrix} = 0$

(b) $\begin{vmatrix} 6 & 2 & 2 \\ 0 & 5 & 5 \\ 2 & -3 & -3 \end{vmatrix} = 0$

PROPERTY 3

If any two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant is changed.

EXAMPLE 3

(a) $\begin{vmatrix} 3 & -2 & 7 \\ 5 & 8 & 4 \\ 0 & 6 & -9 \end{vmatrix} = -\begin{vmatrix} 5 & 8 & 4 \\ 3 & -2 & 7 \\ 0 & 6 & -9 \end{vmatrix}$

(b) $\begin{vmatrix} 0 & 3 & -5 \\ -6 & 2 & 3 \\ 9 & 7 & 1 \end{vmatrix} = -\begin{vmatrix} -5 & 3 & 0 \\ 3 & 2 & -6 \\ 1 & 7 & 9 \end{vmatrix}$

PROPERTY 4

If every element of a row (or a column) is multiplied by the same real number k, the value of the determinant is multiplied by k.

EXAMPLE 4

(a) $\begin{vmatrix} 3 & 5 & -6 \\ 4 & -8 & 12 \\ 3 & 0 & 7 \end{vmatrix} = 4\begin{vmatrix} 3 & 5 & -6 \\ 1 & -2 & 3 \\ 3 & 0 & 7 \end{vmatrix}$

(b) $3\begin{vmatrix} 1 & -4 & 6 \\ 3 & 0 & 8 \\ 2 & 4 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 18 \\ 3 & 0 & 24 \\ 2 & 4 & -9 \end{vmatrix}$

The next property is probably the most useful. As you saw in the previous section, it was nice to have zero elements. This last property allows us to make more zeros before we expand by minors.

PROPERTY 5

If every element of a row (or a column) is multiplied by the same real number k, and if the resulting products are added to another row (or another column), the value of the determinant remains the same.

EXAMPLE 5

Evaluate

$$\begin{vmatrix} 3 & -2 & 6 \\ -6 & 0 & 4 \\ 5 & 1 & -1 \end{vmatrix}$$

Let's multiply the third row by 2 and add the resulting products to the first row to make another zero in the second column.

$$\begin{vmatrix} (2)(5)+3 & (2)(1)+(-2) & (2)(-1)+6 \\ -6 & 0 & 4 \\ 5 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 13 & 0 & 4 \\ -6 & 0 & 4 \\ 5 & 1 & -1 \end{vmatrix}$$

Expanding down the second column, we find that the first two minors result in zero. Thus,

$$\begin{vmatrix} 13 & 0 & 4 \\ -6 & 0 & 4 \\ 5 & 1 & -1 \end{vmatrix} = -1\begin{vmatrix} 13 & 4 \\ -6 & 4 \end{vmatrix} = -1[52-(-24)] = -76$$

Note that if we evaluate this determinant by expansion by minors, we obtain the same result:

$$\begin{vmatrix} 3 & -2 & 6 \\ -6 & 0 & 4 \\ 5 & 1 & -1 \end{vmatrix}$$

Let's expand by minors down the second column as follows:

$$\begin{aligned} &-(-2)\begin{vmatrix} -6 & 4 \\ 5 & -1 \end{vmatrix} + 0\begin{vmatrix} 3 & 6 \\ 5 & -1 \end{vmatrix} - 1\begin{vmatrix} 3 & 6 \\ -6 & 4 \end{vmatrix} \\ &= 2(6-20) \quad + 0(-3-30) - 1(12+36) \\ &= -28 \quad\quad\quad + 0 \quad\quad\quad - 48 \\ &= -76 \end{aligned}$$

EXAMPLE 6

Evaluate

$$\begin{vmatrix} -7 & -6 & 5 & 1 \\ 5 & 3 & 0 & 7 \\ 0 & -1 & 4 & -2 \\ -8 & -3 & 8 & -10 \end{vmatrix}$$

Let's make two more zeros in the third row. First multiply the second column by 4 and add the resulting products to the third column. Then multiply the second column by -2 and add the resulting products to the fourth column.

$$\begin{vmatrix} -7 & -6 & (4)(-6)+5 & (-2)(-6)+1 \\ 5 & 3 & (4)(3)+0 & (-2)(3)+7 \\ 0 & -1 & (4)(-1)+4 & (-2)(-1)+(-2) \\ -8 & -3 & (4)(-3)+8 & (-2)(-3)+(-10) \end{vmatrix} = \begin{vmatrix} -7 & -6 & -19 & 13 \\ 5 & 3 & 12 & 1 \\ 0 & -1 & 0 & 0 \\ -8 & -3 & -4 & -4 \end{vmatrix}$$

Expanding across the third row, we find that the first, third, and fourth minors result in zero. Therefore,

$$\begin{vmatrix} -7 & -6 & -19 & 13 \\ 5 & 3 & 12 & 1 \\ 0 & -1 & 0 & 0 \\ -8 & -3 & -4 & -4 \end{vmatrix} = -(-1)\begin{vmatrix} -7 & -19 & 13 \\ 5 & 12 & 1 \\ -8 & -4 & -4 \end{vmatrix} = (+1)(-4)\begin{vmatrix} -7 & -19 & 13 \\ 5 & 12 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

Let's make two zeros in the third row. First multiply the third column by -2 and add the resulting products to the first column. Then multiply the third column by -1 and add the resulting products to the second column.

$$(-4)\begin{vmatrix} (-2)(13)+(-7) & (-1)(13)+(-19) & 13 \\ (-2)(1)+5 & (-1)(1)+12 & 1 \\ (-2)(1)+2 & (-1)(1)+1 & 1 \end{vmatrix} = (-4)\begin{vmatrix} -33 & -32 & 13 \\ 3 & 11 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding across the third row, we find that the first two minors result in zero. Thus,

$$(-4)\begin{vmatrix} -33 & -32 & 13 \\ 3 & 11 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-4)(+1)\begin{vmatrix} -33 & -32 \\ 3 & 11 \end{vmatrix} = (-4)[-363+96] = 1068$$

Note: You could have chosen to make zeros in any row or column. In any case, the result would be 1068.

Exercises 6.5

Evaluate each determinant using Properties 1–5.

1. $\begin{vmatrix} 4 & 3 & -8 \\ 2 & -7 & 9 \\ 0 & 0 & 0 \end{vmatrix}$

2. $\begin{vmatrix} -3 & 6 & -3 \\ 1 & 5 & 1 \\ 7 & 0 & 7 \end{vmatrix}$

3. $\begin{vmatrix} 3 & 6 & 0 & -8 \\ 2 & 7 & 5 & 4 \\ 0 & -5 & 8 & 1 \\ 2 & 7 & 5 & 4 \end{vmatrix}$

4. $\begin{vmatrix} 6 & 5 & 0 & -9 \\ -3 & -4 & 0 & 2 \\ 2 & 3 & 0 & -7 \\ 7 & -1 & 0 & 12 \end{vmatrix}$

5. $\begin{vmatrix} 4 & 6 & -6 \\ 0 & 0 & -9 \\ 1 & -2 & 7 \end{vmatrix}$

6. $\begin{vmatrix} 2 & 6 & -5 \\ -3 & -5 & 0 \\ 7 & 2 & 0 \end{vmatrix}$

7. $\begin{vmatrix} 3 & 0 & 8 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & -7 & 5 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

8. $\begin{vmatrix} 0 & 0 & 5 & 1 \\ 1 & 0 & 3 & 0 \\ 2 & 6 & 8 & 0 \\ 3 & 0 & 0 & 0 \end{vmatrix}$

9. $\begin{vmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix}$

10. $\begin{vmatrix} 0 & 0 & 0 & -5 \\ 0 & 0 & -7 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{vmatrix}$

11. $\begin{vmatrix} 1 & 0 & 4 \\ 6 & 3 & 2 \\ 5 & 4 & 1 \end{vmatrix}$

12. $\begin{vmatrix} 2 & 7 & 4 \\ 3 & 0 & 1 \\ 6 & 3 & 2 \end{vmatrix}$

13. $\begin{vmatrix} 3 & 1 & -6 \\ -4 & 0 & 7 \\ 5 & 6 & 2 \end{vmatrix}$

14. $\begin{vmatrix} -3 & 2 & 5 \\ 4 & 7 & 0 \\ 6 & -3 & -1 \end{vmatrix}$

15. $\begin{vmatrix} 5 & 3 & 7 \\ 1 & 2 & 3 \\ 6 & 2 & 5 \end{vmatrix}$

16. $\begin{vmatrix} 2 & -4 & 6 \\ 4 & -2 & 1 \\ -8 & 2 & 1 \end{vmatrix}$

17. $\begin{vmatrix} 3 & -7 & 5 \\ 2 & 4 & 6 \\ -9 & 7 & -2 \end{vmatrix}$

18. $\begin{vmatrix} 4 & 3 & -9 \\ 5 & -6 & -4 \\ -7 & 9 & 2 \end{vmatrix}$

19. $\begin{vmatrix} 1 & 0 & 5 & 0 \\ 6 & 2 & -3 & 7 \\ -1 & 2 & 3 & -4 \\ -3 & 2 & -7 & 1 \end{vmatrix}$

20. $\begin{vmatrix} -5 & -6 & 3 & 7 \\ 3 & -1 & 0 & 0 \\ 2 & 1 & 0 & 5 \\ 4 & -4 & -7 & 0 \end{vmatrix}$

21. $\begin{vmatrix} 1 & 3 & -7 & 2 \\ 5 & 6 & 1 & 5 \\ 4 & -8 & 9 & -2 \\ -6 & 7 & -4 & 3 \end{vmatrix}$

22. $\begin{vmatrix} 2 & 5 & 2 & -3 \\ 3 & 4 & -4 & 4 \\ 2 & -8 & 1 & 9 \\ -2 & 6 & 3 & -7 \end{vmatrix}$

23. $\begin{vmatrix} 1 & 0 & 3 & 2 & 5 \\ 3 & 2 & 3 & -2 & 3 \\ -4 & -2 & 4 & 9 & -6 \\ -3 & 7 & 6 & 2 & 3 \\ 0 & 2 & -2 & -4 & 5 \end{vmatrix}$

24. $\begin{vmatrix} 2 & 2 & 3 & 5 & -7 \\ 3 & -6 & 9 & 12 & 15 \\ 4 & -5 & 3 & 7 & 2 \\ 2 & 0 & -4 & 2 & 7 \\ 3 & 3 & 4 & -3 & 3 \end{vmatrix}$

6.6 SOLVING A SYSTEM OF LINEAR EQUATIONS USING DETERMINANTS

Suppose we were to solve the general system of two linear equations in two variables:

$$a_1x + b_1y = c_1 \quad \textbf{(1)}$$

$$a_2x + b_2y = c_2 \quad \textbf{(2)}$$

By the addition-subtraction method we would proceed as follows:

$$a_1b_2x + b_1b_2y = b_2c_1 \quad \text{[Multiply (1) by } b_2\text{.]}$$

$$\underline{a_2b_1x + b_1b_2y = b_1c_2} \quad \text{[Multiply (2) by } b_1\text{.]}$$

$$a_1b_2x - a_2b_1x = b_2c_1 - b_1c_2 \quad \text{(Subtract.)}$$

$$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2 \quad \text{(Factor.)}$$

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$$

By similarly eliminating x, we can show that

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

Note the following:

1. $a_1b_2 - a_2b_1$ is the value of and may be represented by the determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

2. $b_2c_1 - b_1c_2$ is the value of

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

3. $a_1c_2 - a_2c_1$ is the value of

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Therefore, the solution of

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

may be written in determinant form as follows.

DETERMINANT SOLUTION OF TWO LINEAR EQUATIONS

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

If the determinant of the numerator is not zero and the determinant of the denominator is zero, the system is *inconsistent.* If the determinants of numerator and denominator are both zero, the system is *dependent.* If the determinant of the denominator is not zero, there is a unique solution and the system is *independent and consistent.*

Note that the determinant of each denominator is made up of the coefficients of x and y. To find the determinant of the numerator, use these steps:

1. For x, take the determinant of the denominator and replace the coefficients of x, a's, by the corresponding constants, c's.
2. For y, take the determinant of the denominator and replace the coefficients of y, b's, by the corresponding constants, c's.

This method of solution is called **Cramer's rule** for solving a system of two linear equations in two variables. To use this method, the equations must be in the general form shown in the preceding Equations (1) and (2).

EXAMPLE 1

Solve using determinants.

$$2x - 3y = 22$$
$$5x + 4y = -14$$

$$x = \frac{\begin{vmatrix} 22 & -3 \\ -14 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix}} = \frac{(22)(4) - (-14)(-3)}{(2)(4) - 5(-3)} = \frac{46}{23} = 2$$

$$y = \frac{\begin{vmatrix} 2 & 22 \\ 5 & -14 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix}} = \frac{(2)(-14) - (5)(22)}{23} = \frac{-138}{23} = -6$$

The solution is $(2, -6)$.

Check: Substitute the solution in both original equations.

EXAMPLE 2

Solve using determinants.

$$3x + 4y = 10$$
$$6x + 8y = -5$$

$$x = \frac{\begin{vmatrix} 10 & 4 \\ -5 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}} = \frac{(10)(8) - (-5)(4)}{(3)(8) - (6)(4)} = \frac{100}{0}$$

Since the numerator of the determinant is not zero and the denominator is zero, the system is inconsistent (the lines are parallel). There are no solutions.

EXAMPLE 3

Solve using determinants.

$$mx + ny = mn$$
$$nx - my = n^2$$

$$x = \frac{\begin{vmatrix} mn & n \\ n^2 & -m \end{vmatrix}}{\begin{vmatrix} m & n \\ n & -m \end{vmatrix}} = \frac{(mn)(-m) - (n^2)(n)}{m(-m) - (n)(n)}$$

$$= \frac{-m^2n - n^3}{-m^2 - n^2} = \frac{-n(m^2 + n^2)}{-(m^2 + n^2)} = n$$

$$y = \frac{\begin{vmatrix} m & mn \\ n & n^2 \end{vmatrix}}{\begin{vmatrix} m & n \\ n & -m \end{vmatrix}} = \frac{(m)(n^2) - (n)(mn)}{-m^2 - n^2}$$

$$= \frac{mn^2 - mn^2}{-m^2 - n^2} = \frac{0}{-m^2 - n^2} = 0$$

The solution is $(n, 0)$.

The general system of three linear equations in three variables is given as

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

If we used the addition-subtraction method, we would find the following solutions for x, y, and z:

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

$$y = \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

$$z = \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}$$

This general solution may be written in terms of determinants as follows.

DETERMINANT SOLUTION OF THREE LINEAR EQUATIONS

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

If the determinant of the numerator is not zero and the determinant of the denominator is zero, the system is *inconsistent.* If the determinants of numerator and denominator are both zero, the system may be *dependent* or *inconsistent.* If the determinant of the denominator is not zero, there is a unique solution, and the system is *independent and consistent.*

Note that the determinant of each denominator is made up of the coefficients of x, y, and z. To find the determinant of the numerator, use these steps:

1. For x, take the determinant of the denominator and replace the coefficients for x, a's, by the corresponding constants, d's.
2. For y, take the determinant of the denominator and replace the coefficients of y, b's, by the corresponding constants, d's.
3. For z, take the determinant of the denominator and replace the coefficients of z, c's, by the corresponding constants, d's.

This method is called **Cramer's rule** for solving a system of three linear equations in three variables.

EXAMPLE 4

Solve using determinants.

$$\begin{aligned} 3x - y + 4z &= -15 \\ 2x + 5y &= 29 \\ x - 6y - z &= -24 \end{aligned}$$

Using Cramer's rule, we have

$$x = \frac{\begin{vmatrix} -15 & -1 & 4 \\ 29 & 5 & 0 \\ -24 & -6 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 4 \\ 2 & 5 & 0 \\ 1 & -6 & -1 \end{vmatrix}} = \frac{\begin{vmatrix} -111 & -25 & 0 \\ 29 & 5 & 0 \\ -24 & -6 & -1 \end{vmatrix}}{\begin{vmatrix} 7 & -25 & 0 \\ 2 & 5 & 0 \\ 1 & -6 & -1 \end{vmatrix}}$$

(In both numerator and denominator, multiply the third row by 4, and add the result to the first row.)

$$= \frac{(-1)\begin{vmatrix} -111 & -25 \\ 29 & 5 \end{vmatrix}}{(-1)\begin{vmatrix} 7 & -25 \\ 2 & 5 \end{vmatrix}}$$

(Expand down the third column.)

$$= \frac{(-1)(-555 + 725)}{(-1)(35 + 50)}$$

$$= \frac{-170}{-85}$$

$$= 2$$

Next, find y as follows:

$$y = \frac{\begin{vmatrix} 3 & -15 & 4 \\ 2 & 29 & 0 \\ 1 & -24 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 4 \\ 2 & 5 & 0 \\ 1 & -6 & -1 \end{vmatrix}} = \frac{\begin{vmatrix} 7 & -111 & 0 \\ 2 & 29 & 0 \\ 1 & -24 & -1 \end{vmatrix}}{-85}$$

(Multiply the third row by 4 and add the result to the first row.)

$$= \frac{(-1)\begin{vmatrix} 7 & -111 \\ 2 & 29 \end{vmatrix}}{-85}$$

(Expand down the third column.)

$$= \frac{(-1)(203 + 222)}{-85}$$

$$= \frac{-425}{-85} = 5$$

Then find z as follows:

$$z = \frac{\begin{vmatrix} 3 & -1 & -15 \\ 2 & 5 & 29 \\ 1 & -6 & -24 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 4 \\ 2 & 5 & 0 \\ 1 & -6 & -1 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 17 & 57 \\ 0 & 17 & 77 \\ 1 & -6 & -24 \end{vmatrix}}{-85}$$

(Multiply the third row by -3 and add the result to the first row. Then multiply the third row by -2 and add the result to the second row.)

$$= \frac{(1)\begin{vmatrix} 17 & 57 \\ 17 & 77 \end{vmatrix}}{-85}$$

(Expand down the first column.)

$$= \frac{(1)(17)\begin{vmatrix} 1 & 57 \\ 1 & 77 \end{vmatrix}}{-85}$$

(Property 4)

$$= \frac{17(77 - 57)}{-85} = \frac{340}{-85} = -4$$

The solution is $(2, 5, -4)$.

Check: Substitute the solution in all three original equations.

You may also find the value of the third variable by substituting the first two values into any equation and then solving for the third variable. For example, substitute $x = 2$ and $y = 5$ into the third equation and solve for z as follows:

$$\begin{aligned} x - 6y - z &= -24 \\ 2 - 6(5) - z &= -24 \\ 2 - 30 - z &= -24 \\ -z &= 4 \\ z &= -4 \end{aligned}$$

Exercises 6.6

Solve each system of equations using determinants.

1. $3x + 5y = -1$
$2x - 3y = 12$

2. $6x - 2y = 36$
$5x + 4y = 47$

3. $8x - 3y = -43$
$5x - 7y = -73$

4. $4x - 2y = 6$
$-3x + 4y = -17$

5. $6x - 7y = 28$
$-4x + 5y = -20$

6. $3x - 8y = 9$
$-9x - 6y = -27$

7. $3x + 4y = 5$
$6x + 8y = 10$

8. $-x + 4y = 7$
$3x - 12y = -21$

9. $12x - 16y = 24$
$15x - 20y = 36$

10. $-2x + 3y = 6$
$8x - 12y = 24$

11. $15x - 6y = -15$
$9x + 12y = 4$

12. $3x - 5y = 3$
$-5x + 7y = -2$

13. $8x + 7y = 18$
$y = 4x$

14. $20x - 5y = -13$
$x = -3y$

15. $ax + by = 2$
$bx + ay = 4$

16. $ax - 3y = b$
$bx - 2y = a$

17. $5x + ay = b$
$2x - by = a$

18. $4x - by = b$
$6x - ay = a$

19. $ax + by = c$
$y = bx$

20. $ax - by = a$
$x = ay$

21. The sum of two voltages is 210 volts (V). The larger voltage is 15 V less than twice the smaller voltage. Find the voltages.

22. A man has available two different mixtures of solder. One mixture is 25% tin and the other is 65% tin. How much of each must he use to make a 56-kg mixture of 40% tin?

Solve each system of equations using determinants.

23.
$$\begin{aligned} 3x - 4y + 7z &= -26 \\ -2x + y - 3z &= 9 \\ 12x + 15z &= -36 \end{aligned}$$

24.
$$\begin{aligned} 5x - 2y - 3z &= -3 \\ 4y + 3z &= -2 \\ x - y + 9z &= 60 \end{aligned}$$

25.
$$\begin{aligned} 3x + 2y + 5z &= -7 \\ 8x - 3y + 2z &= 10 \\ 7x - 2y + 4z &= 1 \end{aligned}$$

26.
$$\begin{aligned} 3x - 7y - 2z &= -38 \\ 6x + 5y - z &= 63 \\ -2x - 4y + 5z &= -28 \end{aligned}$$

27.
$$\begin{aligned} 7x - 5y - 7z &= 8 \\ 9x + 3y - 6z &= 33 \\ 4x - 2y - 8z &= 28 \end{aligned}$$

28.
$$\begin{aligned} x - 6y - 4z &= 0 \\ 2x - 3y + 5z &= 46 \\ 9x + 7y + 8z &= 3 \end{aligned}$$

29.
$$\begin{aligned} 3x - 4y - 5z &= 0 \\ 9x + 6y + 10z &= 11 \\ 12x + 2y - 20z &= 36 \end{aligned}$$

30.
$$\begin{aligned} 4x - 6y - 8z &= 18 \\ 12x + 15y + 16z &= -13 \\ 20x - 12y - 24z &= 60 \end{aligned}$$

31.
$$\begin{aligned} 2x + 5z &= 29 \\ 5y - 7z &= -40 \\ 8x + y &= 15 \end{aligned}$$

32.
$$\begin{aligned} -4x + 7y &= 37 \\ 6x - 2z &= 24 \\ 3y - 5z &= 36 \end{aligned}$$

33.
$$\begin{aligned} 4x + 6y + 8z &= -8 \\ -x + 5z &= -19 \\ 5y + 7z &= -21 \end{aligned}$$

34.
$$\begin{aligned} 3x - 5y &= 20 \\ 9x + 8y - 2z &= -25 \\ 5y - 8z &= 8 \end{aligned}$$

35. The perimeter of a triangle is 21 cm. The longest side is 2 cm longer than the shortest side. These two sides together are twice the size of the remaining side. Find the lengths of the sides of the triangle.

36. As a result of Kirchhoff's laws, the following system of equations was obtained:

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ 2.2I_1 + 0.5I_2 &= 12.6 \\ 3.4I_1 - 3.8I_2 &= -10.25 \end{aligned}$$

Determine the indicated currents in amperes (A).

Cramer's rule may be extended to solve a system of n linear equations in n unknowns. Use Cramer's rule to solve each system of equations.

37.
$$\begin{aligned} 3x - 2y + z - 3w &= -20 \\ 2x + 5z + w &= -3 \\ 5x + y - z + 4w &= 30 \\ 6x - 3y + 4z &= -11 \end{aligned}$$

38.
$$\begin{aligned} 8x - 2y + z - w &= 38 \\ 4y - 7z + w &= -28 \\ 3x - 5z &= -3 \\ 3x - 2y + 7w &= 23 \end{aligned}$$

39.
$$\begin{aligned} 2x + 2y - z + 3w - 4v &= 13 \\ 3x + 7y - z - 3v &= -8 \\ 3y + 2z + w - v &= 5 \\ -2x + 3z - w + v &= 0 \\ 5x + 7w &= 38 \end{aligned}$$

40.
$$\begin{aligned} 3x - 4y + 2z + 2w + v &= -2 \\ x + 3z + 2v &= -1 \\ 3y + 4z - 8w + 3v &= -13 \\ 7z + 2w - v &= 5 \\ 5x + z + 3w &= 11 \end{aligned}$$

6.7 PARTIAL FRACTIONS

We add two fractions such as

$$\frac{5}{x+1}+\frac{6}{x-2}=\frac{5(x-2)}{(x+1)(x-2)}+\frac{6(x+1)}{(x+1)(x-2)}$$
$$=\frac{5x-10+6x+6}{(x+1)(x-2)}$$
$$=\frac{11x-4}{(x+1)(x-2)}$$

At times, we need to express a fraction as the sum of two or more fractions that are each simpler than the original; that is, we reverse the operation. Such simpler fractions whose numerators are of lower degree than their denominators are called **partial fractions.**

We separate our study of partial fractions into four cases. In each case we assume that the given fraction is expressed in lowest terms and the degree of each numerator is less than the degree of its denominator.

CASE 1: NONREPEATED LINEAR DENOMINATOR FACTORS

For every nonrepeated factor $ax + b$ of the denominator of a given fraction, there corresponds the partial fraction $\frac{A}{ax+b}$, where A is a constant.

EXAMPLE 1

Find the partial fractions of $\frac{11x-4}{(x+1)(x-2)}$.

The possible partial fractions are $\frac{A}{x+1}$ and $\frac{B}{x-2}$. So we have

$$\frac{11x-4}{(x+1)(x-2)}=\frac{A}{x+1}+\frac{B}{x-2}$$

Then multiply each side of this equation by the L.C.D.: $(x + 1)(x - 2)$.

$$11x-4=A(x-2)+B(x+1)$$

Removing parentheses and rearranging terms, we have

$$11x-4=Ax-2A+Bx+B$$
$$11x-4=Ax+Bx-2A+B$$
$$11x-4=(A+B)x-2A+B$$

Next, the coefficients of x must be equal and the constant terms must be equal. This gives the following system of linear equations:

$$A+B=11$$
$$-2A+B=-4$$

Subtracting the two equations gives

$$3A=15$$
$$A=5$$

Substituting $A = 5$ into either of the preceding equations gives

$$B = 6$$

Then

$$\frac{11x - 4}{(x + 1)(x - 2)} = \frac{5}{x + 1} + \frac{6}{x - 2}$$

EXAMPLE 2

Find the partial fractions of $\dfrac{3x^2 - 27x - 12}{x(2x + 1)(x - 4)}$.

The possible partial fractions are

$$\frac{A}{x} \qquad \frac{B}{2x + 1} \quad \text{and} \quad \frac{C}{x - 4}$$

So we have

$$\frac{3x^2 - 27x - 12}{x(2x + 1)(x - 4)} = \frac{A}{x} + \frac{B}{2x + 1} + \frac{C}{x - 4}$$

Now multiply each side of this equation by the L.C.D.: $x(2x + 1)(x - 4)$.

$$3x^2 - 27x - 12 = A(2x + 1)(x - 4) + Bx(x - 4) + Cx(2x + 1)$$

Removing parentheses and rearranging terms, we have

$$3x^2 - 27x - 12 = 2Ax^2 - 7Ax - 4A + Bx^2 - 4Bx + 2Cx^2 + Cx$$
$$3x^2 - 27x - 12 = (2A + B + 2C)x^2 + (-7A - 4B + C)x - 4A$$

Then the coefficients of x^2 must be equal, the coefficients of x must be equal, and the constant terms must be equal. This gives the following system of linear equations:

$$\begin{aligned} 2A + \ B + 2C &= 3 \\ -7A - 4B + \ C &= -27 \\ -4A &= -12 \end{aligned}$$

Note that $A = 3$ from the third equation. Substituting $A = 3$ into the first two equations gives

$$\begin{aligned} 6 + \ B + 2C &= 3 \\ -21 - 4B + \ C &= -27 \end{aligned}$$

or

$$\begin{aligned} B + 2C &= -3 \\ -4B + \ C &= -6 \end{aligned}$$

Multiplying the second equation by 2 gives

$$\begin{aligned} B + 2C &= -3 \\ -8B + 2C &= -12 \end{aligned}$$

Subtracting these two equations gives

$$\begin{aligned} 9B &= 9 \\ B &= 1 \end{aligned}$$

Then $C = -2$ and

$$\frac{3x^2 - 27x - 12}{x(2x + 1)(x - 4)} = \frac{3}{x} + \frac{1}{2x + 1} - \frac{2}{x - 4}$$

CASE 2: REPEATED LINEAR DENOMINATOR FACTORS

For every factor $(ax + b)^k$ of the denominator of a given fraction, there correspond the possible partial fractions

$$\frac{A_1}{ax + b}, \frac{A_2}{(ax + b)^2}, \frac{A_3}{(ax + b)^3}, \ldots, \frac{A_k}{(ax + b)^k}$$

where $A_1, A_2, A_3, \ldots, A_k$ are constants.

EXAMPLE 3

Find the partial fractions of $\dfrac{-x^2 - 8x + 27}{x(x - 3)^2}$.

The possible partial fractions are

$$\frac{A}{x} \quad \frac{B}{x - 3} \quad \text{and} \quad \frac{C}{(x - 3)^2}$$

So we have

$$\frac{-x^2 - 8x + 27}{x(x - 3)^2} = \frac{A}{x} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2}$$

Then multiply each side of this equation by the L.C.D.: $x(x - 3)^2$.

$$-x^2 - 8x + 27 = A(x - 3)^2 + Bx(x - 3) + Cx$$

Removing parentheses and rearranging terms, we have

$$-x^2 - 8x + 27 = Ax^2 - 6Ax + 9A + Bx^2 - 3Bx + Cx$$
$$-x^2 - 8x + 27 = (A + B)x^2 + (-6A - 3B + C)x + 9A$$

Equating coefficients, we have

$$A + B = -1$$
$$-6A - 3B + C = -8$$
$$9A = 27$$

From the third equation, we have $A = 3$. Substituting $A = 3$ into the first equation gives $B = -4$. Then substituting $A = 3$ and $B = -4$ into the second equation gives $C = -2$. Thus,

$$\frac{-x^2 - 8x + 27}{x(x - 3)^2} = \frac{3}{x} - \frac{4}{x - 3} - \frac{2}{(x - 3)^2}$$

EXAMPLE 4

Find the partial fractions of $\dfrac{3x^2 - 12x + 17}{(x - 2)^3}$.

Since $x - 2$ is repeated as a linear factor three times, the possible partial fractions are

$$\frac{A}{x - 2} \quad \frac{B}{(x - 2)^2} \quad \text{and} \quad \frac{C}{(x - 2)^3}$$

So we have

$$\frac{3x^2 - 12x + 17}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

Then multiply each side of this equation by the L.C.D.: $(x-2)^3$.

$$3x^2 - 12x + 17 = A(x-2)^2 + B(x-2) + C$$

Removing parentheses and rearranging terms, we have

$$3x^2 - 12x + 17 = Ax^2 - 4Ax + 4A + Bx - 2B + C$$
$$3x^2 - 12x + 17 = Ax^2 + (-4A + B)x + 4A - 2B + C$$

Equating coefficients, we have the system

$$A = 3$$
$$-4A + B = -12$$
$$4A - 2B + C = 17$$

Substituting $A = 3$ into the second equation, we have $B = 0$. Then substituting $A = 3$ and $B = 0$ into the third equation, we have $C = 5$. Thus,

$$\frac{3x^2 - 12x + 17}{(x-2)^3} = \frac{3}{x-2} + \frac{5}{(x-2)^3}$$

CASE 3: NONREPEATED QUADRATIC DENOMINATOR FACTORS

For every nonrepeated factor $ax^2 + bx + c$ of the denominator of a given fraction, there corresponds the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c}$$ where A and B are constants.

EXAMPLE 5

Find the partial fractions of $\dfrac{11x^2 + 8x - 12}{(2x^2 + x - 4)(x + 1)}$.

The possible partial fractions are

$$\frac{Ax + B}{2x^2 + x - 4} \quad \text{and} \quad \frac{C}{x + 1}$$

So we have

$$\frac{11x^2 + 8x - 12}{(2x^2 + x - 4)(x + 1)} = \frac{Ax + B}{2x^2 + x - 4} + \frac{C}{x + 1}$$

Then multiply each side of this equation by the L.C.D.: $(2x^2 + x - 4)(x + 1)$.

$$11x^2 + 8x - 12 = (Ax + B)(x + 1) + C(2x^2 + x - 4)$$

Removing parentheses and rearranging terms, we have

$$11x^2 + 8x - 12 = Ax^2 + Ax + Bx + B + 2Cx^2 + Cx - 4C$$
$$11x^2 + 8x - 12 = (A + 2C)x^2 + (A + B + C)x + B - 4C$$

Equating coefficients, we have

$$A + 2C = 11$$
$$A + B + C = 8$$
$$B - 4C = -12$$

The solution of this system of linear equations is $A = 5$, $B = 0$, $C = 3$. Then

$$\frac{11x^2 + 8x - 12}{(2x^2 + x - 4)(x + 1)} = \frac{5x}{2x^2 + x - 4} + \frac{3}{x + 1}$$

EXAMPLE 6

Find the partial fractions of $\dfrac{2x^3 - 2x^2 + 8x + 7}{(x^2 + 1)(x^2 + 4)}$.

The possible partial fractions are

$$\frac{Ax + B}{x^2 + 1} \quad \text{and} \quad \frac{Cx + D}{x^2 + 4}$$

So we have

$$\frac{2x^3 - 2x^2 + 8x + 7}{(x^2 + 1)(x^2 + 4)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4}$$

Then multiply each side of this equation by the L.C.D.: $(x^2 + 1)(x^2 + 4)$.

$$2x^3 - 2x^2 + 8x + 7 = (Ax + B)(x^2 + 4) + (Cx + D)(x^2 + 1)$$

Removing parentheses and rearranging terms, we have

$$2x^3 - 2x^2 + 8x + 7 = Ax^3 + Bx^2 + 4Ax + 4B + Cx^3 + Dx^2 + Cx + D$$
$$2x^3 - 2x^2 + 8x + 7 = (A + C)x^3 + (B + D)x^2 + (4A + C)x + 4B + D$$

Equating coefficients, we have

$$A + C = 2$$
$$B + D = -2$$
$$4A + C = 8$$
$$4B + D = 7$$

The solution of this system of linear equations is $A = 2$, $B = 3$, $C = 0$, $D = -5$. Then

$$\frac{2x^3 - 2x^2 + 8x + 7}{(x^2 + 1)(x^2 + 4)} = \frac{2x + 3}{x^2 + 1} - \frac{5}{x^2 + 4}$$

CASE 4: REPEATED QUADRATIC DENOMINATOR FACTORS

For every factor $(ax^2 + bx + c)^k$ of the denominator of a given fraction, there correspond the possible partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c}, \frac{A_2x + B_2}{(ax^2 + bx + c)^2}, \frac{A_3x + B_3}{(ax^2 + bx + c)^3}, \ldots, \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

where $A_1, A_2, A_3, \ldots, A_k, B_1, B_2, B_3, \ldots, B_k$ are constants.

EXAMPLE 7

Find the partial fractions of $\dfrac{5x^4 - x^3 + 44x^2 - 5x + 75}{x(x^2 + 5)^2}$.

The possible partial fractions are

$$\frac{A}{x} \quad \frac{Bx + C}{x^2 + 5} \quad \text{and} \quad \frac{Dx + E}{(x^2 + 5)^2}$$

So we have

$$\frac{5x^4 - x^3 + 44x^2 - 5x + 75}{x(x^2 + 5)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 5} + \frac{Dx + E}{(x^2 + 5)^2}$$

Then multiply each side of this equation by the L.C.D.: $x(x^2 + 5)^2$.

$$5x^4 - x^3 + 44x^2 - 5x + 75 = A(x^2 + 5)^2 + (Bx + C)(x^2 + 5)(x) + (Dx + E)x$$

Removing parentheses and rearranging terms, we have

$$\begin{aligned} 5x^4 - x^3 + 44x^2 - 5x + 75 &= Ax^4 + 10Ax^2 + 25A + Bx^4 + Cx^3 \\ &\quad + 5Bx^2 + 5Cx + Dx^2 + Ex \\ 5x^4 - x^3 + 44x^2 - 5x + 75 &= (A + B)x^4 + Cx^3 + (10A + 5B + D)x^2 \\ &\quad + (5C + E)x + 25A \end{aligned}$$

Equating coefficients, we have

$$\begin{aligned} A + B &= 5 \\ C &= -1 \\ 10A + 5B + D &= 44 \\ 5C + E &= -5 \\ 25A &= 75 \end{aligned}$$

The solution of this system of linear equations is $A = 3$, $B = 2$, $C = -1$, $D = 4$, $E = 0$. Then

$$\frac{5x^4 - x^3 + 44x^2 - 5x + 75}{x(x^2 + 5)^2} = \frac{3}{x} + \frac{2x - 1}{x^2 + 5} + \frac{4x}{(x^2 + 5)^2}$$

If the degree of the numerator is greater than or equal to the degree of the denominator of the original fraction, you must first divide the numerator by the denominator using long division. Then find the partial fractions of the resulting remainder.

EXAMPLE 8

Find the partial fractions of $\dfrac{x^3 + 3x^2 + 7x + 4}{x^2 + 2x}$.

Since the degree of the numerator is greater than the degree of the denominator, divide as follows:

$$\begin{array}{r} x + 1 \\ x^2 + 2x \overline{)\, x^3 + 3x^2 + 7x + 4} \\ \underline{x^3 + 2x^2} \\ x^2 + 7x \\ \underline{x^2 + 2x} \\ 5x + 4 \end{array}$$

or

$$\frac{x^3 + 3x^2 + 7x + 4}{x^2 + 2x} = x + 1 + \frac{5x + 4}{x^2 + 2x}$$

Now factor the denominator and find the partial fractions of the remainder.

$$\frac{5x + 4}{x(x + 2)} = \frac{A}{x} + \frac{B}{x + 2}$$
$$5x + 4 = A(x + 2) + Bx$$
$$5x + 4 = Ax + 2A + Bx$$
$$5x + 4 = (A + B)x + 2A$$

Then

$$A + B = 5$$
$$2A = 4$$

So $A = 2$ and $B = 3$ and

$$\frac{x^3 + 3x^2 + 7x + 4}{x^2 + 2x} = x + 1 + \frac{2}{x} + \frac{3}{x + 2}$$

Exercises 6.7

Find the partial fractions of each expression.

1. $\dfrac{8x - 29}{(x + 2)(x - 7)}$

2. $\dfrac{10x - 34}{(x - 4)(x - 2)}$

3. $\dfrac{-x - 18}{2x^2 - 5x - 12}$

4. $\dfrac{17x - 18}{3x^2 + x - 2}$

5. $\dfrac{61x^2 - 53x - 28}{x(3x - 4)(2x + 1)}$

6. $\dfrac{11x^2 - 7x - 42}{(2x + 3)(x^2 - 2x - 3)}$

7. $\dfrac{x^2 + 7x + 10}{(x + 1)(x + 3)^2}$

8. $\dfrac{3x^2 - 18x + 9}{(2x - 1)(x - 1)^2}$

9. $\dfrac{48x^2 - 20x - 5}{(4x - 1)^3}$

10. $\dfrac{x^2 + 8x}{(x + 4)^3}$

11. $\dfrac{11x^2 - 18x + 3}{x(x - 1)^2}$

12. $\dfrac{6x^2 + 4x + 4}{x^3 + 2x^2}$

13. $\dfrac{-x^2 - 4x + 3}{(x^2 + 1)(x^2 - 3)}$

14. $\dfrac{-6x^3 + 2x^2 - 3x + 10}{(2x^2 + 1)(x^2 + 5)}$

15. $\dfrac{4x^3 - 21x - 6}{(x^2 + x + 1)(x^2 - 5)}$

16. $\dfrac{x^3 + 6x^2 + 2x - 2}{(3x^2 - x - 1)(x^2 + 4)}$

17. $\dfrac{4x^3 - 16x^2 - 93x - 9}{(x^2 + 5x + 3)(x^2 - 9)}$

18. $\dfrac{12x^2 + 8x - 72}{(x^2 + x - 1)(x^2 - 16)}$

19. $\dfrac{8x^4 - x^3 + 13x^2 - 6x + 5}{x(x^2 + 1)^2}$

20. $\dfrac{-4x^4 + 6x^3 + 8x^2 - 19x + 17}{(x - 1)(x^2 - 3)^2}$

21. $\dfrac{x^5 - 2x^4 - 8x^2 + 4x - 8}{x^2(x^2 + 2)^2}$

22. $\dfrac{3x^5 + x^4 + 24x^3 + 10x^2 + 48x + 16}{x^2(x^2 + 4)^2}$

23. $\dfrac{6x^2 + 108x + 54}{x^4 - 81}$

24. $\dfrac{x^6 + 2x^4 + 3x^2 + 1}{x^2(x^2 + 1)^3}$

25. $\dfrac{x^3}{x^2 - 1}$

26. $\dfrac{x^4 + x^2}{(x + 1)(x - 2)}$

27. $\dfrac{x^3 - x^2 + 8}{x^2 - 4}$

28. $\dfrac{2x^3 - 2x^2 + 8x - 3}{x(x - 1)}$

29. $\dfrac{3x^4 - 2x^3 - 2x + 5}{x(x^2 + 1)}$

30. $\dfrac{x^5 - x^4 - 3x^3 + 7x^2 + 3x + 20}{(x + 2)(x^2 + 2)}$

CHAPTER SUMMARY

1. *Graphs of linear systems of equations with two variables:*
   (a) The two lines may intersect at a common, single point. This point, in ordered pair form (x, y), is the solution of the system.
   (b) The two lines may be parallel with no points in common; hence, the system has no common solution.
   (c) The two lines may coincide; the solution of the system is the set of all points on the common line.
2. *Solving a pair of linear equations by the addition-subtraction method:*
   (a) If necessary, multiply each side of one or both equations by some number so that the numerical coefficients of one of the variables are of equal absolute value.
   (b) If these coefficients of equal absolute value have like signs, subtract one equation from the other. If they have unlike signs, add the equations. That is, do whatever is necessary to eliminate that variable.
   (c) Solve the resulting equation for the remaining variable.
   (d) Substitute the solution for the variable found in Step (c) in either of the original equations, and solve this resulting equation for the second variable.
   (e) Check by substituting the solution in both original equations.
3. *Solving a pair of linear equations by the method of substitution:*
   (a) From either of the two given equations, solve for one variable in terms of the other.
   (b) Substitute this result from Step (a) in the *other* equation. Note that this step eliminates a variable.
   (c) Solve the equation obtained from Step (b) for the remaining variable.
   (d) From the equation obtained in Step (a), substitute the solution for the variable found in Step (c), and solve this resulting equation for the second variable.
   (e) Check by substituting the solution in both original equations.
4. *Steps for problem solving:*
   (a) Read the problem carefully at least two times.
   (b) If possible, draw a picture or a diagram.
   (c) Write what facts are given and what unknown quantities are to be found.
   (d) Choose a symbol to represent each quantity to be found (there will be more than one). Be sure to label each symbol to indicate what it represents.
   (e) Write appropriate equations relating these variables from the information given in the problem. Watch for information that is not stated but which should be assumed.
   (f) Solve for the unknown variables by the methods presented in this chapter.
   (g) Check your solution in the original equations.
   (h) Check your solution in the original verbal problem.
5. *Graphs of linear systems of equations with three variables:*
   (a) The three planes may intersect at a common, single point. This point, in ordered triple form (x, y, z), is then the solution of the system.
   (b) The three planes may intersect along a common line. The infinite set of points that satisfy the equation of the line is the solution of the system.
   (c) The three planes may not have any points in common; the system has no solution. For example, the planes may be parallel, or they may intersect triangularly with no points common to all three planes.

(d) The three planes may coincide; the solution of the system is the set of all points in the common plane.

6. *The addition-subtraction method for a system of three linear equations with three variables:*
   (a) Choose a variable to be eliminated. Eliminate it from any pair of equations by using the techniques of the addition-subtraction method.
   (b) Eliminate this same variable from *any other pair* of equations.
   (c) The result of Steps (a) and (b) is a pair of linear equations in two unknowns. Solve this pair for the two variables.
   (d) Solve for the third variable by substituting the results from Step (c) in any one of the original equations.
   (e) Check by substituting the solution in all three original equations.

7. A **determinant** is a square array of numbers such as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

8. *Value of second-order determinant:*

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

9. The value of a third-order determinant is defined as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Each 2×2 determinant is called a **minor** of an element in the 3×3 determinant. The minor of a given element is the resulting determinant after the row and the column which contain the element have been deleted, as shown in Table 6.4.

The value of any determinant of any order may be found by evaluating the sums and differences of *any* row or column of the products of the elements and the corresponding minors. The following diagram shows the signs (sum or difference) of the various products of elements and minors:

$$\begin{vmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix}$$

TABLE 6.4

Element	Minor	Original Determinant
a_1	$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$
a_2	$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$
a_3	$\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

10. *Properties of determinants:*

(a) *Property 1:* If every element in a row (or a column) of a determinant is zero, the value of the determinant is zero.

(b) *Property 2:* If two rows (or two columns) of a determinant are identical, the value of the determinant is zero.

(c) *Property 3:* If any two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant is changed.

(d) *Property 4:* If every element of a row (or a column) is multiplied by the same real number k, the value of the determinant is multiplied by k.

(e) *Property 5:* If every element of a row (or a column) is multiplied by the same real number k, and if the resulting products are added to another row (or another column), the value of the determinant remains the same.

11. *Cramer's rule:*

(a) The solution of two linear equations in two variables

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

may be written in determinant form as follows:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

(b) The solution of three linear equations in three variables

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

may be written in determinant form as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

12. *Partial fractions:*

(a) *Case 1: Nonrepeated linear denominator factors.* For every nonrepeated factor $ax + b$ of the denominator of a given fraction, there corresponds the partial fraction $\frac{A}{ax + b}$, where A is a constant.

(b) *Case 2: Repeated linear denominator factors.* For every factor $(ax + b)^k$ of the denominator of the given fraction, there correspond the possible partial fractions

$$\frac{A_1}{ax + b}, \frac{A_2}{(ax + b)^2}, \frac{A_3}{(ax + b)^3}, \ldots, \frac{A_k}{(ax + b)^k}$$

where $A_1, A_2, A_3, \ldots, A_k$ are constants.

(c) *Case 3: Nonrepeated quadratic denominator factors.* For every nonrepeated factor $ax^2 + bx + c$ of the denominator of the given fraction, there corresponds the partial fraction

$$\frac{Ax + B}{ax^2 + bx + c}$$ where A and B are constants.

(d) *Case 4: Repeated quadratic denominator factors.* For every factor $(ax^2 + bx + c)^k$ of the denominator of the given fraction, there correspond the possible partial fractions

$$\frac{A_1x + B_1}{ax^2 + bx + c}, \frac{A_2x + B_2}{(ax^2 + bx + c)^2}, \frac{A_3x + B_3}{(ax^2 + bx + c)^3}, \ldots, \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

where $A_1, A_2, A_3, \ldots, A_k, B_1, B_2, B_3, \ldots, B_k$ are constants.

CHAPTER 6 REVIEW

Solve each system of equations graphically and check by substituting the solution ordered pair in both original equations.

1. $x + y = 3$
$x - y = -1$

2. $y - 3x = 2$
$y - x = 2$

3. $2x - y = -10$
$4x + y = 4$

4. $2x - 3y = 6$
$-4x + 6y = 8$

Solve each system of equations by the addition-subtraction method and check.

5. $3x - 4y = 5$
$x + 7y = 10$

6. $2x + 3y = 7$
$4x - 6y = 11$

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7. $x - 2y = 6$
$4x + y = 6$

8. $3x - 2y = 5$
$2x + 2y = 15$

Solve each system by the method of substitution and check.

9. $x = -4y$
$3x - 5y = 17$

10. $y = 3x$
$2x - 15y = 86$

11. $5x - y = 7$
$2x + 3y = 13$

12. $3x + 4y = -7$
$-5x + 12y = 14$

Solve each system of equations.

13. $8x + by = 4$
$16x - ay = 12$

14. $\frac{3}{x} - \frac{4}{y} = -5$
$\frac{6}{x} + \frac{5}{y} = 16$

15. $x + y + z = 2$
$x - y - 2z = 3$
$x + 2y - z = \frac{3}{2}$

16. $2x + 2y - z = 4$
$2x - y + 2z = -2$
$x - 5y + z = 8$

Evaluate each determinant.

17. $\begin{vmatrix} 2 & -3 \\ 6 & 2 \end{vmatrix}$

18. $\begin{vmatrix} -1 & 7 \\ 4 & -3 \end{vmatrix}$

19. $\begin{vmatrix} 1 & 3 & 5 \\ 7 & 2 & -1 \\ -4 & 2 & -8 \end{vmatrix}$

20. $\begin{vmatrix} 2 & 0 & -3 \\ 1 & 4 & 2 \\ 5 & 1 & -1 \end{vmatrix}$

Solve each system using determinants and check.

21. $4x + 3y = 46$
$2x - 3y = 14$

22. $3x - 2y = 13$
$-6x + 4y = -26$

23. $2x + 9y = 4$
$5x - 9y = 10$

24. $5x - 2y = 2$
$-3x + 3y = 1$

25. $2x + 2y - z = 11$
$3x + 4y + z = -16$
$4x - 8y + 3z = -113$

26. $x + y + z = 9$
$2x - y + z = 3$
$x + 3y - z = 3$

Solve each system by any method and check.

27. $2x + 7y = 3$
$10x + 3y = -1$

28. $-5x + 2y = 5$
$-3x + 7y = 32$

29. $2x - 3y = 10$
$2x + 5y = 2$

30. $x + y = -25$
$2x - 3y = 70$

31. $x + y + z = 7$
$x + z = 2$
$y - z = 4$

32. $7x - 2y + 5z = 9$
$3x + 4y - 9z = 27$
$10x - 3y + 7z = 7$

33. An alloy contains 10% lead. Another alloy contains 20% lead. How many kg of each must be used to make a 30-kg alloy containing 12% lead?

Day: Date:

34. A person deposits \$3600 into two different savings accounts. One account earns interest at $5\frac{1}{2}\%$ and the other at $6\frac{1}{2}\%$. The total interest earned from both accounts at the end of one year is \$220.00. Find the amount deposited at $5\frac{1}{2}\%$ and the amount deposited at $6\frac{1}{2}\%$.

35. It took a plane 1 h 15 min to make a trip of 100 km flying into a head wind. On the return trip with the same wind velocity, it took only 1 h. Find the wind velocity and the speed of the plane (in still air).

36. Jack, John, and Bob work on an assembly line producing hand-cut aluminum parts. Jack and John together can process, on average, three times as many pieces per hour as Bob. Jack can process 2 more pieces per h than John, and together all three can process 32 pieces per h. How many pieces per h does each man process, on average?

Evaluate each determinant.

37. $\begin{vmatrix} 1 & 0 & 3 & 6 \\ 4 & -2 & 1 & -5 \\ -3 & 2 & 5 & 2 \\ 1 & 0 & 7 & -9 \end{vmatrix}$

38. $\begin{vmatrix} 5 & 1 & 6 & -3 \\ -2 & 2 & 5 & 1 \\ 3 & -1 & 7 & 5 \\ 3 & 2 & 3 & -2 \end{vmatrix}$

Find the partial fractions of each expression.

39. $\dfrac{6x + 14}{(x - 3)(x + 5)}$

40. $\dfrac{-x^2 - 6x - 1}{x(x^2 - 1)}$

41. $\dfrac{3x^2 + 5x + 1}{x^2(x + 1)}$

42. $\dfrac{10x + 4}{(x + 1)^2}$

43. $\dfrac{7x^2 - x + 2}{(x^2 + 1)(x - 1)}$

44. $\dfrac{5x^2 + 2x + 21}{(x^2 + 4)^2}$

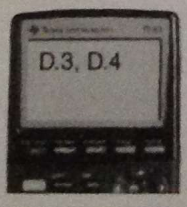
INTRODUCTION

The path of a golf ball in flight, the relationship between the strength of a beam and its length, and the cost of manufacturing a particular item can all be described using the concept of a function. In this chapter we study functions and relate their equations to a graph. This graphic representation which connects algebra and geometry is extremely helpful in solving problems.

Objectives

- Identify relations that are functions.
- Determine the domain and the range of a function.
- Work with function notation.
- Solve equations graphically.
- Find the slope of a line.
- Find the equation of a line given appropriate information.
- Know the characteristics of parallel and perpendicular lines.
- Know the distance and midpoint formulas.

4.1 FUNCTIONS



In common usage, a relation means that two or more things have something in common. We say that a brother and a sister are related because they have the same parents or that a person's career potential is related to his or her education and work experience.

In mathematics a **relation** is defined as a set of ordered pairs of numbers in the form (x, y). Sometimes an equation, a rule, a data chart, or some other type of description is given that states the relationship between x and y. In an ordered pair the first element or variable, called the **independent variable,** may be represented by any letter, but x is nor-

mally used. The second element or variable is normally represented by the letter y and is called the **dependent variable** because its value depends on the particular choice of the independent variable.

All of the numbers that can be used as the first element of an ordered pair or as replacements for the independent variable of a given relation form a set of numbers called the **domain.** The domain is often referred to as the set of all x's. We can think of these x-values as "inputs." The **range** of a relation is the set of numbers that can be used as the second element of an ordered pair or as replacements for the dependent variable. The range is often referred to as the set of all y's. We can think of these y-values as "outputs."

EXAMPLE 1

Given the relation described in ordered pair form $A = \{(1, 2), (3, 5), (7, 9), (6, 3)\}$, find its domain and its range.

The domain is the set of first elements: $\{1, 3, 6, 7\}$. The range is the set of second elements: $\{2, 3, 5, 9\}$.

Note: Braces { } are normally used to group elements of sets.

EXAMPLE 2

Given the relation in equation form $y = x^2$, find its domain and its range.

The domain is the set of possible replacements for the independent variable x. Note that there are no restrictions on the numbers that you may substitute for x. That is, we may replace x by any real number. We say that the domain is the set of real numbers.

After each replacement of x, there is no possible way that we can obtain a negative value for y because the square of any real number is always positive or zero. Thus, the range is the set of nonnegative real numbers, or $y \geq 0$.

EXAMPLE 3

Find the domain and the range of the relation $y = \sqrt{x - 4}$.

Note that no value of x less than 4 may be used because the square root of any negative number is not a real number. Thus, the domain is the set of real numbers greater than or equal to 4, or $x \geq 4$.

After each possible x-replacement, the square root of the resulting value is never negative, so the range is $y \geq 0$.

FUNCTION

A **function** is a special relation: a set of ordered pairs in which no two distinct ordered pairs have the same first element.

In equation form, a relation is a function when for each possible value of the first or independent variable, there is only one corresponding value of the second or dependent variable. In brief, for a relation to be a function, each value of x must correspond to one, and only one, value of y.

EXAMPLE 4

Is the relation $B = \{(3, 2), (6, 7), (5, 3), (1, 1), (3, 7)\}$ a function? Find its domain and its range.

B is not a function because it contains two different ordered pairs that have the same first element: (3, 2) and (3, 7). In other words, the fact that both 2 and 7 correspond to 3 causes the relation B not to be a function. The domain of B is {1, 3, 5, 6}. The range of B is {1, 2, 3, 7}.

Does the set $A = \{(1, 2), (3, 5), (7, 9), (6, 3)\}$ from Example 1 describe a function? Yes, because no two ordered pairs have the same first element.

EXAMPLE 5

Is the relation $x = y^2$ a function? Find its domain and its range.

Can we find two ordered pairs that have the same first element? Yes, for example, (9, 3) and (9, −3) as well as (16, 4) and (16, −4) and many others. Therefore, $x = y^2$ is not a function because for at least one x-value, there corresponds more than one y-value.

To find the domain, note that each x-value is the square of a real number and can never be negative. Thus, the domain is $x \geq 0$.

There are no restrictions on replacements for y; therefore, the range is the set of all real numbers.

Consider the relations in Examples 2 and 3. Are they functions? Note that in the relation $y = x^2$, for each value of x there is only one corresponding value of y. For example, (2, 4), (−2, 4), (3, 9), (−3, 9), (4, 16), (−4, 16), and so forth. Therefore, $y = x^2$ is a function.

Example 3 was the relation $y = \sqrt{x - 4}$. Here we find that for each x-value there corresponds only one y-value [for example, (5, 1), (8, 2), (10, $\sqrt{6}$), . . .]. Therefore, $y = \sqrt{x - 4}$ is a function.

In summary, a function is a relationship between two sets of numbers, the domain and the range, which relates each number, x, in the domain to one and only one number, y, in the range.

Next let's consider the following, more intuitive function: On a summer vacation trip, you are driving 65 mi/h using cruise control on an interstate highway. You want to relate the distance and the time you are traveling. First, we know that distance equals rate times time, or $d = rt$. Also, since $r = 65$, we have the relation $d = 65t$. As you drive along, you begin to think how far you can drive in one hour:

$$d = 65t = 65(1) = 65 \text{ mi}$$

How far can you drive in three hours?

$$d = 65t = 65(3) = 195 \text{ mi}$$

Is this relation a function? Yes, because for each value of t, there is one and only one value of d; that is, during each driving time period, there is one and only one distance traveled. What are the domain and the range? First, note that $t \geq 0$ and $d \geq 0$. While there are no theoretical upper limits on t and d, the practical limits depend on the amount of time and the distance that you want to travel.

Functional Notation

To say that y is a function of x means that for each value of x from the domain of the function, we can find exactly one value of y from the range. This statement is said so of-

ten that we have developed the following notation, called **functional notation,** to write that y is a function of x:

$$y = f(x)$$

with $f(x)$ read "f of x." *Note:* $f(x)$ does **not** mean f times x.

In each of the following equations, y can be replaced by $f(x)$, and the resulting equation is written in functional notation.

Equation	*Functional notation form*
$y = 3x - 4$	$f(x) = 3x - 4$
$y = 5x^2 - 8x + 7$	$f(x) = 5x^2 - 8x + 7$
$y = \sqrt{6 - 2x}$	$f(x) = \sqrt{6 - 2x}$

Functional notation can be used to simplify statements. For example, find the value of $y = 3x^2 + 5x - 6$ for $x = 2$. Using substitution, we replace x with 2 as follows:

$$y = 3x^2 + 5x - 6$$
$$y = 3(2)^2 + 5(2) - 6 = 16$$

The statement "Find the value of $y = 3x^2 + 5x - 6$ for $x = 2$" may be abbreviated using functional notation as follows:

$$\text{Given } f(x) = 3x^2 + 5x - 6, \text{ find } f(2).$$

EXAMPLE 6

Given the function $f(x) = 5x - 4$, find each of the following:

(a) $f(0)$

Replace x with 0 as follows:

$$f(0) = 5(0) - 4 = 0 - 4 = -4$$

(b) $f(7)$

Replace x with 7 as follows:

$$f(7) = 5(7) - 4 = 35 - 4 = 31$$

A function is usually named by a specific letter, such as $f(x)$, where f names the function. Other letters, such as g in $g(x)$ and h in $h(x)$, are often used to represent or name functions.

EXAMPLE 7

Given the function $g(x) = \sqrt{x + 4} + 3x^2$, find each of the following:

(a) $g(5)$

Replace x with 5 as follows

$$g(5) = \sqrt{5 + 4} + 3(5)^2 = 3 + 75 = 78$$

(b) $g(-3)$

Replace x with -3 as follows:

$$g(-3) = \sqrt{-3 + 4} + 3(-3)^2 = 1 + 27 = 28$$

(c) $g(-10)$

Replace x with -10 as follows:

$$g(-10) = \sqrt{-10 + 4} + 3(-10)^2 = \sqrt{-6} + 300$$

which is not a real number because $\sqrt{-6}$ is not a real number. Another way of responding to Part (c) is to say, "Since -10 is not in the domain of $g(x)$, $g(-10)$ has no real value."

Letters may also be used with functional notation as illustrated by the following example.

EXAMPLE 8

Given the function $f(x) = x^2 - 4x$, find each of the following:

(a) $f(a)$

Replace x with a as follows:

$$f(a) = a^2 - 4a$$

(b) $f(3c^2)$

Replace x with $3c^2$ as follows:

$$f(3c^2) = (3c^2)^2 - 4(3c^2) = 9c^4 - 12c^2$$

(c) $f(a + 5)$

Replace x with $a + 5$ as follows:

$$\begin{aligned} f(a + 5) &= (a + 5)^2 - 4(a + 5) \\ &= a^2 + 10a + 25 - 4a - 20 \\ &= a^2 + 6a + 5 \end{aligned}$$

Other letters, such as t in $f(t)$ and r in $f(r)$, are used in applications to name independent variables.

EXAMPLE 9

Given the function $f(t) = 0.50t + 5.4$, find each of the following:

(a) $f(3.2)$

Replace t with 3.2 as follows:

$$\begin{aligned} f(t) &= 0.50t + 5.4 \\ f(3.2) &= 0.50(3.2) + 5.4 \\ &= 1.6 + 5.4 \\ &= 7.0 \end{aligned}$$

(b) $f(t_0)$

Replace t with t_0 as follows:

$$\begin{aligned} f(t) &= 0.50t + 5.4 \\ f(t_0) &= 0.50t_0 + 5.4 \end{aligned}$$

Exercises 4.1

Determine whether or not each relation is a function. Write its domain and its range.

1. $A = \{(2, 4), (3, 7), (9, 2)\}$

2. $B = \{(5, 2), (3, 3), (1, 2)\}$

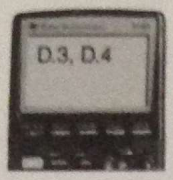
Day: Date:

3. $C = \{(2, 5), (7, 3), (2, 1), (1, 3)\}$
4. $D = \{(0, 2), (5, -1), (2, 7), (5, 1)\}$
5. $E = \{(3, 2), (5, 2), (2, 2), (-2, 2)\}$
6. $F = \{(3, 4), (3, -4), (-3, -4), (-3, 4)\}$
7. $y = 2x + 5$
8. $y = -3x$
9. $y = x^2 + 1$
10. $y = 2x^2 - 3$
11. $x = y^2 - 2$
12. $x = 3y^2 + 4$
13. $y = \sqrt{x + 3}$
14. $y = \sqrt{3 - 6x}$
15. $y = 6 + \sqrt{2x - 8}$
16. $y = 16 - \sqrt{x + 5}$
17. Given the function $f(x) = 8x - 12$, find
    (a) $f(4)$ **(b)** $f(0)$ **(c)** $f(-2)$
18. Given the function $g(x) = 20 - 4x$, find
    (a) $g(6)$ **(b)** $g(0)$ **(c)** $g(-3)$
19. Given $g(x) = 10x + 15$, find
    (a) $g(2)$ **(b)** $g(0)$ **(c)** $g(-4)$
20. Given $f(x) = x^2 - 4$, find
    (a) $f(6)$ **(b)** $f(0)$ **(c)** $f(-6)$
21. Given $h(x) = 3x^2 + 4x$, find
    (a) $h(5)$ **(b)** $h(0)$ **(c)** $h(-2)$
22. Given $f(x) = -2x^2 + 6x - 7$, find
    (a) $f(3)$ **(b)** $f(0)$ **(c)** $f(-1)$
23. Given $f(t) = \dfrac{5 - t^2}{2t}$, find
    (a) $f(1)$ **(b)** $f(-3)$ **(c)** $f(0)$
24. Given $g(t) = \sqrt{21 - 5t}$, find
    (a) $g(1)$ **(b)** $g(-3)$ **(c)** $g(2)$ **(d)** $g(8)$
25. Given $f(x) = 6x + 8$, find
    (a) $f(a)$ **(b)** $f(4a)$ **(c)** $f(c^2)$
26. Given $g(x) = 8x^2 - 7x$, find
    (a) $g(z)$ **(b)** $g(2y)$ **(c)** $g(3t^2)$
27. Given $h(x) = 4x^2 - 12x$, find
    (a) $h(x + 2)$ **(b)** $h(x - 3)$ **(c)** $h(2x + 1)$
28. Given $f(y) = y^2 - 3y + 6$, find
    (a) $f(y - 1)$ **(b)** $f(y^2 + 1)$ **(c)** $f(1 - 4y)$
29. Given $f(x) = 3x - 1$ and $g(x) = x^2 - 6x + 1$, find
    (a) $f(x) + g(x)$ **(b)** $f(x) - g(x)$ **(c)** $[f(x)][g(x)]$ **(d)** $f(x + h)$
30. Given $f(t) = 5 - 2t + t^2$ and $g(t) = t^2 - 4t + 4$, find
    (a) $f(t) + g(t)$ **(b)** $g(t) - f(t)$ **(c)** $[f(t)][g(t)]$ **(d)** $g(t + h)$

Find the domain of each function.

31. $f(x) = \dfrac{3x + 4}{x - 2}$
32. $f(t) = \dfrac{8}{6t + 3}$
33. $g(t) = \dfrac{2t + 4t^2}{(t - 6)(t + 3)}$
34. $g(x) = \dfrac{3x - 10}{x^2 + 4}$
35. $f(x) = \dfrac{12}{\sqrt{15 - 3x}}$
36. $g(t) = \dfrac{9}{\sqrt{5t + 20}}$

4.2 GRAPHING EQUATIONS



Consider a plane in which two number lines intersect at right angles. Let the point of intersection be the zero point of each line and call it the **origin.** Each line is called an **axis.** The horizontal number line is usually called the ***x*-axis,** and the vertical line is usually called the ***y*-axis.** On each axis the same scale (unit length) is preferred but not always possible in all applications. Such a system is called the **rectangular coordinate system,** or the **Cartesian coordinate system.** (The name *Cartesian* is after Descartes, a seventeenth-century French mathematician. He first conceived this idea of combining algebra and geometry together in such a way that each could aid the study of the other.) The plane is divided by the axes into four regions called **quadrants.** The quadrants are numbered as shown in Fig. 4.1.

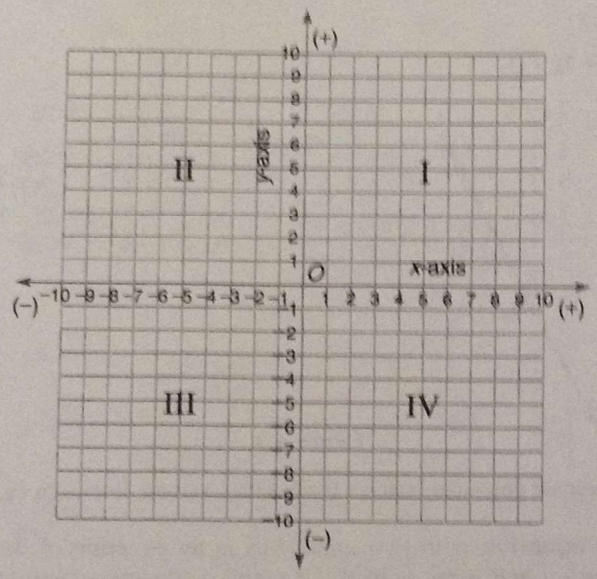


Figure 4.1 The rectangular coordinate system

In the plane there is a point that corresponds to each ordered pair of real numbers (x, y). Likewise, there is an ordered pair (x, y) that corresponds to each point in the plane. Together x and y are called the **coordinates** of the point; x is called the **abscissa,** and y is called the **ordinate.** This relationship is called a **one-to-one correspondence.** The location, or position, of a point in the plane corresponding to a given ordered pair is found by first counting right or left from O (origin) the number of units along the x-axis indicated by the first number of the ordered pair (right if positive, left if negative). Then from this point reached on the x-axis, count up or down the number of units indicated by the second number of the ordered pair (up if positive, down if negative).

EXAMPLE 1

Plot the point corresponding to each ordered pair in the number plane:

$A(3, 1)$ $B(2, -3)$ $C(-4, -2)$ $D(-3, 0)$ $E(-6, 2)$ $F(0, 2)$ (See Fig. 4.2.)

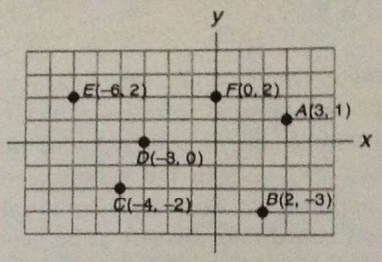
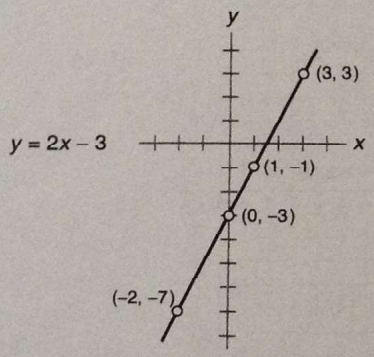


Figure 4.2

To graph equations we plot a sample of ordered pairs and connect them with a smooth curve. To obtain the sample, we need to generate ordered pairs from a given equation. One way to generate these ordered pairs is by randomly choosing a value for x, replacing this value for x in the equation, and solving for y.

EXAMPLE 2

Graph $y = 2x - 3$.



x	y	$y = 2x - 3$ or $f(x) = 2x - 3$
1	−1	$y = 2(1) - 3 = -1$
3	3	$y = 2(3) - 3 = 3$
−2	−7	$y = 2(-2) - 3 = -7$
0	−3	$y = 2(0) - 3 = -3$

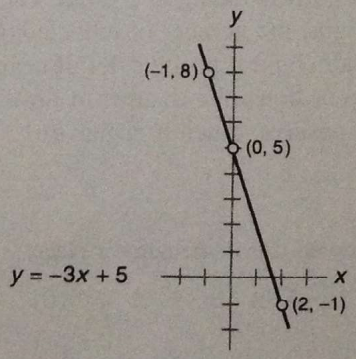
Figure 4.3

Plot the ordered pairs and connect them with a smooth line as in Fig. 4.3.

A **linear equation** with two unknowns is an equation of degree one in the form $ax + by = c$ with a and b not both 0. Its graph is always a straight line. Therefore, two ordered pairs are sufficient to graph a linear function, since two points determine a straight line. However, finding a third point provides good insurance against a careless error.

EXAMPLE 3

Graph $y = -3x + 5$.



x	y	$y = -3x + 5$ or $g(x) = -3x + 5$
0	5	$y = -3(0) + 5 = 5$
2	−1	$y = -3(2) + 5 = -1$
−1	8	$y = -3(-1) + 5 = 8$

See Fig. 4.4.

Figure 4.4

The graph of an equation that is not linear is usually a curve of some kind and hence requires several points to sketch a smooth curve.

EXAMPLE 4

Graph $y = x^2 - 4$.

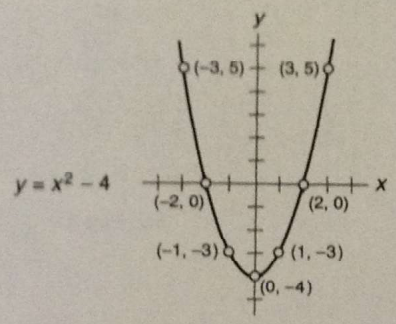


Figure 4.5

x	y	$y = x^2 - 4$
0	−4	$y = (0)^2 - 4 = -4$
1	−3	$y = (1)^2 - 4 = -3$
2	0	$y = (2)^2 - 4 = 0$
3	5	$y = (3)^2 - 4 = 5$
−1	−3	$y = (-1)^2 - 4 = -3$
−2	0	$y = (-2)^2 - 4 = 0$
−3	5	$y = (-3)^2 - 4 = 5$

See Fig. 4.5.

EXAMPLE 5

Graph $y = 2x^2 + x - 5$.

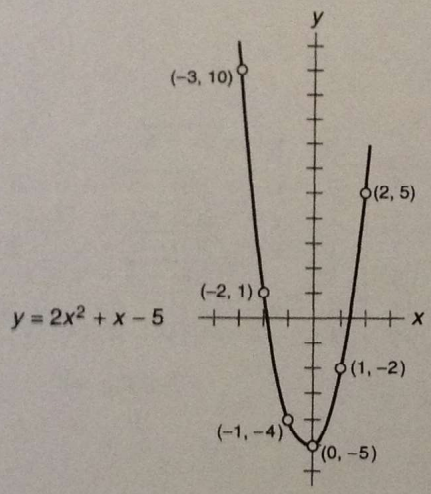


Figure 4.6

x	y	$y = 2x^2 + x - 5$
0	−5	$y = 2(0)^2 + (0) - 5 = -5$
1	−2	$y = 2(1)^2 + (1) - 5 = -2$
2	5	$y = 2(2)^2 + (2) - 5 = 5$
−1	−4	$y = 2(-1)^2 + (-1) - 5 = -4$
−2	1	$y = 2(-2)^2 + (-2) - 5 = 1$
−3	10	$y = 2(-3)^2 + (-3) - 5 = 10$

See Fig. 4.6.

For a more complicated function, more ordered pairs are usually required to obtain a smooth curve. It may also be necessary to change the scale of the graph in order to plot enough ordered pairs to obtain a smooth curve. To change the scale means to enlarge or reduce the unit length on the axes according to a specified ratio. This ratio is chosen on the basis of fitting the necessary values in a given space allowed for the graph.

EXAMPLE 6

Graph $y = x^3 + 4x^2 - x - 4$.

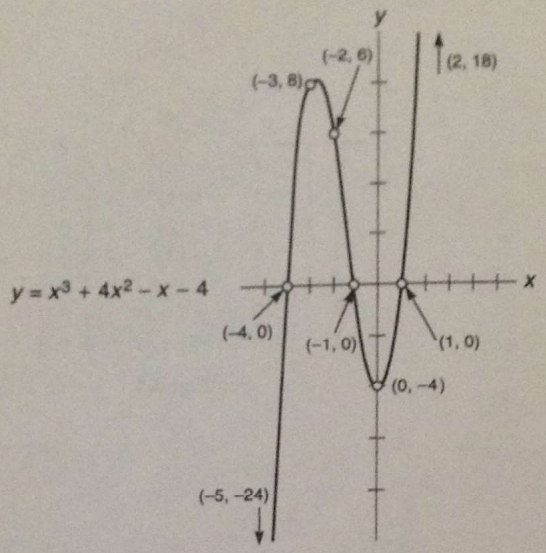


Figure 4.7

x	y	$y = x^3 + 4x^2 - x - 4$
0	−4	$y = (0)^3 + 4(0)^2 - (0) - 4 = -4$
1	0	$y = (1)^3 + 4(1)^2 - (1) - 4 = 0$
2	18	$y = (2)^3 + 4(2)^2 - (2) - 4 = 18$
3	56	$y = (3)^3 + 4(3)^2 - (3) - 4 = 56$
−1	0	$y = (-1)^3 + 4(-1)^2 - (-1) - 4 = 0$
−2	6	$y = (-2)^3 + 4(-2)^2 - (-2) - 4 = 6$
−3	8	$y = (-3)^3 + 4(-3)^2 - (-3) - 4 = 8$
−4	0	$y = (-4)^3 + 4(-4)^2 - (-4) - 4 = 0$
−5	−24	$y = (-5)^3 + 4(-5)^2 - (-5) - 4 = -24$

See Fig. 4.7.

EXAMPLE 7

Graph $y = \sqrt{2x - 6}$.

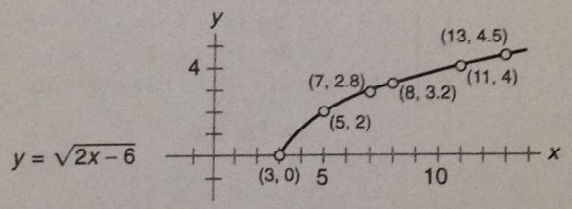


Figure 4.8

x	y	$y = \sqrt{2x - 6}$	
3	0	$y = \sqrt{2(3) - 6} = \sqrt{0} = 0$	
5	2	$y = \sqrt{2(5) - 6} = \sqrt{4} = 2$	
7	2.8	$y = \sqrt{2(7) - 6} = \sqrt{8} = 2.8$	(approx.)
8	3.2	$y = \sqrt{2(8) - 6} = \sqrt{10} = 3.2$	(approx.)
11	4	$y = \sqrt{2(11) - 6} = \sqrt{16} = 4$	
13	4.5	$y = \sqrt{2(13) - 6} = \sqrt{20} = 4.5$	(approx.)

See Fig. 4.8.

Solving Equations by Graphing

Equations may be solved graphically. This method is particularly useful when an algebraic method is very cumbersome, cannot be recalled, or does not exist; it is especially useful in technical applications.

Solving for y = 0 To solve the equation $y = x^2 - x - 6$ for $y = 0$ graphically means to find the point or points, if any, where the graph crosses the line $y = 0$ (the x-axis).

EXAMPLE 8

Solve $y = x^2 - x - 6$ for $y = 0$ graphically.

First, graph the equation $y = x^2 - x - 6$ (see Fig. 4.9).

x	y
1	−6
2	−4
3	0
4	6
0	−6
−1	−4
−2	0
−3	6

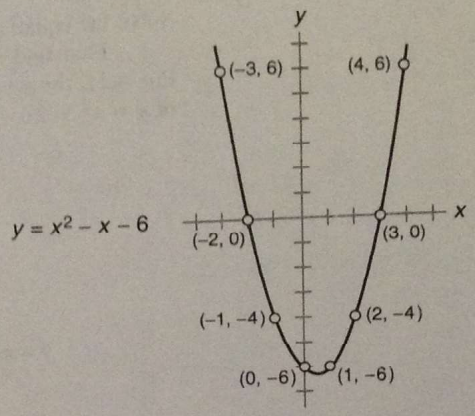


Figure 4.9

Then note the values of x where the curve crosses the x-axis: $x = -2$ and $x = 3$. Therefore, from the graph the solutions of $y = x^2 - x - 6$ for $y = 0$ are $x = -2$ and $x = 3$.

Sometimes the curve crosses the x-axis between the unit marks on the x-axis. In this case we must estimate as closely as possible the point of intersection of the curve and the x-axis. If a particular problem requires greater accuracy, we can scale the graph to allow a more accurate estimation.

EXAMPLE 9

Solve $y = x^2 + 2x - 4$ for $y = 0$ graphically.

First, graph the equation $y = x^2 + 2x - 4$ (see Fig. 4.10).

x	y
0	−4
1	−1
2	4
−1	−5
−2	−4
−3	−1
−4	4

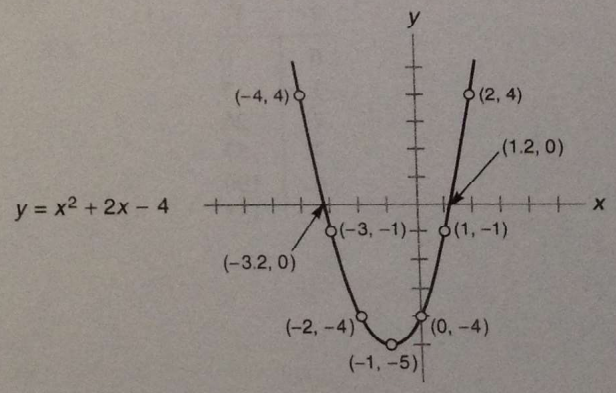


Figure 4.10

The values of x where the curve crosses the x-axis are approximately 1.2 and −3.2. Therefore, the approximate solutions of $y = x^2 + 2x - 4$ for $y = 0$ are $x = 1.2$ and $x = -3.2$.

Solving for y = k

EXAMPLE 10

Solve the equation from Example 9, which was $y = x^2 + 2x - 4$, for $y = 4$ and $y = -3$.

First find the values of x where the curve crosses the line $y = 4$. From the graph in Fig. 4.11, the x-values are 2 and -4. Therefore, solving for $y = 4$, we find that the solutions of $y = x^2 + 2x - 4$ are $x = 2$ and $x = -4$.

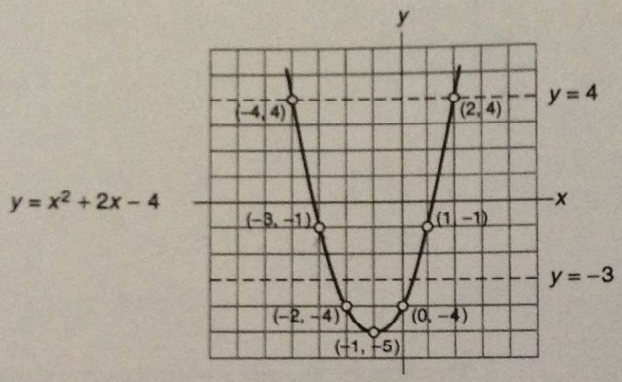


Figure 4.11

Next find the values of x where the curve crosses the line $y = -3$. From the graph in Fig. 4.11, the approximate x-values are 0.4 and -2.4. That is, for $y = -3$, the solutions of $y = x^2 + 2x - 4$ are approximately $x = 0.4$ and $x = -2.4$.

Note that $y = x^2 + 2x - 4$ has no solutions for $y = -7$.

EXAMPLE 11

The voltage, V in volts, in a given circuit varies with time, t in ms, according to the equation $V = 6t^2 + t$. Solve for t when $V = 0$, 70, and 120.

t	V
0	0
1	7
2	26
3	57
4	100
5	155

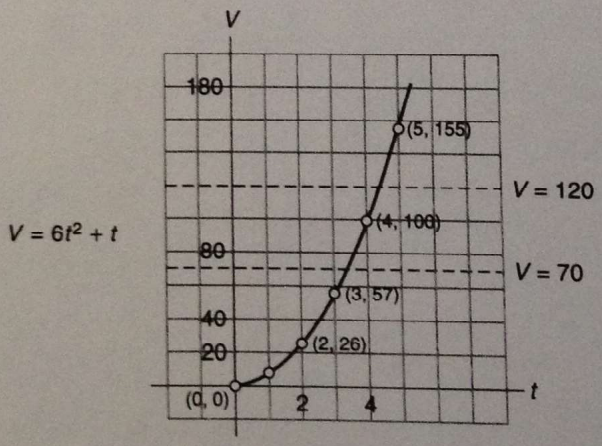


Figure 4.12

In Fig. 4.12, we used the following scales.

t: 1 square = 1 ms

V: 1 square = 20 V

From the graph,

at $V = 0$ V, $t = 0$ ms
at $V = 70$ V, $t = 3.3$ ms
at $V = 120$ V, $t = 4.4$ ms

Negative values of time, t, are not meaningful in this example.

EXAMPLE 12

The work, w, done in a circuit varies with time, t, according to the equation $w = 8t^2 + 4t$. Solve for t when $w = 60$, 120, and 250.

t	w
0	0
1	12
2	40
3	84
4	144
5	220
6	312

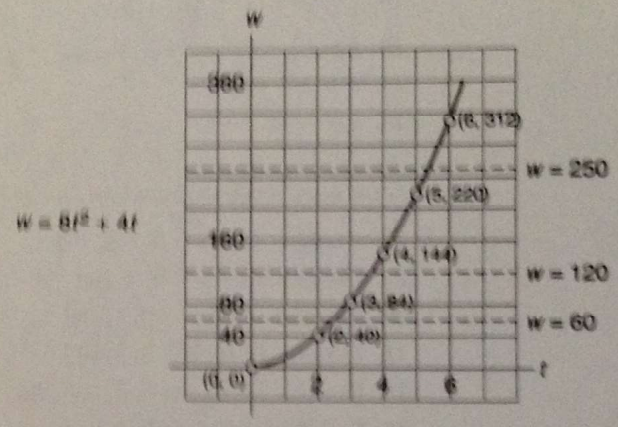


Figure 4.13

In Fig. 4.13, we used these scales

t: 1 square = 1 unit
w: 1 square = 40 units

From the graph,

at $w = 60$, $t = 2.5$
at $w = 120$, $t = 3.6$
at $w = 250$, $t = 5.4$

Exercises 4.2

1. Plot the point corresponding to each ordered pair in the number plane.

$A(2, 3)$	$B(-3, -6)$	$C(-4, 0)$
$D(2, -1)$	$E(3, 4)$	$F(0, 3)$
$G(0, -3)$	$H(5, 0)$	$I(-4, -1)$
$J(-2, 2)$	$K(3, -5)$	$L(-3, 4)$

2. Plot the point corresponding to each ordered pair in the number plane.

$P_1(3, 2)$	$P_2(-2, 5)$	$P_3(-1, -6)$
$P_4(4, -7)$	$P_5(0, 4)$	$P_6(-5, 0)$
$P_7(-4, 4)$	$P_8(6, 0)$	$P_9(-8, 5)$
$P_{10}(7, -2)$	$P_{11}(-5, -3)$	$P_{12}(0, -8)$

Day: Date:

Graph each equation.

3. $y = 2x + 1$
4. $y = 3x - 4$
5. $-2x - 3y = 6$
6. $2y = -4x - 3$
7. $y = x^2 - 9$
8. $y = x^2 + x - 6$
9. $y = x^2 - 5x + 4$
10. $y = x^2 + 3$
11. $y = 2x^2 + 3x - 2$
12. $y = -x^2 + 2x + 4$
13. $y = x^2 + 2x$
14. $y = x^2 - 4x$
15. $y = -2x^2 + 4x$
16. $y = -\frac{1}{4}x^2 - \frac{3}{2}x + 2$
17. $y = x^3 - x^2 - 10x + 8$
18. $y = x^3 - 4x^2 + x + 6$
19. $y = x^3 + 2x^2 - 7x + 4$
20. $y = x^3 - 8x - 3$
21. $y = \sqrt{x + 4}$
22. $y = \sqrt{3x - 12}$
23. $y = \sqrt{12 - 6x}$
24. $y = \sqrt{3 - x}$

Solve each equation graphically for the given values.

25. Exercise 7 for $y = 0, -5$, and 2.
26. Exercise 8 for $y = 0, 6$, and -3.
27. Exercise 9 for $y = 0, 2$, and -4.
28. Exercise 10 for $y = 0, 4$, and 6.
29. Exercise 11 for $y = 0, 3$, and 5.
30. Exercise 12 for $y = 0, 4$, and -2.
31. Exercise 13 for $y = 0, 3$, and 6.
32. Exercise 14 for $y = 0, -2$, and 3.
33. Exercise 15 for $y = 0, 5, -4$, and $-1\frac{1}{2}$.
34. Exercise 16 for $y = 0, -1$, and 1.5.
35. Exercise 17 for $y = 0, 2$, and -2.
36. Exercise 18 for $y = 0, 2$, and 8.
37. Exercise 19 for $y = 0, 4$, and 8.
38. Exercise 20 for $y = 0, 2$, and -3.

Solve each equation graphically for the given values.

39. $y = x^2 + 3x - 4$ for $y = 0, 6$, and -2.
40. $y = 2x^2 - 5x - 3$ for $y = 2, 0$, and -3.
41. $y = -\frac{1}{2}x^2 + 2$ for $y = 0, 4$, and -4.
42. $y = -\frac{1}{4}x^2 + x$ for $y = 0, \frac{1}{2}$, and -4.
43. $y = x^3 - 3x^2 + 1$ for $y = 0, -2$, and -0.5.
44. $y = -x^3 + 3x + 2$ for $y = 2, 0$, and 3.
45. The resistance, r, of a resistor in a circuit of constant current varies with time, t in ms, according to the equation

$$r = 10t^2 + 20$$

Solve for t when $r = 90\ \Omega$, $180\ \Omega$, and $320\ \Omega$.

46. An object dropped from an airplane 2500 m above the ground falls according to the equation $h = 2500 - 4.95t^2$, where h is the height in metres above the ground and t is the time in seconds. Find the times for the object to fall to a height of 2000 m, 1200 m, and 600 m above the ground. Also find the time it takes to hit the ground.
47. The energy dissipated (work lost), w, by a resistor varies with the time, t in ms, according to the equation $w = 5t^2 + 6t$. Solve for t when $w = 2, 4$, and 10.

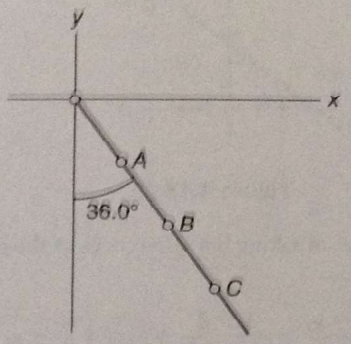
48. The resistance r in ohms, in a given circuit varies with time, t in ms, according to the equation $r = 10 + \sqrt{t}$. Find t when $r = 14.1\ \Omega$, $14.3\ \Omega$, and $14.7\ \Omega$. (*Hint:* Choose a suitable scale for the graph and graph only the part you need.)

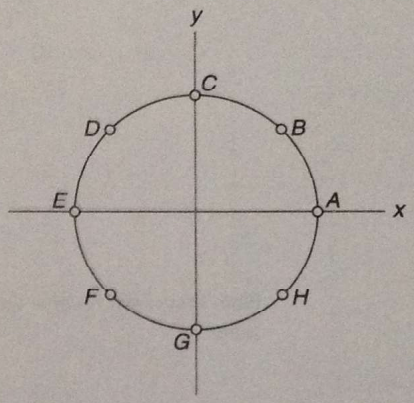
49. A given inductor carries a current expressed by the equation $i = t^3 - 15$ where i is the current in amperes and t is the time in seconds. Find t when i is 5 A and 15 A.

50. A charge, q in coulombs, flowing in a given circuit varies with the time, t in μs, according to the equation $q = t^2 - \frac{t^3}{3}$. Find t when q is 5 coulombs (C) and 10 C.

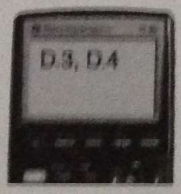
51. A machinist needs to drill four holes 2.00 in. apart in a straight line in a metal plate as shown below. The first hole is placed at the origin, and the line forms an angle of 36.0° with the vertical axis. Find the coordinates of the other three holes.

52. A machinist often uses a coordinate system to drill holes by placing the origin at the most convenient location. A bolt circle is the circle formed by completing an arc through the centers of the bolt holes in a piece of metal. Find the coordinates of the centers of eight equally spaced, $\frac{1}{2}$-in. holes on a bolt circle of radius 4.00 in. as shown below.





4.3 THE STRAIGHT LINE



Analytic geometry is the study of the relationships between aigebra and geometry. The concepts of analytic geometry provide us with ways of algebraically analyzing a geometrical problem. Likewise, with these concepts we can often solve an algebraic problem by viewing it geometrically.

We now develop several basic relations between equations and their graphs. The **slope** of a nonvertical line is the ratio of the difference of the y-coordinates of any two points on the line to the difference of their x-coordinates when the differences are taken in the same order (see Fig. 4.14).

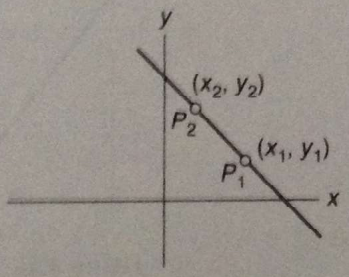


Figure 4.14

Day: Date:

SLOPE OF A LINE

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ represent any two points on a straight line, then the slope m of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

EXAMPLE 1

Find the slope of the line passing through $(-2, 1)$ and $(3, 5)$.

If we let $x_1 = -2$, $y_1 = 1$, $x_2 = 3$, and $y_2 = 5$ as in Fig. 4.15, then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 1}{3 - (-2)} = \frac{4}{5}$$

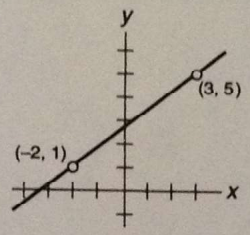


Figure 4.15

Note that if we reverse the order of taking the differences of the coordinates, the result is the same.

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{1 - 5}{-2 - 3} = \frac{-4}{-5} = \frac{4}{5} = m$$

EXAMPLE 2

Find the slope of the line passing through $(-2, 4)$ and $(6, -6)$.

If we let $x_1 = -2$, $y_1 = 4$, $x_2 = 6$, and $y_2 = -6$ as in Fig. 4.16, then

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6 - 4}{6 - (-2)} = \frac{-10}{8} = -\frac{5}{4}$$

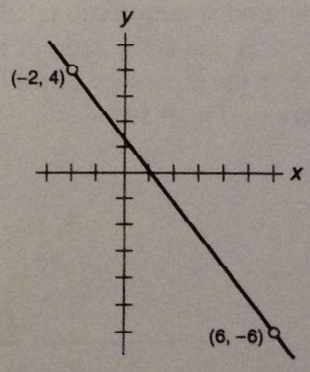


Figure 4.16

Note that in Example 1, the line slopes upward from left to right, while in Example 2 the line slopes downward. In general, we have the following:

1. If a line has positive slope, then the line slopes upward from left to right ("rises").
2. If a line has negative slope, then the line slopes downward from left to right ("falls").
3. If the line has zero slope, then the line is horizontal ("flat").
4. If the line is vertical, then the line has no slope as $x_1 = x_2$, or $x_2 - x_1 = 0$. In this case, the ratio $\frac{y_2 - y_1}{x_2 - x_1}$ is undefined because division by zero is undefined.

We can use these facts to assist us in graphing a line if we know the slope of the line and one point P on the line. The line can be sketched by drawing a line through the given point P and a point Q which is plotted by moving one unit to the right of P, then moving vertically m units. That is, a point moving along a line will move vertically an amount equal to m, the slope, for every unit move to the right as in Fig. 4.17.

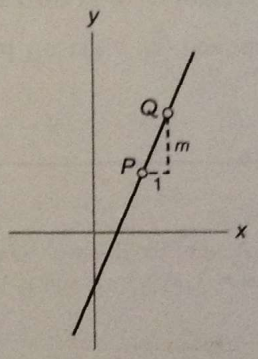


Figure 4.17

EXAMPLE 3

Graph a line with slope -2 that passes through the point $(1, 3)$.

Since the slope is -2, points on the line drop 2 units for every unit move to the right. The line passes through $(1, 3)$ and $(2, 1)$ as in Fig. 4.18.

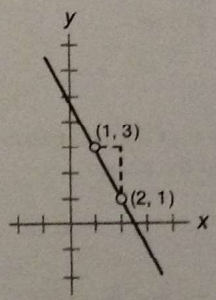


Figure 4.18

Knowing the slope and one point on the line will also determine the equation of the straight line. Let m be the slope of a given nonvertical straight line, and let (x_1, y_1) be the

coordinates of a point on this line. If (x, y) is any other point on the line as in Fig. 4.19, then we have

$$\frac{y - y_1}{x - x_1} = m$$

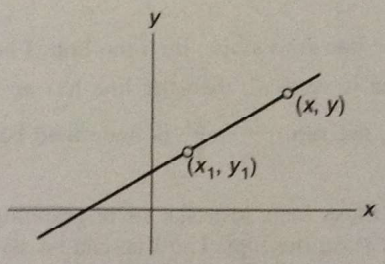


Figure 4.19

By multiplying each side of the equation by $(x - x_1)$, we obtain the following.

POINT-SLOPE FORM OF A STRAIGHT (OBLIQUE) LINE

If m is the slope and (x_1, y_1) is any point on a nonvertical straight line, its equation is

$$y - y_1 = m(x - x_1)$$

EXAMPLE 4

Find the equation of the line with slope 3 that passes through the point $(-1, 2)$.

Here $m = 3$, $x_1 = -1$, and $y_1 = 2$. Using the point-slope form, we have the equation

$$y - y_1 = m(x - x_1)$$
$$y - 2 = 3(x - (-1))$$

Simplifying, we have

$$y - 2 = 3x + 3$$
$$y = 3x + 5$$

The point-slope form can also be used to find the equation of a straight line that passes through two points.

EXAMPLE 5

Find the equation of the line passing through the points $(2, -3)$ and $(-2, 5)$.

First, find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-3)}{-2 - 2} = \frac{5 + 3}{-4} = \frac{8}{-4} = -2$$

Substitute $m = -2$ and the point $(2, -3)$ in the point-slope form.

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = -2(x - 2)$$
$$y + 3 = -2x + 4$$
$$2x + y - 1 = 0$$

Note: We could have used the other point $(-2, 5)$ in the point-slope form to obtain the equation

$$y - 5 = -2(x - (-2))$$

which also simplifies to

$$2x + y - 1 = 0$$

A nonvertical line will intersect the y-axis at some point in the form $(0, b)$ as in Fig. 4.20. This ordinate (y-value) b is called the **y-intercept** of the line. If the slope of the line is m, then

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - b &= m(x - 0) \\ y - b &= mx \\ y &= mx + b \end{aligned}$$

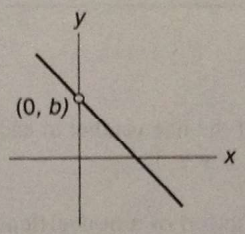


Figure 4.20

SLOPE-INTERCEPT FORM OF A STRAIGHT (OBLIQUE) LINE

If m is the slope and $(0, b)$ is the y-intercept of a nonvertical straight line, its equation is

$$y = mx + b$$

EXAMPLE 6

Find the equation of the line with slope $\frac{1}{2}$ that crosses the y-axis at $b = -3$.

Using the slope-intercept form, we have

$$\begin{aligned} y &= mx + b \\ y &= \frac{1}{2}x + (-3) \\ y &= \frac{1}{2}x - 3 \end{aligned}$$

or

$$x - 2y - 6 = 0$$

A line parallel to the x-axis has slope $m = 0$ (see Fig. 4.21). Its equation is

$$\begin{aligned} y &= mx + b \\ y &= (0)x + b \\ y &= b \end{aligned}$$

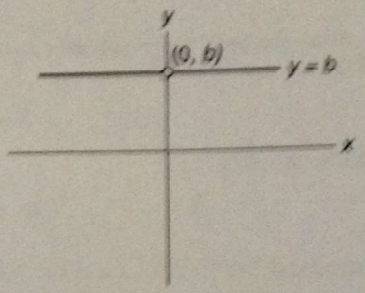


Figure 4.21 Horizontal line

EQUATION OF A HORIZONTAL LINE

If a horizontal line passes through the point (a, b), its equation is

$$y = b$$

EXAMPLE 7

Find the equation of the line parallel to and 3 units above the x-axis.

The equation is $y = 3$.

By writing the equation of a nonvertical straight line in the slope-intercept form, we can quickly determine the line's slope and a point on the line (the point where it crosses the y-axis).

EXAMPLE 8

Find the slope and the y-intercept of $3y - x + 6 = 0$. Graph the line.

Write the equation in slope-intercept form; that is, solve for y.

$$3y - x + 6 = 0$$
$$3y = x - 6$$
$$y = \frac{x}{3} - 2$$
$$y = \left(\frac{1}{3}\right)x + (-2)$$

So $m = \frac{1}{3}$ and $b = -2$ (see Fig. 4.22).

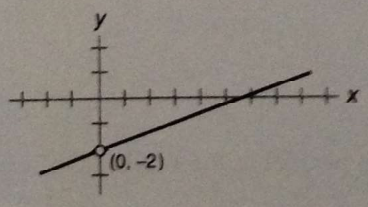


Figure 4.22

EXAMPLE 9

Describe and graph the line whose equation is $y = -5$.

This is a line parallel to and 5 units below the x-axis (see Fig. 4.23).

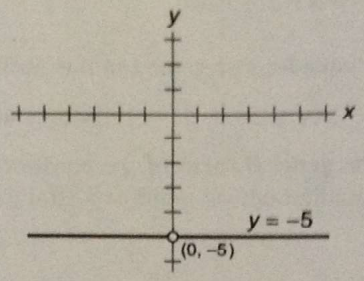


Figure 4.23

If a line is vertical, then we cannot use any of these equations since the line has no slope. However, note that in this case, as shown in Fig. 4.24, the line crosses the x-axis at some point in the form $(a, 0)$. All points on the line have the same abscissa as the point $(a, 0)$. This characterizes the line, giving us the following equation.

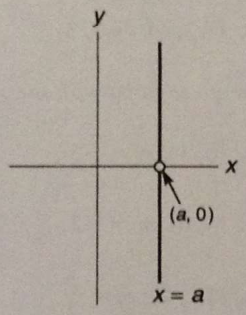


Figure 4.24 Vertical Line

EQUATION OF A VERTICAL LINE

If a vertical line passes through the point (a, b), its equation is

$$x = a$$

EXAMPLE 10

Describe and graph the line whose equation is $x = 2$.

This is a line perpendicular to the x-axis that crosses the x-axis at the point (2, 0) (see Fig. 4.25).

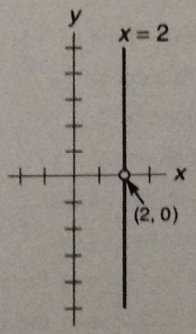


Figure 4.25

EXAMPLE 11

Write the equation of the line perpendicular to the x-axis that crosses the x-axis at the point $(-3, 0)$.

The equation is $x = -3$.

Note: All the equations presented in this section can be put in the form

$$Ax + By + C = 0 \quad \text{with } A \text{ and } B \text{ not both } 0$$

This is known as the **general form** of the equation of the line and agrees with our definition of a linear equation whose graph is a straight line.

Exercises 4.3

Find the slope of each line passing through the given points.

1. $(4, 2), (3, 1)$ **2.** $(-3, 2), (-1, -2)$ **3.** $(4, -5), (2, 3)$

4. $(-6, -4), (5, -3)$ **5.** $(-3, 2), (6, 2)$ **6.** $(4, -7), (4, 3)$

7. $(5, 7), (-3, 2)$ **8.** $(-3, 6), (-1, 3)$

Graph each line passing through the given point with the given slope.

9. $(2, -1)$, $m = 2$ **10.** $(0, 1)$, $m = -3$ **11.** $(-3, -2)$, $m = \frac{1}{2}$

12. $(4, 4)$, $m = -\frac{1}{3}$ **13.** $(4, 0)$, $m = -2$ **14.** $(-3, 1)$, $m = 4$

15. $(0, -3)$, $m = -\frac{3}{4}$ **16.** $(5, -2)$, $m = \frac{3}{2}$

Find the equation of the line with the given properties.

17. Passes through $(-2, 8)$ with slope of -3.

18. Passes through $(3, -5)$ with slope of 2.

19. Passes through $(-3, -4)$ with slope of $\frac{1}{2}$.

20. Passes through $(6, -7)$ with slope of $-\frac{3}{4}$.

21. Passes through $(-2, 7)$ and $(1, 4)$.

22. Passes through $(1, 6)$ and $(4, -3)$.

23. Passes through $(6, -8)$ and $(-4, -3)$.

24. Passes through $(-2, 2)$ and $(7, -1)$.

25. Crosses the y-axis at -2 with slope of -5.

26. Crosses the y-axis at 8 with slope of $\frac{1}{3}$.

27. Has y-intercept of 7 and slope of 2.

28. Has y-intercept of -4 and slope of $-\frac{3}{4}$.

29. Parallel to and 5 units above the x-axis.

30. Parallel to and 2 units below the x-axis.

31. Perpendicular to the x-axis and crosses the x-axis at $(-2, 0)$.

32. Perpendicular to the x-axis and crosses the x-axis at $(5, 0)$.

33. Parallel to the x-axis containing the point $(2, -3)$.
34. Parallel to the y-axis containing the point $(-5, -4)$.
35. Perpendicular to the x-axis containing the point $(-7, 9)$.
36. Perpendicular to the y-axis containing the point $(4, 6)$.

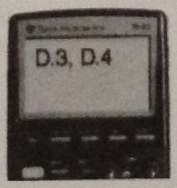
Find the slope and the y-intercept of each straight line.

37. $x + 4y = 12$
38. $-2x + 3y + 9 = 0$
39. $6x - 2y + 14 = 0$
40. $3x - 6y = 0$
41. $y = 6$
42. $x = -4$

Graph each equation.

43. $y = 3x - 2$
44. $y = -2x + 5$
45. $5x - 2y + 4 = 0$
46. $4x + 3y + 6 = 0$
47. $x = 7$
48. $x = -2$
49. $y = -3$
50. $y = 2$
51. $6x + 8y = 24$
52. $3x - 5y = 30$
53. $x - 3y = -12$
54. $x + 6y = 8$
55. A certain metal rod with temperature $-15.0°C$ is 43.0 cm long and at $55.0°C$ is 43.2 cm long. These data can be listed in (x, y) form as $(-15.0, 43.0)$ and $(55.0, 43.2)$. Find the slope (as a simplified fraction) of the straight line passing through these two points.

4.4 PARALLEL AND PERPENDICULAR LINES



PARALLEL LINES

Two lines are parallel if either one of the following conditions holds:

1. They are both perpendicular to the x-axis [see Fig. 4.26(a)].
2. They both have the same slope [see Fig. 4.26(b)]. That is, if the equations of the lines are

$$L_1: \quad y = m_1x + b_1 \quad \text{and} \quad L_2: \quad y = m_2x + b_2$$

then

$m_1 = m_2$

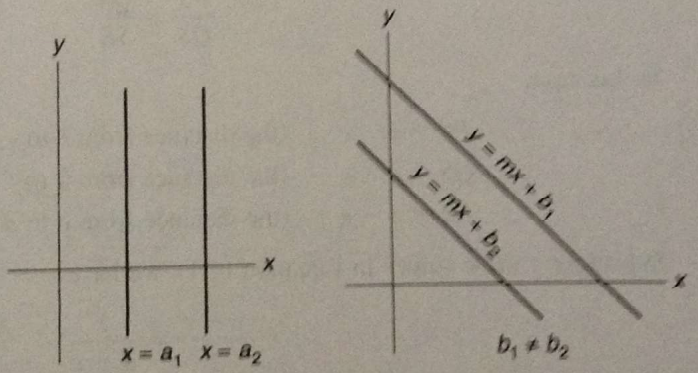


Figure 4.26

Compute slopes m_1 and m_2 as follows:

$$m_1 = \frac{f-0}{e-d} = \frac{f}{e-d}$$

$$m_2 = \frac{f-0}{e-c} = \frac{f}{e-c}$$

$$(m_1)(m_2) = \frac{f}{e-d} \cdot \frac{f}{e-c} = \frac{f^2}{(e-d)(e-c)}$$

Substituting from Equation 4.2 we have

$$(m_1)(m_2) = \frac{(d-e)(e-c)}{(e-d)(e-c)} = \frac{d-e}{e-d} = -\left(\frac{e-d}{e-d}\right) = -1$$

or

$$(m_1)(m_2) = -1$$

Dividing each side of this equation by m_2 we have

$$m_1 = \frac{-1}{m_2}$$

EXAMPLE 1

Determine whether the lines given by the equations $3y + 6x - 5 = 0$ and $2y - x + 7 = 0$ are perpendicular.

Change each equation into slope-intercept form; that is, solve for y.

$$y = -2x + \frac{5}{3} \qquad \text{(Slope is } -2.\text{)}$$

and

$$y = \frac{1}{2}x - \frac{7}{2} \qquad \text{(Slope is } \frac{1}{2}.\text{)}$$

Since

$$-2 = \frac{-1}{\frac{1}{2}} \qquad \left(-2 \text{ is the negative reciprocal of } \frac{1}{2}.\right)$$

the lines are perpendicular.

EXAMPLE 2

Find the equation of the line through $(-3, 2)$ and perpendicular to $2y - 3x + 5 = 0$.

We can find the slope of the desired line by finding the negative reciprocal of the slope of the given line. First find the slope of the line $2y - 3x + 5 = 0$. Writing this equation in slope-intercept form, we have

$$y = \frac{3}{2}x - \frac{5}{2}$$

The slope of this line is $m = \frac{3}{2}$. The slope of the line perpendicular to this line is then equal to $-\frac{2}{3}$, the negative reciprocal of $\frac{3}{2}$. Now using the point-slope form, we have

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{2}{3}(x - (-3))$$

$$y - 2 = -\frac{2}{3}(x + 3)$$

or

$$2x + 3y = 0$$

EXAMPLE 3

Find the equation of the line through $(2, -5)$ and parallel to $3x + y = 7$.

First, find the slope of the given line by solving its equation for y.

$$y = -3x + 7$$

Its slope is -3. The slope of any line parallel to this line has the same slope. Now, write the equation of the line with slope -3 passing through $(2, -5)$.

$$y - y_1 = m(x - x_1)$$
$$y - (-5) = -3(x - 2)$$
$$y + 5 = -3x + 6$$
$$y = -3x + 1 \quad \text{or} \quad 3x + y = 1$$

Exercises 4.4

Determine whether each given pair of equations represents lines that are parallel, perpendicular, or neither.

1. $x + 3y - 7 = 0$; $-3x + y + 2 = 0$

2. $x + 2y - 11 = 0$; $x + 2y + 4 = 0$

3. $-x + 4y + 7 = 0$; $x + 4y - 5 = 0$

4. $2x + 7y + 4 = 0$; $7x - 2y - 5 = 0$

5. $y - 5x + 13 = 0$; $y - 5x + 9 = 0$

6. $-3x + 9y + 22 = 0$; $x + 3y - 17 = 0$

Find the equation of the line that satisfies each set of conditions.

7. Passes through $(-1, 5)$ and is parallel to $-2x + y + 13 = 0$.

8. Passes through $(2, -2)$ and is perpendicular to $3x - 2y - 14 = 0$.

9. Passes through $(-7, 4)$ and is perpendicular to $5y = x$.

10. Passes through $(2, -10)$ and is parallel to $2x + 3y - 7 = 0$.

11. Passes through the origin and is parallel to $3x - 4y = 12$.

12. Passes through the origin and is perpendicular to $4x + 5y = 17$.

13. Has an x-intercept of 6 and is perpendicular to $4x + 6y = 9$.

14. Has a y-intercept of -2 and is parallel to $6x - 4y = 11$.

15. Has a y-intercept of 8 and is parallel to $y = 2$.

16. Has an x-intercept of -4 and is perpendicular to $y = 6$.

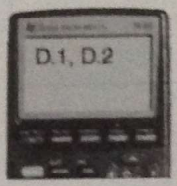
17. Has an x-intercept of 7 and is parallel to $x = -4$.

18. Has a y-intercept of -9 and is perpendicular to $x = 5$.

19. The vertices of a quadrilateral are $A(-2, 3)$, $B(2, 2)$, $C(9, 6)$, and $D(5, 7)$.
   (a) Is the quadrilateral a parallelogram? Why or why not?
   (b) Is the quadrilateral a rectangle? Why or why not?

20. The vertices of a quadrilateral are $A(-4, 1)$, $B(0, -2)$, $C(6, 6)$, and $D(2, 9)$.
   (a) Is the quadrilateral a parallelogram? Why or why not?
   (b) Is the quadrilateral a rectangle? Why or why not?

4.5 THE DISTANCE AND MIDPOINT FORMULAS



We now wish to find the distance between two points on a straight line. Suppose P has the coordinates (x_1, y_1) and Q has the coordinates (x_2, y_2). Then a triangle similar to that in Fig. 4.28 can be constructed. Note that R must have the coordinates (x_2, y_1). (Point R has the same x-coordinate as Q and the same y-coordinate as P.)

Using the Pythagorean theorem, we have

$$PQ^2 = PR^2 + QR^2 \tag{4.3}$$

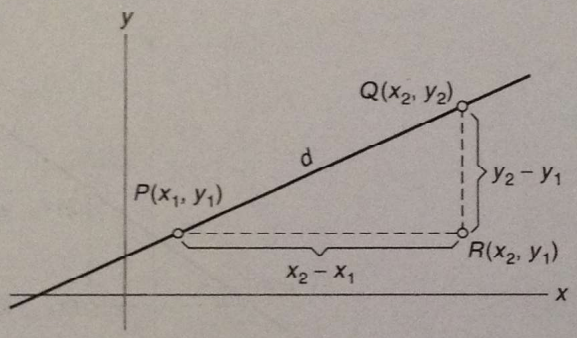


Figure 4.28 d is the distance between points P and Q.

Observe that

$PR = x_2 - x_1$ (the horizontal distance between x_1 and x_2 on the x-axis)

$QR = y_2 - y_1$ (the vertical distance between y_1 and y_2 on the y-axis)

Substituting these values for PR and QR in Equation (4.3) gives

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

DISTANCE FORMULA

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula

$$d = PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE 1

Find the distance, d, between $(3, 4)$ and $(-2, 7)$.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(-2 - 3)^2 + (7 - 4)^2}$$
$$= \sqrt{(-5)^2 + (3)^2} = \sqrt{25 + 9} = \sqrt{34}$$

Note that we can reverse the order of $(3, 4)$ and $(-2, 7)$ in the formula for computing d without affecting the result.

$$d = \sqrt{(3 - (-2))^2 + (4 - 7)^2}$$
$$= \sqrt{(5)^2 + (-3)^2} = \sqrt{25 + 9} = \sqrt{34}$$

MIDPOINT FORMULA

The coordinates of point $Q(x_m, y_m)$ which is midway between two points $P(x_1, y_1)$ and $R(x_2, y_2)$ are given by

$$x_m = \frac{x_1 + x_2}{2} \qquad y_m = \frac{y_1 + y_2}{2}$$

Figure 4.29 illustrates the midpoint formula. First look at points P, Q, and R. Triangles PSQ and QTR are congruent. This means that

$$PS = QT$$

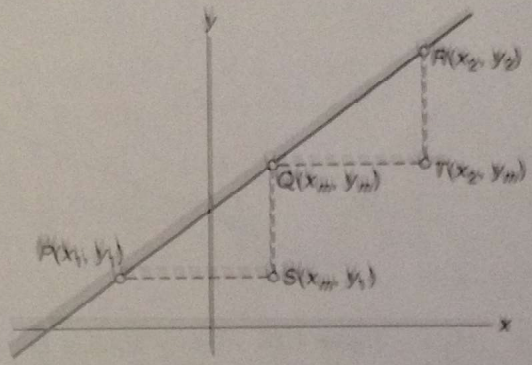


Figure 4.29 Point Q is the midpoint between points P and R.

Since

$$PS = x_m - x_1$$

and

$$QT = x_2 - x_m$$

then

$$x_m - x_1 = x_2 - x_m$$

or

$$2x_m - x_1 = x_2$$
$$2x_m = x_1 + x_2$$
$$x_m = \frac{x_1 + x_2}{2}$$

The formula for y_m is found in the same manner.

EXAMPLE 2

Find the point midway between $(2, -3)$ and $(-4, 6)$.

$$x_m = \frac{x_1 + x_2}{2} = \frac{2 + (-4)}{2} = \frac{-2}{2} = -1$$

$$y_m = \frac{y_1 + y_2}{2} = \frac{-3 + 6}{2} = \frac{3}{2}$$

The midpoint is $(-1, \frac{3}{2})$.

Exercises 4.5

Find the distance between each pair of points.

1. $(4, -7)$; $(-5, 5)$ **2.** $(4, 3)$; $(-2, -1)$ **3.** $(3, -2)$; $(10, -2)$

4. $(6, -2)$; $(6, 4)$ **5.** $(5, -2)$; $(1, 2)$ **6.** $(2, -3)$; $(-1, 1)$

7. $(3, -5)$; $(3, 2)$ **8.** $(2, -4)$; $(6, -4)$

Find the coordinates of the point midway between each pair of points.

9. $(2, 3)$; $(5, 7)$ **10.** $(0, 5)$; $(2, -4)$ **11.** $(3, -2)$; $(0, 0)$

12. $(2, -3)$; $(4, -3)$ **13.** $(11, 4)$; $(-11, -9)$ **14.** $(4, 10)$; $(-6, -8)$

The vertices of each $\triangle ABC$ are given below. For each triangle, find **(a)** *the perimeter,* **(b)** *whether it is a right triangle,* **(c)** *whether it is isosceles, and* **(d)** *its area if it is a right triangle.*

15. $A(2, 8)$; $B(10, 2)$; $C(10, 8)$ **16.** $A(0, 0)$; $B(3, 3)$; $C(3, -3)$

17. $A(-3, 6)$; $B(5, 0)$; $C(4, 9)$ **18.** $A(-6, 3)$; $B(-3, 7)$; $C(1, 4)$

19. Given $\triangle ABC$ with vertices $A(7, -1)$, $B(9, 1)$, and $C(-3, 5)$, find the distance from A to the midpoint of side BC.

20. Find the distance from B to the midpoint of side AC in Exercise 19.

21. Find the equation of the line parallel to the line $3x - 6y = 10$ and through the midpoint of AB, where $A(4, 2)$ and $B(8, -6)$.

22. Find the equation of the line perpendicular to the line $2x + 5y = 12$ and through the midpoint of AB, where $A(-3, -4)$ and $B(7, -8)$.

23. Find the equation of the line perpendicular to the line $4x + 8y = 16$ and through the midpoint of AB, where $A(-8, 12)$ and $B(6, 10)$.

24. Find the equation of the line parallel to the line $5x - 6y = 30$ and through the midpoint of AB, where $A(3, 11)$ and $B(7, -5)$.

In Exercises 25–28, start with a graph and then use the distance formula and the slopes to confirm the given geometric figure.

25. Show that $ABCD$ is a rectangle, where $A(-2, 2)$; $B(1, 3)$; $C(2, 0)$; and $D(-1, -1)$.

26. Show that $ABCD$ is a parallelogram, where $A(2, 6)$; $B(7, 2)$; $C(8, 5)$; and $D(3, 9)$.

27. Show that $ABCD$ is a trapezoid with one right angle, where $A(-12, 8)$; $B(3, 2)$; $C(5, 7)$; $D(-5, 11)$.

28. Show that if the coordinates of the vertices of a triangle are (a, b), $(a + c, b)$, and $(a + c, b + c)$, the triangle is a right triangle.

CHAPTER SUMMARY

1. *Basic terms:*
   (a) *Relation:* Set of ordered pairs, usually in the form (x, y).
   (b) *Independent variable:* First element of an ordered pair, usually x.
   (c) *Dependent variable:* Second element of an ordered pair, usually y.
   (d) *Domain:* Set of all first elements of the ordered pairs in a relation, or set of all x's.
   (e) *Range:* Set of all second elements of the ordered pairs in a relation, or set of all y's.
   (f) *Function:* Set of ordered pairs in which no two distinct ordered pairs have the same first element.
   (g) *Functional notation:* To write an equation in variables x and y in functional notation, solve for y and replace y with $f(x)$.
   (h) *Linear equation with two unknowns:* An equation of degree one in the form $ax + by = c$ where a and b are not both 0.
2. *Slope of a line:* If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ represent any two points on a line, then the slope m of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

   (a) If a line has positive slope, the line slopes upward from left to right.
   (b) If a line has negative slope, the line slopes downward from left to right.
   (c) If a line has zero slope, the line is horizontal.
   (d) If a line has no slope, the line is vertical.
3. *Point-slope form of a line:* If m is the slope and $P_1(x_1, y_1)$ is any point on a nonvertical line, its equation is

$$y - y_1 = m(x - x_1)$$

4. *Slope-intercept form of a line:* If m is the slope and $(0, b)$ is the y-intercept of a nonvertical line, its equation is

$$y = mx + b$$

5. *Equation of a horizontal line:* If a horizontal line passes through the point (a, b), its equation is

$$y = b$$

6. *Equation of a vertical line:* If a vertical line passes through the point (a, b), its equation is

$$x = a$$

7. *General form of the equation of a straight line:*

$$Ax + By + C = 0 \quad \text{where } A \text{ and } B \text{ are not both } 0$$

8. *Parallel lines:* Two lines are parallel if either one of the following conditions holds:
   (a) They are both perpendicular to the x-axis.
   (b) They both have the same slope. That is, if the equations of the lines are

$$L_1: \quad y = m_1x + b_1 \quad \text{and} \quad L_2: \quad y = m_2x + b_2$$

then

$$m_1 = m_2$$

9. *Perpendicular lines:* Two lines are perpendicular if either one of the following conditions holds:
   (a) One line is vertical with equation $x = a$, and the other is horizontal with equation $y = b$.
   (b) Neither is vertical and the slope of one line is the negative reciprocal of the other. That is, if the equations of the lines are

$$L_1: \quad y = m_1x + b_1 \quad \text{and} \quad L_2: \quad y = m_2x + b_2$$

then

$$m_1 = -\frac{1}{m_2}$$

10. *Distance formula:* The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula

$$d = PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

11. *Midpoint formula:* The coordinates of the point $Q(x_m, y_m)$ which is midway between two points $P(x_1, y_1)$ and $R(x_2, y_2)$ are given by

$$x_m = \frac{x_1 + x_2}{2} \quad \text{and} \quad y_m = \frac{y_1 + y_2}{2}$$

CHAPTER 4 REVIEW

Determine whether or not each relation is a function. Write its domain and its range.

1. $A = \{(2, 3), (3, 4), (4, 5), (5, 6)\}$
2. $B = \{(2, 6), (6, 4), (2, 1), (4, 3)\}$
3. $y = -4x + 3$
4. $y = x^2 - 5$
5. $x = y^2 + 4$
6. $y = \sqrt{4 - 8x}$
7. Given $f(x) = 5x + 14$, find
   (a) $f(2)$ **(b)** $f(0)$ **(c)** $f(-4)$
8. Given $g(t) = 3t^2 + 5t - 12$, find
   (a) $g(2)$ **(b)** $g(0)$ **(c)** $g(-5)$
9. Given $h(x) = \dfrac{4x^2 - 3x}{2\sqrt{x - 1}}$, find
   (a) $h(2)$ **(b)** $h(5)$ **(c)** $h(-15)$ **(d)** $h(1)$
10. Given $g(x) = x^2 - 6x + 4$, find
    (a) $g(a)$ **(b)** $g(2x)$ **(c)** $g(z - 2)$

Graph each equation.

11. $y = 4x + 5$
12. $y = x^2 + 4$
13. $y = x^2 + 2x - 8$
14. $y = 2x^2 + x - 6$
15. $y = -x^2 - x + 4$
16. $y = \sqrt{2x}$
17. $y = \sqrt{-2 - 4x}$
18. $y = x^3 - 6x$

Solve each graphically.

19. Exercise 12 for $y = 5$, 7, and 2.

20. Exercise 13 for $y = 0$, -2, and 3.

21. Exercise 15 for $y = 2$, 0, and -2.

22. Exercise 18 for $y = 0$, 2, and -3.

23. The current, i, in a given circuit varies with the time, t, according to $i = 2t^2$. Find t when $i = 2$, 6, and 8.

24. A given capacitor receives between its terminals a voltage, V, where $V = 4t^3 + t$ when t is in s. Find t when V is 40 and 60.

Use the points (3, -4) and (-6, -2) in Exercises 25–27.

25. Find the slope of the line through the two points.

26. Find the distance between the two points.

27. Find the coordinates of the point midway between the two points.

Find the equation of the line that satisfies each condition in Exercises 28–31.

28. Passes through (4, 7) and (6, -4).

29. Passes through (-3, 1) with a slope of $\frac{2}{3}$.

30. Crosses the y-axis at -3 with a slope of $-\frac{1}{3}$.

31. Is parallel to and 3 units to the left of the y-axis.

32. Find the slope and the y-intercept of $3x - 2y - 6 = 0$.

33. Graph $3x - 4y = 12$.

Using the slope and the intercept of each line, determine whether each given pair of equations represents lines that are parallel, perpendicular, or neither.

34. $x - 2y + 3 = 0$; $8x + 4y - 9 = 0$

35. $2x - 3y + 4 = 0$; $-8x + 12y = 16$

36. $3x - 2y + 5 = 0$; $2x - 3y + 9 = 0$

37. $x = 2$; $y = -3$

38. $x = 4$; $x = 7$

39. Find the equation of the line parallel to the line $2x - y + 4 = 0$ that passes through the point (5, 2).

40. Find the equation of the line perpendicular to the line $3x + 5y - 6 = 0$ that passes through the point (-4, 0).

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Chapter 10

Quadratic Equations

INTRODUCTION

A baseball player at bat pops a ball straight up. The ball's height, h, in feet, after t seconds may be expressed by $h = 64t - 16t^2$. Does the ball ever reach a height of 80 feet? How long is the ball in the air?

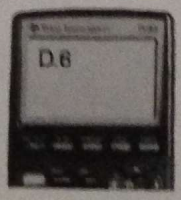
The formula that relates the height of the ball in t seconds is called a *quadratic equation*. Variable electric current, specification of box sizes, and the path of a projectile shot or thrown in the air can all be represented by a quadratic equation.

In this chapter we learn how to solve quadratic equations and how to interpret some of their characteristics. (The preceding problem is number 25 in the chapter review exercises.)

Objectives

- Solve quadratic equations by factoring.
- Solve quadratic equations by completing the square.
- Solve quadratic equations by using the quadratic formula.

7.1 SOLVING A QUADRATIC EQUATION BY FACTORING



A **quadratic equation** in one variable is an equation with at least one term of second degree and no other term of higher degree. A quadratic equation is usually expressed in the form $ax^2 + bx + c = 0$ $(a \neq 0)$.

In Chapter 4 equations, including quadratic equations, were solved graphically, which was an approximation method. Methods for obtaining exact solutions for quadratic equations are now presented.

One way to solve quadratic equations is by factoring. When using this method, we shall use the following principle.

> If $ab = 0$, then either $a = 0$ or $b = 0$, or both.

That is, if the product of two factors is zero, one or both of the factors must also be zero.

EXAMPLE 1

Solve $(x + 4)(x - 3) = 0$.

By the preceding principle,

$$x + 4 = 0 \quad \text{or} \quad x - 3 = 0$$

so

$$x = -4 \quad \text{or} \quad x = 3$$

SOLVING A QUADRATIC EQUATION BY FACTORING

1. If necessary, write the equation in the form $ax^2 + bx + c = 0$; that is, "set the equation equal to zero."
2. Factor the nonzero side of the equation.
3. Using the preceding principle, set each factor that contains a variable equal to zero.
4. Solve each resulting linear equation.
5. Check.

EXAMPLE 2

Solve $2x^2 = 7x + 15$.

Step 1: $2x^2 - 7x - 15 = 0$

Step 2: $(2x + 3)(x - 5) = 0$

Step 3: $2x + 3 = 0 \quad \text{or} \quad x - 5 = 0$

Step 4: $2x = -3$

$$x = -\frac{3}{2} \quad \text{or} \quad x = 5$$

Check:

Step 5:

$$2\left(-\frac{3}{2}\right)^2 = 7\left(-\frac{3}{2}\right) + 15 \qquad 2(5)^2 = 7(5) + 15$$

$$\frac{9}{2} = -\frac{21}{2} + 15 \qquad 50 = 35 + 15$$

$$\frac{9}{2} = \frac{9}{2} \qquad 50 = 50$$

EXAMPLE 3

Solve $6x^2 - 13x - 5 = 0$.

$$(3x + 1)(2x - 5) = 0$$
$$3x + 1 = 0 \quad \text{or} \quad 2x - 5 = 0$$
$$3x = -1 \qquad\qquad 2x = 5$$
$$x = -\frac{1}{3} \quad \text{or} \quad x = \frac{5}{2}$$

Check: Use the same procedure as in Example 2.

EXAMPLE 4

Solve $16x^2 + 4x = 0$.

$$4x(4x + 1) = 0$$
$$4x = 0 \quad \text{or} \quad 4x + 1 = 0$$
$$4x = -1$$
$$x = 0 \quad \text{or} \quad x = -\frac{1}{4}$$

EXAMPLE 5

Solve $4x^2 = 49$.

$$4x^2 - 49 = 0$$
$$(2x + 7)(2x - 7) = 0$$
$$2x + 7 = 0 \quad \text{or} \quad 2x - 7 = 0$$
$$2x = -7 \qquad\qquad 2x = 7$$
$$x = -\frac{7}{2} \quad \text{or} \quad x = \frac{7}{2}$$

SOLVING $ax^2 = c$

1. Divide each side by a.
2. Take the square root of each side.
3. Simplify the result, if possible.

EXAMPLE 6

Solve $2x^2 = 32$.

$$2x^2 = 32$$

Step 1: $x^2 = 16$

Step 2: $x = \pm\sqrt{16}$

Step 3: $x = \pm 4$

Note that in Step 2 we introduced the $\pm$ sign because there are **two** numbers whose square is 16, namely, 4 and -4.

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EXAMPLE 7

Solve $9y^2 - 13 = 7$.

$$9y^2 - 13 = 7$$
$$9y^2 = 20$$
$$y^2 = \frac{20}{9}$$
$$y = \pm\sqrt{\frac{20}{9}}$$
$$y = \pm\frac{2\sqrt{5}}{3}$$

You may wish to review simplifying radicals in Section 2.2.

Exercises 7.1

Solve each equation.

1. $(x + 4)(x - 7) = 0$

2. $(3x + 1)(x + 5) = 0$

3. $x^2 - 6x + 8 = 0$

4. $x^2 - 9x + 18 = 0$

5. $x^2 + 3x = 10$

6. $x^2 = x + 42$

7. $2x^2 = 3x + 9$

8. $2x^2 + 9x = 5$

9. $14x^2 + 17x + 5 = 0$

10. $36x^2 + 4 = 25x$

11. $18x^2 + 56 = 69x$

12. $14x^2 + 5x = 24$

13. $8x^2 + x = 0$

14. $9x^2 - 6x = 0$

15. $-7x^2 + 21x = 0$

16. $x - 2x^2 = 0$

17. $x^2 - 36 = 0$

18. $16 - x^2 = 0$

19. $16x^2 - 25 = 0$

20. $4x^2 = 9$

21. $5x^2 - 12 = 0$

22. $3x^2 - 8 = 0$

23. $4x^2 = 21$

24. $7x^2 = 4$

25. $5a^2 - 40 = 0$

26. $2c^2 - 36 = 0$

27. $2 = \frac{1}{3}x^2$

28. $\frac{1}{4} = 3x^2$

29. $30x^2 + 16x = 24$

30. $24x^2 + 24 = 50x$

31. $40x^2 + 100x + 40 = 0$

32. $27x^2 + 45x = 18$

33. $\frac{x - 1}{2x} = \frac{5}{x + 12}$

34. $\frac{3x - 1}{\frac{11}{2}} = \frac{2}{5 - x}$

35. $\frac{3x}{x - 4} = x + 2$

36. $\frac{3x + 16}{9x} = x + \frac{1}{3}$

37. $\frac{x + 1}{x - 2} + \frac{x - 1}{x + 1} = \frac{9}{2}$

38. $\frac{4}{x^2 - 1} + \frac{1}{x - 1} = \frac{7}{3}$

7.2 SOLVING QUADRATIC EQUATIONS BY COMPLETING THE SQUARE

Factoring is usually the easiest and fastest method of solving a quadratic equation *if* the equation can be factored. If the equation cannot be factored, then one of two methods is normally used. We shall study one of these methods, **completing the square,** in this section. The other method, the *quadratic formula,* is developed using the method of completing the square. This formula and its development are included in the next section.

SOLVING A QUADRATIC EQUATION BY COMPLETING THE SQUARE

1. The coefficient of the second-degree term *must* equal (positive) 1. If not, divide each side of the equation by its coefficient.
2. Write an equivalent equation with the variable terms on the left side and the constant term on the right side; that is, in the form $x^2 + px = q$.
3. Add the square of one-half of the coefficient of the first-degree term to each side; that is, $\left(\frac{1}{2}p\right)^2$.

$$x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4}$$

4. The left side is now a perfect square trinomial—as the method's name implies. Rewrite the left side as a square.

$$\left(x + \frac{p}{2}\right)^2 = q + \frac{p^2}{4}$$

5. Take the square root of each side.

$$x + \frac{p}{2} = \pm\sqrt{q + \frac{p^2}{4}}$$

6. Solve for the variable x and simplify, if possible.

$$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}$$

7. Check.

EXAMPLE 1

Solve $x^2 + 3x - 10 = 0$ by completing the square.

Step 1: The coefficient of x^2 is 1, so add 10 to each side.

Step 2:
$$x^2 + 3x = 10$$

Add the square of one-half of the coefficient of x, $\left(\frac{3}{2}\right)^2$ or $\frac{9}{4}$, to each side.

Step 3:
$$x^2 + 3x + \frac{9}{4} = 10 + \frac{9}{4}$$

Rewrite the left side as a square and simplify the right side.

$$\left(x+\frac{3}{2}\right)\left(x+\frac{3}{2}\right)=\frac{40}{4}+\frac{9}{4}$$

Step 4:
$$\left(x+\frac{3}{2}\right)^2=\frac{49}{4}$$

Take the square root of each side.

Step 5:
$$x+\frac{3}{2}=\pm\frac{7}{2}$$

Solve for x and simplify.

Step 6:
$$x=-\frac{3}{2}\pm\frac{7}{2}$$

Note: $x=-\frac{3}{2}\pm\frac{7}{2}$ is a short way of writing

$$x=-\frac{3}{2}+\frac{7}{2} \quad \text{or} \quad x=-\frac{3}{2}-\frac{7}{2}$$

That is, read the equation all the way through a first time by using only the + sign of the $\pm$ sign; then read it through a second time using only the − sign. Therefore,

$$x=\frac{4}{2}=2 \quad \text{or} \quad x=\frac{-10}{2}=-5$$

Check:

Step 7:

$(2)^2+3(2)-10=0$	$(-5)^2+3(-5)-10=0$
$4+6-10=0$	$25-15-10=0$
$0=0$	$0=0$

EXAMPLE 2

Solve $4x^2-24x+7=0$ by completing the square.

First, divide each side of the equation by the coefficient of x^2, which is 4.

$$x^2-6x+\frac{7}{4}=0$$

Add $-\frac{7}{4}$ to each side.

$$x^2-6x=-\frac{7}{4}$$

Add the square of one-half of the coefficient of x, $\left(\frac{-6}{2}\right)^2$ or 9, to each side.

$$x^2-6x+9=-\frac{7}{4}+9$$

Rewrite the left side as a perfect square and simplify the right side.

$$(x-3)^2=\frac{29}{4}$$

Take the square root of each side.

$$x - 3 = \pm\frac{\sqrt{29}}{2}$$

Solve for x and simplify.

$$x = 3 \pm \frac{\sqrt{29}}{2}$$

Or if we combine the terms by using the lowest common denominator,

$$x = \frac{6 \pm \sqrt{29}}{2}$$

EXAMPLE 3

Solve $-2x^2 + 3x + 1 = 0$ by completing the square.

$$x^2 - \frac{3}{2}x - \frac{1}{2} = 0 \qquad \text{(Divide each side by } -2.\text{)}$$

$$x^2 - \frac{3}{2}x = \frac{1}{2}$$

$$x^2 - \frac{3}{2}x + \frac{9}{16} = \frac{1}{2} + \frac{9}{16} \qquad \left[\textit{Note: } \left(\frac{1}{2} \cdot -\frac{3}{2}\right)^2 = \frac{9}{16}\right]$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$$

$$x - \frac{3}{4} = \pm\frac{\sqrt{17}}{4}$$

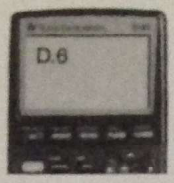
$$x = \frac{3}{4} \pm \frac{\sqrt{17}}{4} \quad \text{or} \quad x = \frac{3 \pm \sqrt{17}}{4}$$

Exercises 7.2

Solve each equation by completing the square.

1. $x^2 + 6x - 16 = 0$
2. $x^2 + 15x + 54 = 0$
3. $x^2 + 35 = 12x$
4. $x^2 = 6x + 27$
5. $2x^2 + x = 1$
6. $2x^2 + 3 = 7x$
7. $25x^2 = 15x + 18$
8. $3x^2 + 17x + 20 = 0$
9. $x^2 + 4x - 7 = 0$
10. $x^2 - 5x + 3 = 0$
11. $2x^2 + 2x - 9 = 0$
12. $5x^2 + 4x = 1$
13. $3x^2 + 9x + 5 = 0$
14. $-3x^2 - 5x + 4 = 0$
15. $-2x^2 + 3x + 9 = 0$
16. $-5x^2 + 8x - 2 = 0$
17. $4x^2 + 11x = 3$
18. $3a^2 + 5a = 2$
19. $6m^2 + 15m + 3 = 0$
20. $8y^2 - 7y - 10 = 0$

7.3 THE QUADRATIC FORMULA



Instead of using the method of completing the square for each quadratic equation, we may use the method to solve the general quadratic equation ($ax^2 + bx + c = 0$) and obtain a formula for solving any quadratic equation in that form. Let's now solve

$$ax^2 + bx + c = 0$$

by the method of completing the square.

First, divide each side by a, the coefficient of x^2.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Add $-\frac{c}{a}$ to each side.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Add the square of one-half of the coefficient of x, $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ or $\frac{b^2}{4a^2}$, to each side.

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

Rewrite the left side as a perfect square and simplify the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of each side.

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

Solve for x by adding $-\frac{b}{2a}$ to each side, and simplify.

SOLVING A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To solve a quadratic equation by using the quadratic formula, we need only to identify a, b, and c from $ax^2 + bx + c = 0$ and substitute those values in the preceding quadratic formula.

EXAMPLE 1

Solve $x^2 + 2x - 35 = 0$ using the quadratic formula.

First, identify a, b, and c.

$$a = 1$$
$$b = 2$$
$$c = -35$$

Then substitute these values in the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-35)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 + 140}}{2}$$

$$= \frac{-2 \pm \sqrt{144}}{2}$$

$$= \frac{-2 \pm 12}{2}$$

$$x = \frac{-2 + 12}{2} \quad \text{or} \quad x = \frac{-2 - 12}{2}$$

$$x = 5 \quad \text{or} \quad x = -7$$

Check:

$$(5)^2 + 2(5) - 35 = 0$$
$$25 + 10 - 35 = 0$$
$$0 = 0$$

$$(-7)^2 + 2(-7) - 35 = 0$$
$$49 - 14 - 35 = 0$$
$$0 = 0$$

EXAMPLE 2

Solve $3x^2 = 4x + 8$ using the quadratic formula.

First, identify a, b, and c. To find a, b, and c, the equation *must* be in the form

$$ax^2 + bx + c = 0$$

That is,

$$3x^2 - 4x - 8 = 0$$

Therefore, $a = 3$, $b = -4$, and $c = -8$. Second, substitute these values in the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-8)}}{2(3)}$$

$$= \frac{4 \pm \sqrt{16 + 96}}{6}$$

$$= \frac{4 \pm \sqrt{112}}{6}$$

$$= \frac{4 \pm 4\sqrt{7}}{6} \qquad (\sqrt{112} = \sqrt{16 \cdot 7} = 4\sqrt{7})$$

$$= \frac{2 \pm 2\sqrt{7}}{3}$$

An examination of the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives an indication about the type of roots, or solutions, we can expect from the equation $ax^2 + bx + c = 0$, where a, b, and c are integers.

The quantity under the radical sign ($b^2 - 4ac$) is called the **discriminant** because it discriminates, or determines, whether the roots are real or imaginary, rational or irrational, or one or two in number.

1. If $b^2 - 4ac > 0$, there are two real solutions.
   - **(a)** If $b^2 - 4ac$ is also a perfect square, then the two solutions are rational.
   - **(b)** If $b^2 - 4ac$ is not a perfect square, then both solutions are irrational.
2. If $b^2 - 4ac = 0$, there is only one real, rational solution.
3. If $b^2 - 4ac < 0$, there are two imaginary solutions (see Chapter 14).

In Section 4.2, we solved equations graphically. The equation $y = x^2 + 2x - 4$ was solved graphically for $y = 0$, $y = 4$, $y = -3$, and $y = -7$; the graph is repeated in Fig. 7.1. Let's now find each of these solutions either by using the quadratic formula or by factoring.

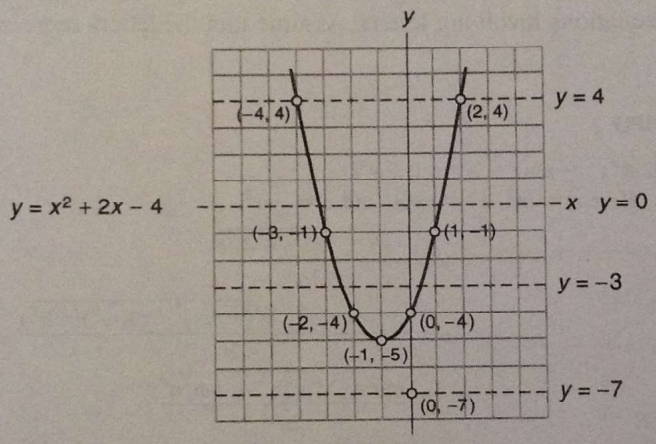


Figure 7.1

Quadratic formula

$y = x^2 + 2x - 4$ for $y = 0$

$0 = x^2 + 2x - 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{20}}{2}$$

$$= \frac{-2 \pm 2\sqrt{5}}{2}$$

$= -1 \pm \sqrt{5}$ or $1.24, -3.24$

Factoring

$y = x^2 + 2x - 4$ for $y = 4$

$4 = x^2 + 2x - 4$

$0 = x^2 + 2x - 8$

$0 = (x + 4)(x - 2)$

$x + 4 = 0$ or $x - 2 = 0$

$x = -4$ or $x = 2$

Quadratic formula

$y = x^2 + 2x - 4$ for $y = -3$

$-3 = x^2 + 2x - 4$

$0 = x^2 + 2x - 1$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$= -1 \pm \sqrt{2}$ or 0.414, −2.41

Factoring

$y = x^2 + 2x - 4$ for $y = -7$

$-7 = x^2 + 2x - 4$

$0 = x^2 + 2x + 3$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-8}}{2}$$

The discriminant is negative; therefore, there are no real solutions. *Note:* In Fig. 7.1, the curve does not intersect the line $y = -7$.

The quadratic formula may also be used to solve literal quadratic equations; that is, quadratic equations involving letters. Assume that the letters represent positive real numbers.

EXAMPLE 3

Solve $m^2x^2 - mnx - n^2 = 0$ for x.

Here $a = m^2$, $b = -mn$, and $c = -n^2$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-mn) \pm \sqrt{(-mn)^2 - 4(m^2)(-n^2)}}{2(m^2)}$$

$$= \frac{mn \pm \sqrt{m^2n^2 + 4m^2n^2}}{2m^2}$$

$$= \frac{mn \pm \sqrt{5m^2n^2}}{2m^2}$$

$$= \frac{mn \pm mn\sqrt{5}}{2m^2}$$

$$= \frac{n \pm n\sqrt{5}}{2m}$$

Exercises 7.3

Solve each equation using the quadratic formula.

1. $x^2 - 4x - 32 = 0$
2. $x^2 + 10x + 21 = 0$
3. $2x^2 = 3x + 9$
4. $2x^2 + 9x = 5$
5. $12x^2 + 5x = 28$
6. $6x^2 = 5x + 25$
7. $15x^2 + 75x + 90 = 0$
8. $20x^2 - 30x = 200$
9. $8x^2 + 27 = 30x$

10. $12x^2 - 72x = 135$
11. $6x^2 + 15x = 0$
12. $8x^2 - 24 = 0$
13. $x^2 + 5x = 7$
14. $4x^2 = 4x + 11$
15. $x^2 = 4x + 14$
16. $x^2 = 6x + 19$
17. $4x^2 = 8x + 23$
18. $20x + 3 = 4x^2$
19. $9x^2 + 25 = 42x$
20. $9x^2 = 15x + 59$
21. $x^2 - 4x + 1 = 0$
22. $2x^2 + 7x = -1$
23. $-3x^2 + 7x = 2$
24. $-6x^2 - 4x + 3 = 0$
25. $x^2 + 10x = 24$
26. $x^2 + 7 = 8x$
27. $4x^2 + 12x + 9 = 0$
28. $2x^2 + 5x = 4$
29. $2x^2 - 5x + 1 = 0$
30. $9x^2 + 25 = 30x$
31. $5x^2 + 6x + 4 = 0$
32. $3x^2 - 5x + 8 = 0$

Solve each equation for the indicated values using any method.

33. $y = -2x^2 + 2x$
for $y = 0, -12$, and 7

34. $y = x^2 + x - 12$
for $y = 0$, 8, and 6

35. $y = 3x^2 + x$
for $y = 0$, 4, and 1

36. $y = 3x^2 + 2x$
for $y = 0$, 40, 3, and -3

Solve for x.

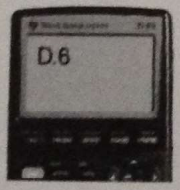
37. $m^2x^2 - 2mx + 1 = 0$
38. $4m^2x^2 + 6mnx - n^2 = 0$
39. $n^2x^2 + 4nx + 4 = 0$
40. $3x^2 + 4x - mn = 0$
41. $x^2 + 5x = 16a^2 - 20a$
42. $x^2 + 6nx + 9n^2 = 0$

Solve for r.

43. $A = \pi r^2$
44. $S = 2\pi r^2 + 2\pi rh$

7.4 APPLICATIONS

Let us now present some applications that involve quadratic equations. For consistency, all final results are rounded to three significant digits in this section.



EXAMPLE 1

Design a rectangular metal plate to meet the following specifications: (a) the length is 4.00 cm less than twice its width, and (b) its area is 96.0 cm^2.

First, draw a diagram as in Fig. 7.2 and let

$$x = \text{width}$$
$$2x - 4 = \text{length}$$
$$A = lw$$
$$96 = (2x - 4)x$$
$$96 = 2x^2 - 4x$$
$$0 = 2x^2 - 4x - 96$$
$$0 = x^2 - 2x - 48$$
$$0 = (x + 6)(x - 8)$$
$$x + 6 = 0 \quad \text{or} \quad x - 8 = 0$$
$$x = -6 \quad \text{or} \quad x = 8$$

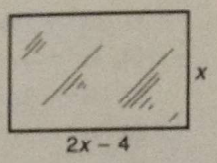


Figure 7.2

The solution $x = -6$ is not meaningful, as x refers to a length measurement which must be a positive quantity. Therefore,

$$x = \text{width} = 8.00 \text{ cm}$$
$$2x - 4 = \text{length} = 12.0 \text{ cm}$$

Example 1 represents a situation where we must be very cautious of our interpretation of algebraic solutions. We must be aware that in some cases a numerical solution may not be meaningful to the given physical problem. Such a solution is algebraically correct, but makes no sense when it is applied to the problem.

EXAMPLE 2

A variable voltage in a given electrical circuit is given by the formula $V = t^2 - 12t + 40$. At what values of t (in seconds) is the voltage V equal to 8.00 volts? To 25.0 volts?

If $V = 8.00$,

$$8 = t^2 - 12t + 40$$
$$0 = t^2 - 12t + 32$$
$$0 = (t - 4)(t - 8)$$
$$t - 4 = 0 \quad \text{or} \quad t - 8 = 0$$
$$t = 4.00 \text{ s} \quad \text{or} \quad t = 8.00 \text{ s}$$

If $V = 25.0$,

$$25 = t^2 - 12t + 40$$
$$0 = t^2 - 12t + 15 \qquad \text{(Does not factor)}$$
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(15)}}{2(1)}$$
$$= \frac{12 \pm \sqrt{144 - 60}}{2}$$
$$= \frac{12 \pm \sqrt{84}}{2} \qquad (\sqrt{84} = \sqrt{4 \cdot 21} = 2\sqrt{21})$$
$$= \frac{12 \pm 2\sqrt{21}}{2}$$
$$= 6 \pm \sqrt{21}$$

If decimal solutions are needed ($\sqrt{21}$ is approximately 4.58),

$$t = 6 + \sqrt{21} \quad \text{or} \quad t = 6 - \sqrt{21}$$
$$t = 6 + 4.58 \qquad t = 6 - 4.58$$
$$t = 10.6 \text{ s} \quad \text{or} \quad t = 1.42 \text{ s}$$

EXAMPLE 3

The perimeter of a rectangle is 24.0 m and the area is 20.0 m^2. Find the dimensions (the length and the width).

First, draw a diagram as in Fig. 7.3 and let

$$x = \text{width}$$
$$12 - x = \text{length}$$

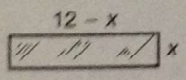


Figure 7.3

(*Note:* The perimeter of a rectangle is the sum of the measurements of all four sides. Therefore, if the perimeter is 24.0 m, one width plus one length is 12.0 m.)

$$A = lw$$
$$20 = (12 - x)x$$
$$20 = 12x - x^2$$
$$x^2 - 12x + 20 = 0$$
$$(x - 2)(x - 10) = 0$$
$$x - 2 = 0 \quad \text{or} \quad x - 10 = 0$$
$$x = 2 \qquad\qquad x = 10$$

If $x = 2$	If $x = 10$
$x = \text{width} = 2.00$ m	$x = \text{width} = 10.0$ m
$12 - x = \text{length} = 10.0$ m	$12 - x = \text{length} = 2.00$ m

Since the length is greater than the width, the width is 2.00 m and the length is 10.0 m.

EXAMPLE 4

A square is cut out of each corner of a rectangular sheet of metal 40.0 cm × 60.0 cm. The sides are folded up to form a rectangular container. What are the dimensions of the square if the area of the bottom of the container is 1500 cm^2? Also, find the volume of the container.

Draw a diagram as in Fig. 7.4 and let

$$x = \text{side of square cut out}$$
$$40 - 2x = \text{width of rectangular container}$$
$$60 - 2x = \text{length of rectangular container}$$

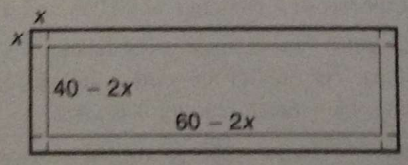


Figure 7.4

Then

$$
\begin{aligned}
A &= lw \\
1500 &= (60 - 2x)(40 - 2x) \\
1500 &= 2400 - 200x + 4x^2 \\
0 &= 4x^2 - 200x + 900 \\
0 &= x^2 - 50x + 225 \\
0 &= (x - 5)(x - 45) \\
x - 5 = 0 \quad &\text{or} \quad x - 45 = 0 \\
x = 5 \quad &\text{or} \quad x = 45 \qquad (\textit{Note: } x = 45 \text{ is physically impossible!})
\end{aligned}
$$

Therefore, the length of the side of square cut out is 5.00 cm, and

$$
\begin{aligned}
V &= lwh \\
&= (50.0 \text{ cm})(30.0 \text{ cm})(5.00 \text{ cm}) = 7\bar{5}00 \text{ cm}^3
\end{aligned}
$$

Exercises 7.4

Express the final results using three significant digits.

1. Separate 13 into two parts whose product is 40.
2. Separate 10 into two parts whose product is 24.
3. The length of a rectangular metal plate is 3.00 cm greater than twice the width. Find the dimensions if its area is 35.0 cm^2.
4. A rectangular plot of ground contains $54\bar{0}0$ m^2. If the length exceeds the width by 30.0 m, what are the dimensions?
5. The area of a triangle is 66.0 m^2. Find the lengths of the base and the height if the base is one metre longer than the height ($A = \frac{1}{2}bh$).
6. A rectangle is 2.00 ft longer than it is wide. Find its dimensions if the area is 48.0 ft^2.
7. A variable electric current is given by the formula $i = t^2 - 7t + 12$. If t is given in s, at what times is the current equal to 2.00 amperes (A)? To 0 A? To 4.00 A?
8. A variable voltage is given by the formula $V = t^2 - 14t + 48$. At what times, t in s, is the voltage equal to 3.00 volts (V)? To 35.0 V? To 0 V?
9. A charge in coulombs flows in a given circuit according to the formula $q = 2t^2 - 4t + 4$. Find t in μs when $q = 2.00$ and when $q = 3.00$.
10. The work done in a circuit varies with time according to the formula $w = 8t^2 - 12t + 20$. Find t in ms when $w = 16.0$, $w = 18.0$, and $w = 0$.
11. A winding links a magnetic field that varies according to the formula $\phi = 0.4t - 2t^2$. Find t in ms when ϕ is 0.0200 weber.
12. The perimeter of a rectangle is 22.0 cm and the area is 24.0 cm^2. Find the dimensions.
13. The perimeter of a rectangle is 80.0 m and the area is 375 m^2. Find the dimensions.
14. A rectangular yard is made using fencing for three sides of the yard and the house for one of the widths. The area of the yard is 288 m^2. Find its length and its width if the length is twice the width. How much fencing is needed?
15. A square 4.00 in. on a side is cut out of each corner of a square sheet of aluminum. The sides are folded up to form a rectangular container. If the volume is $40\bar{0}$ in^3, what was the size of the original sheet of aluminum?

16. A square is cut out of each corner of a rectangular sheet of aluminum 20.0 cm × 30.0 cm. The sides are folded up to form a rectangular container. What are the dimensions of the square if the area of the bottom of the container is 416 cm^2? Also find the volume of the container.

17. A rectangular field is fenced in by using a river as one side. If 2500 m of fencing are used for the 720,000-m^2 field, what are its dimensions?

18. A projectile is shot vertically upward. Its height, h in feet, after time t in s may be expressed by the formula $h = 96t - 16t^2$.
    (a) Find t when $h = 80.0$ ft.
    (b) Find t when $h = 128$ ft.
    (c) Find t when $h = 0$.
    (d) Find the maximum height reached by the projectile.

19. How wide a strip must be mowed around a rectangular grass plot 40.0 m by 60.0 m for one-half of the grass to be mowed?

20. A rectangular sheet of metal 24.0 in. wide is formed into a rectangular trough with an open top and no ends. If the cross-sectional area is 70.0 in^2, find the depth of the trough.

21. A rectangular sheet of metal 48.0 in. wide is formed into a rectangular closed tube. If the cross-sectional area is 108 in^2, find the length and the width of the cross-section of the tube.

22. The cross-sectional area of the L-shaped beam in Fig. 7.5 is 24.0 in^2. What is the thickness, x, of the metal?

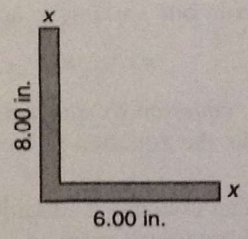


Figure 7.5

23. The area of a rectangular lot, 80.0 m × 100 m, is increased by 4000 m^2. If the length and the width are increased by the same amount, what are the dimensions of the larger lot?

24. The area of a rectangular lot, 240 ft × 400 ft, is increased by 15,800 ft^2. If the width and the length are increased at a ratio of 2:3, what are the dimensions of the larger lot?

25. The equivalent resistance of two resistors connected in parallel is 45.0 ohms (Ω). If the resistance of one resistor is three times the other, what is the value of each? The formula for two resistors in parallel is $R = \dfrac{R_1 R_2}{R_1 + R_2}$, where R is the equivalent resistance and R_1 and R_2 are the individual resistors.

26. A 100-W lamp and a 20.0-Ω resistor are connected in series with a 120-V power supply. By Kirchhoff's law, we have

$$\varepsilon = IR + \frac{P}{I}$$

After substituting the known values, we have

$$120 = 20I + \frac{100}{I}$$

Solve for I (in amperes).

27. A parking lot $20\bar{0}$ m $\times$ $10\bar{0}$ m is to be tripled in area by adding the region as shown in Fig. 7.6. Find x.

28. A parking lot $20\bar{0}$ m $\times$ $10\bar{0}$ m is to be tripled in area by adding the region as shown in Fig. 7.7. Find x.

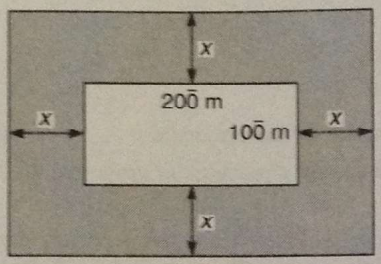


Figure 7.6

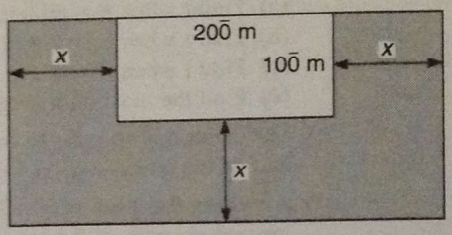


Figure 7.7

CHAPTER SUMMARY

1. A quadratic equation in one variable is an equation in the form

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

2. *To solve a quadratic equation by factoring:*
   (a) If necessary, write the equation in the form $ax^2 + bx + c = 0$.
   (b) Factor.
   (c) Set each factor that contains a variable equal to zero.
   (d) Solve each resulting linear equation.
   (e) Check.

3. *To solve an equation in the form $ax^2 = c$:*
   (a) Divide each side by a.
   (b) Take the square root of each side.
   (c) Simplify the result, if possible.

4. *To solve a quadratic equation by completing the square:*
   (a) The coefficient of the second-degree term *must* equal 1. If not, divide each side by its coefficient.
   (b) Write the equation with the variable terms on one side and the constant term on the other.
   (c) Add the square of one-half of the coefficient of the first-degree term to each side.
   (d) Rewrite the variable side, now a perfect square, as the square of a binomial.
   (e) Take the square root of each side.
   (f) Solve for the variable and simplify, if possible.
   (g) Check.

5. *The quadratic formula:* To solve a quadratic equation in the form $ax^2 + bx + c = 0$, substitute the values for a, b, and c into the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

6. The **discriminant** is $b^2 - 4ac$, the quantity under the radical in the quadratic formula.
   (a) If $b^2 - 4ac > 0$, there are two real solutions.
      (i) If $b^2 - 4ac$ is also a perfect square, then the two solutions are rational.
      (ii) If $b^2 - 4ac$ is not a perfect square, then both solutions are irrational.
   (b) If $b^2 - 4ac = 0$, there is only one real, rational solution.
   (c) If $b^2 - 4ac < 0$, there are two imaginary solutions.

CHAPTER 7 REVIEW

Solve each equation.

1. $x^2 + 4x = 21$
2. $6x^2 + 40 = 31x$
3. $20x^2 + 21x + 4 = 0$
4. $36x^2 - 1 = 0$
5. $18x^2 + 45x + 18 = 0$
6. $6x^2 - 36x = 0$
7. $-8x^2 + 6x + 9 = 0$
8. $36 - 4x^2 = 0$
9. $x^2 + 10x + 25 = 0$
10. $9x^2 + 16 = 24x$

Solve each equation by completing the square.

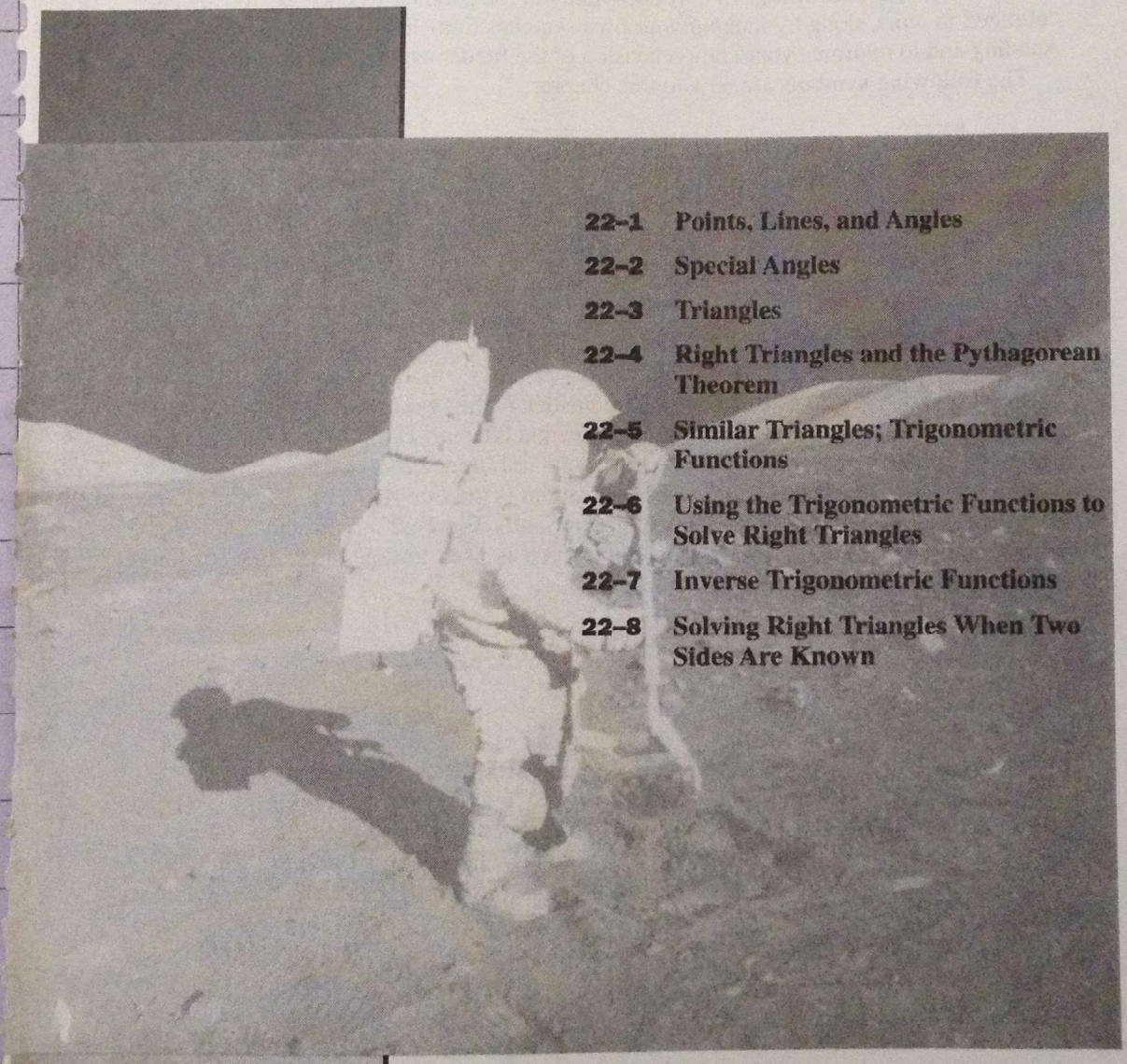
11. $2x^2 + 13x + 20 = 0$
12. $3x^2 = 4x + 7$
13. $x^2 - 8x + 9 = 0$
14. $-5x^2 + 6x + 1 = 0$

Solve each equation for x using the quadratic formula.

15. $3x^2 - 16x + 20 = 0$
16. $4x^2 = 6x + 3$
17. $-2x^2 + 3x + 1 = 0$
18. $6x^2 + 8x - 5 = 0$
19. $mnx^2 - mx + n = 0$
20. $n^4x^2 - 2m^2n^2x + m^4 = 0$
21. Solve $s = vt + \frac{1}{2}at^2$ for t.
22. Divide 15 into two parts whose product is 36.
23. Three thousand feet of fence are used to enclose a rectangular plot of ground. Find the dimensions if the area is $540{,}000\ \text{ft}^2$.
24. A square is cut out of each corner of a square sheet of aluminum 50.0 cm on a side. The sides are folded up to form a rectangular container. What are the dimensions of the square cut out if the area of the bottom of the container is $900\ \text{cm}^2$? Also, find the volume of the container.
25. A ball is thrown vertically upward. Its height, h in feet, after time t in s may be expressed by $h = 64t - 16t^2$.
   (a) At what times is the ball 48.0 ft above the ground?
   (b) Does the ball ever reach a height of 80.0 ft?
   (c) Find the time that the ball is in the air.
   (d) What is the maximum height reached?
26. Find the resistance R (in ohms).

$$\frac{1}{R} + \frac{1}{550 - R} = \frac{1}{120}$$

Day: Date:



22–1 Points, Lines, and Angles

22–2 Special Angles

22–3 Triangles

22–4 Right Triangles and the Pythagorean Theorem

22–5 Similar Triangles; Trigonometric Functions

22–6 Using the Trigonometric Functions to Solve Right Triangles

22–7 Inverse Trigonometric Functions

22–8 Solving Right Triangles When Two Sides Are Known

Apollo 17 Scientist-Astronaut Harrison H. Schmitt collects lunar rock samples on the surface of the moon at the Taurus-Littrow landing site. Notice the backpack life support system worn by the astronaut and the lunar rake being used to collect rock samples (1.3–2.5 cm). (Courtesy of NASA/Johnson Space Center)

This chapter starts a series of chapters that will enable you to understand alternating current (ac) principles and circuits. Once you master this material, you will be able to comprehend advanced topics in electronics, including active devices (amplifiers), filters, and frequency domains.

The key to your understanding of ac circuit principles starts with a knowledge of angles, triangles, and trigonometric functions, which, in turn, allow you to master the concepts of circular functions and phasors.

This chapter uses drawings to introduce many of the geometric concepts. You are encouraged to work along by making your own sketches (drawings) to aid in your understanding and to reinforce your comprehension of the fundamental ideas being presented.

The following symbols are new to this chapter:

SIN	sine	SIN^{-1}	inverse sine
COS	cosine	COS^{-1}	inverse cosine
TAN	tangent	TAN^{-1}	inverse tangent
DEG	degree	RAD	radian

22–1 POINTS, LINES, AND ANGLES

The first geometric concept that we will consider is the *point*. We have already used points when we were constructing graphs. In this chapter, capital letters are used to designate points. Thus, A and O represent points in Figure 22–1.

The second geometric concept to be considered is the *line segment*. A line segment is the portion of a straight line connecting two points. Figure 22–2 presents a line segment joining the points A and B. The symbol $\overline{AB}$ is used to denote the line segment connecting points A and B.

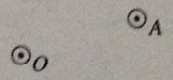


FIGURE 22–1 Points O and A.

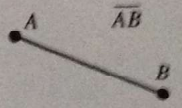


FIGURE 22–2 Line segment $\overline{AB}$.

Building on the concepts of points and line segments, we can define the concept of an angle. An **angle** is the geometric figure formed when one end point of a line segment is held fixed while the second end point is moved to a new location without changing the length of the line segment. Thus, in Figure 22–3, the figure represented by $\overline{OA}$ and $\overline{OA'}$ is the angle "swept out" by $\overline{OA}$ as A moves to A', while O is held fixed.

FIGURE 22–3 Sweeping out $\angle AOA'$.

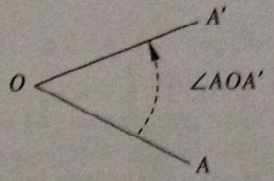
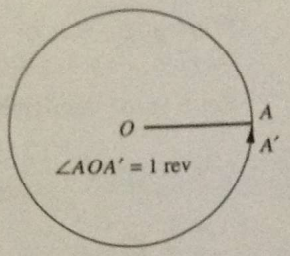


FIGURE 22–4 Sweeping out an angle of one revolution.



The symbol $\angle AOA'$ is used for the angle AOA'. The fixed end point, O, is called the **vertex** of the angle.

Like other mathematical quantities, angles have size and direction. One unit of size is the **revolution.** The angle swept out by moving one end point all the way around the fixed end point and back to its original position is one revolution in size. This is illustrated in Figure 22–4. The abbreviation for revolution is *rev.*

EXAMPLE 22–1 How many revolutions are swept out by a second hand in 2 min 45 s?

Solution Each minute, the second hand sweeps out an angle of one revolution. Let n represent the number of revolutions.

$$n = (2 \text{ min } 45 \text{ s})\left(\frac{1 \text{ rev}}{\text{min}}\right)$$

Convert 45 s to 0.75 min:

$$n = (2.75 \cancel{\text{min}}) \left(\frac{1 \text{ rev}}{\cancel{\text{min}}}\right) = 2.75 \text{ rev}$$

$\therefore$ The second hand sweeps out 2.75 rev in 2 min 45 s.

EXAMPLE 22–2 Engine speed is measured in revolutions per minute (rev/min). If an automobile engine is operated at 5500 rev/min for 6.0 min, how many revolutions will have been swept out by the crankshaft?

Solution Let n be the number of revolutions:

$$n = 5500\left(\frac{\text{rev}}{\cancel{\text{min}}}\right) \times 6.0 \ (\cancel{\text{min}})$$

$\therefore$ $n = 33\,000$ rev

Often we want to measure fractions of revolutions. For this a common unit of measure is the degree. One revolution is equal to 360 *degrees,* or 360°, where the superscript ° stands for degree. The following example explores the relation between revolutions and degrees.

EXAMPLE 22–3 What angle size in degrees is swept out by a second hand in 20 s?

Solution Let n equal the number of revolutions:

$$n = (20 \cancel{s}) \left(\frac{1 \text{ rev}}{\cancel{\text{min}}}\right)\left(\frac{1 \cancel{\text{min}}}{60 \cancel{s}}\right)$$

$$n = \frac{20}{60} \text{ rev}$$

$$n = \frac{1}{3} \text{ rev}$$

Multiply by 360 to convert to degrees:

$$n = \frac{1}{3} \cancel{\text{rev}} \frac{360^\circ}{1 \cancel{\text{rev}}}$$

$$\therefore \quad n = 120^\circ$$

A second unit of size for measuring fractions of revolutions is the **radian**. The distance around a circle is 2π times the length of the radius (see Figure 22–5). In like manner, the angle swept out in one revolution is 2π radians, or 2π.

Although the abbreviation for radians is *rad*, it is not necessary to use any unit indication with angles measured in radians since the radian is not a unit in the same sense as the meter or the ampere. Instead, a radian has no dimension since it is defined as a ratio of the arc length to the radius length of a circle (a ratio of two lengths). Since each is measured in the same unit (m, cm, etc.), the units factor, leaving the radian as a dimensionless quantity. When an angle is substituted into a formula, the word radian (rad) may be used or it may be omitted in expressing the units of the physical quantities. In this chapter, we introduce the superscript ^{r} as a convenience for indicating radians. When an angular quantity contains π, then no unit indication is used.

Although degrees are "nicer" numbers than radians, advanced mathematics and engineering frequently use radians. So we will practice with both units. Table 22–1 shows conversion factors for revolutions, degrees, and radians. Remember, the number π is approximately 3.14159.

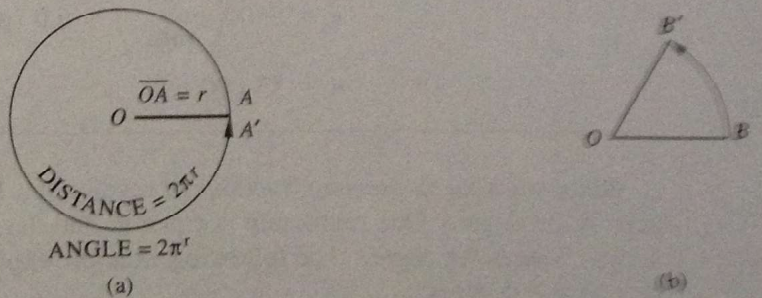


FIGURE 22–5 Geometric properties of radians. (a) The circumference of a circle is related to the radians in one revolution. (b) When the arc length BB' is equal to the sides OB and OB', the angle is one radian.

TABLE 22–1 Conversion Factors for Revolutions, Degrees, and Radians

Revolutions	Radians (Using π)	Radians (Decimal)	Degrees
1	2π	6.28	360
0.5	π	3.14	180
0.159	1	1	57.3
0.002 78	$\pi/180$	0.0175	1

EXAMPLE 22–4 How many degrees are in one radian?

Solution Start with 1 rev = 360° and 1 rev = 2π; solve for 1^r:

$$2\pi = 360°$$

$$1 = 360/2\pi$$

$$\therefore \quad 1^r \approx 57.3°$$

EXAMPLE 22–5 Convert 75.2° to radians.

Solution Look up 1° in Table 22–1; multiply 75.2° by the corresponding value under radians (0.0175):

$$75.2 = 75.2(0.0175)$$

$$\therefore \quad 75.2° = 1.32^r$$

EXAMPLE 22–6 Convert 5.25^r to revolutions.

Solution Look up 1^r in Table 22–1 and multiply 5.25^r by the corresponding value under revolutions (0.159):

$$5.25 = 5.25(0.159)$$

$$\therefore \quad 5.25^r = 0.835 \text{ rev}$$

Observation Some calculators have keystrokes that will convert degrees to radians and radians to degrees. Check your owner's guide to see if your calculator has this feature.

EXERCISE 22–1

1. How many revolutions does a second hand sweep out in 1 h?
2. The earth spins on its axis at the rate of 1 rev/day. How many revolutions does it sweep out in 60 h?
3. What angle in revolutions is swept out by a second hand during 48 s? In degrees?
4. How many revolutions does a CD make while playing a 3-min 15-s recording if it rotates 300 rev/min?

Convert the following angles to degrees. (Remember that angles containing π are in radians.)

5. $\frac{1}{4}$ rev	**6.** 2.5^r	**7.** 1.76π	**8.** 0.15 rev
9. 1.24 rev	**10.** 0.81 rev	**11.** 0.45^r	**12.** 5.75^r

Convert the following angles to revolutions.

13. 300°	**14.** 544°	**15.** 4.8π	**16.** 172π
17. 29.6^r	**18.** 31.4^r	**19.** 5.27π	**20.** 1700°

Convert the following angles to decimal radians.

21. 114.6°	**22.** 29.3°	**23.** 0.75 rev	**24.** 1.54 rev
25. 175°	**26.** 1.4π	**27.** 0.75π	**28.** $\pi/3$

22–2 SPECIAL ANGLES

Several angles are so important that they have their own names. The first special angle is 180° or $\frac{1}{2}$ rev. This angle, which is a straight line, is called a **straight angle.** Angle β in Figure 22–6 is a straight angle. Two angles that sum to a straight angle are called **supplementary angles.** In Figure 22–7, $\angle\theta$ and $\angle\phi$ are supplementary.

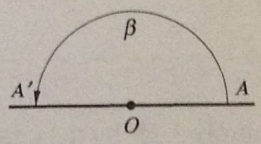


FIGURE 22–6 An angle of 180° is a straight angle.

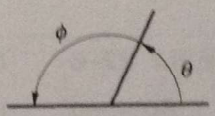


FIGURE 22–7 Two supplementary angles, θ and ϕ, sum to a straight angle.

The second special angle is 90° or $\frac{1}{4}$ rev. This angle is called a **right angle.** The special symbol ⌞ is used to indicate a right angle, as in Figure 22–8. Two angles that sum to a right angle are called **complementary angles.** In Figure 22–9, $\angle\alpha$ and $\angle\beta$ are complementary.

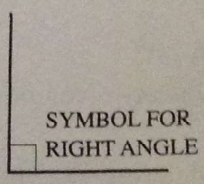


FIGURE 22–8 An angle of 90° is a right angle.

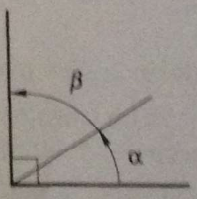


FIGURE 22–9 The complementary angles α and β sum to a right angle.

Other angles may be classified in terms of straight angles and right angles. If an angle is between a straight angle and a right angle, it is called an *obtuse angle*. If an angle is between zero and a right angle, it is called an *acute angle*. More special angles are presented in Table 22–2. The size of each angle is given in revolutions, degrees, and radians.

TABLE 22–2 Fractional Parts of a Revolution

Revolutions	Radians (Using π)	Radians (Decimal)	Degrees
0	0	0	0
$\frac{1}{12}$	$\pi/6$	0.524	30
$\frac{1}{8}$	$\pi/4$	0.785	45
$\frac{1}{6}$	$\pi/3$	1.05	60
$\frac{1}{4}$	$\pi/2$	1.57	90
$\frac{1}{2}$	π	3.14	180
$\frac{3}{4}$	$3\pi/2$	4.71	270
1	2π	6.28	360

Most devices used to measure angles are similar in operation to the protractor pictured in Figure 22–10. Almost all such devices measure angles in degrees.

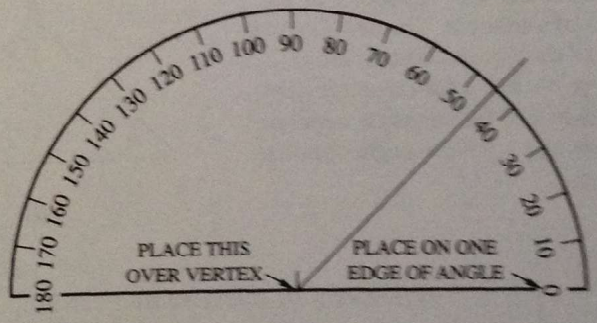


FIGURE 22–10 Protractor measuring an angle.

EXERCISE 22–2

Decide whether the following pairs of angles are supplementary, complementary, or neither.

1. 33°, 57°

2. 0.374 rev, 0.125 rev

3. 135°, 45°

4. 75°, 85°

5. $\pi/3$, $\pi/6$ **6.** 0.125 rev, 0.125 rev

7. $\frac{3}{8}$ rev, $\frac{1}{4}$ rev **8.** 1.07^r, 0.50^r

9. 2.00^r, 1.14^r **10.** 0.65π, 0.45π

Find the supplement of the following angles.

11. 144° **12.** 37.5° **13.** 80.7°

14. 105° **15.** 90° **16.** 120°

17. 3.01^r **18.** 1.57^r **19.** 2.61^r

Find the complement of the following angles.

20. 38° **21.** 57.3° **22.** 22.1°

23. 45° **24.** 83.4° **25.** 49.9°

26. 0.785^r **27.** 1.07^r **28.** 0.52^r

22–3 TRIANGLES

The fourth geometric concept is the triangle. A triangle is a figure formed by joining three line segments endpoint to endpoint as in Figure 22–11. A triangle has three angles, three vertices, and three sides. In Figure 22–11(a), the vertices are labeled *A*, *B*, and *C*. The angles are labeled α, β, and γ. The sides are not labeled in Figure 22–11(a). We can use the line segment notation to refer to them as $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. This notation is very cumbersome, so we will use the notation of Figure 22–11(b), where the sides are labeled as *a*, *b*, and *c*. Notice that side *a* is opposite vertex *A*, side *b* is opposite vertex *B*, and side *c* is opposite vertex *C*.

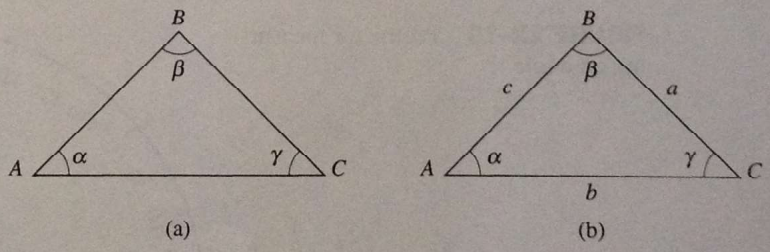


FIGURE 22–11 Labeling of the parts of a triangle. The first three letters of the Greek alphabet are α, β, and γ. We use α for the angle opposite side *a*, β for the angle opposite side *b*, and γ for the angle opposite side *c*.

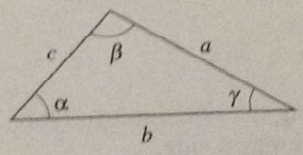
A property of triangles that can help you to check your work is stated as Rule 22–1.

RULE 22–1. Relative Size of the Sides of a Triangle

The longest side is opposite the largest angle.
The shortest side is opposite the smallest angle.

EXAMPLE 22–7 Examine the triangle in Figure 22–12 to determine the smallest angle and the shortest side.

FIGURE 22–12 Triangle for Example 22–7.



Solution By inspection, the smallest angle is $\angle\gamma$, and the shortest side is c.

A remarkable property of triangles is that the sum of the angles is a straight angle. This fact can be used to find the size of the third angle if two angles are known. The procedure for finding the third angle is presented as Rule 22–2.

RULE 22–2. Finding the Third Angle of a Triangle

To find the third angle of a triangle:

1. Form the sum of the first two angles.
2. The third angle is the supplement of the sum.

EXAMPLE 22–8 Two angles of a triangle are 35° and 47°; what is the third angle?

Solution Use the steps of Rule 22–2:

Step 1: Form the sum:

$$35° + 47° = 82°$$

Step 2: The supplement of 82° is 180° − 82°, which is 98°.

$\therefore$ The third angle is 98°.

EXAMPLE 22–9 In a triangle, $\angle\alpha = 1.20^r$ and $\angle\beta = 0.90^r$; find $\angle\gamma$.

Solution Use Rule 22–2.

Step 1: Form the sum:

$$1.20 + 0.90 = 2.10$$

Step 2: Find the supplement:

$$\angle\gamma = 3.14 - 2.10$$

$$\therefore \quad \angle\gamma = 1.04^{r}$$

EXERCISE 22–3

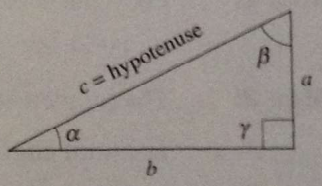
Using the notation of Figure 22–11(b), find the missing angle and determine which is the longest side.

1. $\angle\alpha = 57°, \angle\beta = 22°$
2. $\angle\beta = 51°, \angle\gamma = 78°$
3. $\angle\alpha = 101°, \angle\gamma = 30°$
4. $\angle\alpha = 38°, \angle\beta = 90°$
5. $\angle\alpha = 29.5°, \angle\beta = 50.5°$
6. $\angle\alpha = 115°, \angle\gamma = 15.4°$
7. $\angle\alpha = 1.5^{r}, \angle\beta = 0.5^{r}$
8. $\angle\alpha = 1.57^{r}, \angle\beta = 0.52^{r}$
9. $\angle\beta = 1.00^{r}, \angle\gamma = 1.00^{r}$
10. $\angle\alpha = 1.21^{r}, \angle\beta = 1.0^{r}$
11. $\angle\alpha = \pi/2, \angle\gamma = \pi/4$
12. $\angle\beta = \pi/3, \angle\gamma = \pi/6$
13. $\angle\alpha = \pi/3, \angle\beta = \pi/3$
14. $\angle\alpha = \pi/2, \angle\gamma = \pi/6$
15. $\angle\alpha = 57.3°, \angle\beta = 57.3°$
16. $\angle\beta = 90°, \angle\gamma = 85°$
17. $\angle\alpha = 79.5°, \angle\gamma = 79.6°$
18. $\angle\alpha = 70°, \angle\beta = 42°$
19. $\angle\beta = 45°, \angle\gamma = 45°$
20. $\angle\alpha = 15.2°, \angle\gamma = 15.2°$

22–4 RIGHT TRIANGLES AND THE PYTHAGOREAN THEOREM

Triangles containing a right angle make up an important class of triangles. A triangle containing a right angle is called a **right triangle.** The side opposite the right angle is called the **hypotenuse,** as shown in Figure 22–13.

FIGURE 22–13 Parts of a right triangle. The side opposite the right angle is called the hypotenuse.



The ancient Greek mathematician Pythagoras is given credit for discovering that, if the lengths of any two sides of a right triangle are known, the length of the third side can be easily determined. This relation, known as the Pythagorean theorem, is stated as Rule 22–3.

Day: Date:

RULE 22–3. Pythagorean Theorem

In every right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides. In Figure 22–13,

$$c^2 = a^2 + b^2$$

EXAMPLE 22–10 Find the length of the hypotenuse in a right triangle if the other two sides are 3 m and 4 m.

Solution Use the Pythagorean theorem, Rule 22–3. Substitute 3 for a and 4 for b, then solve for c:

$$c^2 = a^2 + b^2$$
$$c^2 = 3^2 + 4^2$$
$$c^2 = 9 + 16$$
$$\boxed{\sqrt{x}} \quad c^2 = 25$$
$$\therefore \quad c = 5 \text{ m}$$

EXAMPLE 22–11 If the hypotenuse is 27 m and a second side is 17 m, how long is the third side in a right triangle?

Solution Use the Pythagorean theorem, Rule 22–3. Substitute 27 for c and 17 for a; solve for b:

$$c^2 = a^2 + b^2$$
$$27^2 = 17^2 + b^2$$
$$b^2 = 729 - 289$$
$$\boxed{\sqrt{x}} \quad b^2 = 440$$
$$\therefore \quad b = 21 \text{ m}$$

The angle γ is a right angle in the right triangle of Figure 22–13. We know that the sum of $\angle\alpha$ and $\angle\beta$ is the supplement of $\angle\gamma$. Thus, the sum of $\angle\alpha$ and $\angle\beta$ is a right angle. And $\angle\alpha$ and $\angle\beta$ are complements. This important fact is restated as Rule 22–4.

RULE 22–4. Sum of Two Acute Angles in a Right Triangle

The sum of the two acute angles in a right triangle is a right angle. In Figure 22–13, $\angle\alpha$ and $\angle\beta$ are complements, and

$$\alpha + \beta = 90°$$

EXAMPLE 22–12 Find $\angle\beta$ when $\angle\alpha = 37°$ in a right triangle.

Solution Use Rule 22–4:

$$\alpha + \beta = 90$$
$$37 + \beta = 90$$
$$\therefore \quad \beta = 53°$$

EXERCISE 22–4

Find the missing sides and angles in the following right triangles. Use the notation of Figure 22–13.

1.	$a = 1.5$ m	$b = 2.0$ m	$\alpha = 36.9°$
2.	$a = 13$ m	$b = 13$ m	$\alpha = 45°$
3.	$c = 39$ m	$a = 16$ m	$\beta = 65.8°$
4.	$c = 109$ m	$b = 74$ m	$\beta = 43.5°$
5.	$a = 78.5$ m	$b = 44.2$ m	$\beta = 29.4°$
6.	$a = 0.152$ m	$c = 0.305$ m	$\alpha = 29.9°$
7.	$b = 2.59$ m	$c = 4.00$ m	$\alpha = 49.6°$
8.	$a = 7$ m	$b = 24$ m	$\alpha = 16.3°$
9.	$a = 60$ m	$c = 61$ m	$\alpha = 79.6°$
10.	$b = 12$ m	$c = 13$ m	$\beta = 67.4°$

22–5 SIMILAR TRIANGLES; TRIGONOMETRIC FUNCTIONS

Similar Triangles

The two triangles in Figure 22–14 have the same proportions. The ratios of the corresponding sides, a'/a, b'/b, and c'/c, are all equal. When these three ratios are equal, the two triangles are said to be **similar.** If two triangles are similar, their angles are the same size. Thus, in Figure 22–14, $\angle\alpha = \angle\alpha'$, $\angle\beta = \angle\beta'$, and $\angle\gamma = \angle\gamma'$.

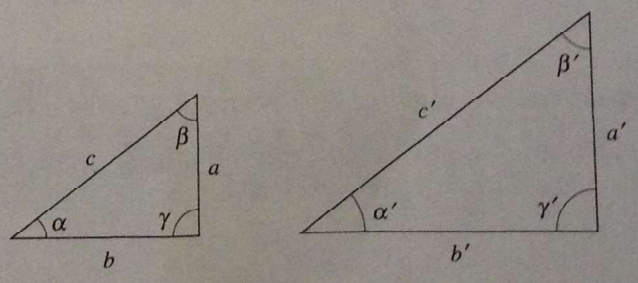


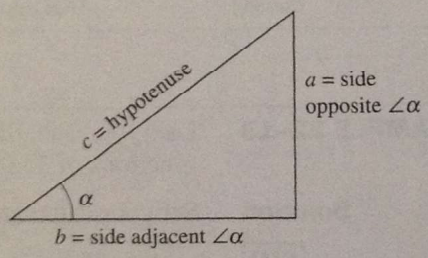
FIGURE 22–14 Similar triangles have equal angles: $\angle\alpha = \angle\alpha'$, $\angle\beta = \angle\beta'$, and $\angle\gamma = \angle\gamma'$.

In two similar triangles, the ratio of two sides in one triangle is equal to the ratio of the corresponding sides in the second triangle. Thus, in Figure 22–14, $a/b = a'/b'$, $a/c = a'/c'$, and $b/c = b'/c'$. These ratios become the function of one of the acute angles in a right triangle. This is because in a right triangle we know that one angle is a right angle, and knowing a second angle gives us enough information to determine the third angle. Knowing all three angles, we know the proportions of the sides. Thus, the ratios a/c, b/c, and a/b can be considered functions of the angle.

Trigonometric Functions

In a right triangle we define the **sine function** of an angle as the *ratio* of the length of the **side opposite** the angle **to** the length of the **hypotenuse.** It is customary to use the symbol sin ( ) to indicate the sine function. Thus, in Figure 22–15, sin (α) = side opposite/hypotenuse = a/c. Sin (α) is read "sine of alpha." Remember that the parentheses indicate function, not multiplication.

FIGURE 22–15 Right triangle used in definition of the trigonometric functions.



Check your calculator and your owner's guide for calculating the sine function. Look for a key like [sin x] or [SIN]. Pay special attention to how you tell your calculator whether the angle is in degrees or in radians. Two systems are in common use. The first is an angular mode key [DRG] with which you can select degrees or radians. The second is a series of keystrokes ([RAD] or [DEG]). The keystrokes allow you to change the system, and an indicator tells you in which system the calculator is working. Consult your owner's guide to determine if there is a restriction on the size of the angle your calculator will accept.

In a right triangle we define the **cosine function** of an angle as the *ratio* of the length of the **side adjacent** to the angle **to** the length of the **hypotenuse.** It is customary to use the symbol cos ( ) to indicate the cosine function. Thus, in Figure 22–15, cos (α) = side adjacent/hypotenuse = b/c. Cos (α) is read "cosine of alpha." Again, the parentheses indicate function and not multiplication.

Check your calculator and your owner's guide for how to calculate the cosine function. Look for a key like [cos x] or [COS]. How do you select degrees or radians? Are there any constraints on the size of the angle?

In a right triangle we define the **tangent function** of an angle as the *ratio* of the length of the **side opposite** the angle **to** the length of the **side adjacent** to the angle. The common symbol for the tangent function is tan ( ). Thus, in Figure 22–15, tan (α) = side opposite/side adjacent = a/b. Tan (α) is read "tangent alpha." The parentheses are used to indicate function, not multiplication.

Check your calculator and your owner's guide for how to calculate the tangent function. Look for a key like [tan x] or [TAN]. How do you select degrees or radians? What limits are placed on the size of the angle?

The definitions of the three functions—sine, cosine, and tangent—are summarized in Table 22–3. Refer also to Figure 22–15.

TABLE 22–3 Definition of the Trigonometric Functions

Function	Symbol	Definition	Figure 22–15
sine (α)	sin (α)	Side opposite/hypotenuse	a/c
cosine (α)	cos (α)	Side adjacent/hypotenuse	b/c
tangent (α)	tan (α)	Side opposite/side adjacent	a/b

EXAMPLE 22–13 Use your calculator to evaluate the sine, cosine, and tangent functions of 35.0°:

Solution Set your calculator to work in degrees [DEG]:

[SIN] sin (35.0°) = 0.574

[COS] cos (35.0°) = 0.819

[TAN] tan (35.0°) = 0.700

EXAMPLE 22–14 Use your calculator to evaluate the sine, cosine, and tangent functions of 1.00^r.

Solution Set your calculator to work in radians [RAD]:

[SIN] sin (1.00^r) = 0.841

[COS] cos (1.00^r) = 0.540

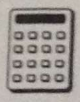
[TAN] tan (1.00^r) = 1.56

It is helpful to learn the brief list of trigonometric functions in Table 22–4. The list will aid you in finding approximate solutions to problems and in detecting errors in keystrokes while using your calculator.

TABLE 22–4 A Brief Table of Trigonometric Functions

Angle (Degrees)	Angle (Radians)	Sine	Cosine	Tangent
0	0.000	0.000	1.000	0.000
30	0.524	0.500	0.866	0.577
45	0.785	0.707	0.707	1.000
60	1.047	0.866	0.500	1.732
90	1.571	1.000	0.000	—

EXERCISE 22–5



Calculator Drill

Evaluate the sine, cosine, and tangent functions of the following angles in degrees.

1. 1.00°	**2.** 25.0°	**3.** 79.0°
4. 15.0°	**5.** 89.0°	**6.** 0°
7. 51.5°	**8.** 70.9°	**9.** 11.4°
10. 17.3°	**11.** 0.05°	**12.** 5.37°
13. 81.6°	**14.** 77.7°	**15.** 49.2°
16. 63.9°	**17.** 50.2°	**18.** 71.4°
19. 21.3°	**20.** 38.4°	

Evaluate the sine, cosine, and tangent of the following angles in radians.

21. 1.50^r	**22.** 0.250^r	**23.** 0.790^r
24. 0.0170^r	**25.** 0^r	**26.** 0.500^r
27. 0.100^r	**28.** 1.20^r	**29.** 1.450^r
30. 0.670^r	**31.** 0.159^r	**32.** 0.318^r
33. 1.170^r	**34.** 0.200^r	**35.** 1.570^r
36. 1.230^r	**37.** 0.972^r	**38.** 0.808^r
39. 0.355^r	**40.** 0.621^r	

22–6 USING THE TRIGONOMETRIC FUNCTIONS TO SOLVE RIGHT TRIANGLES

In the previous section we learned about three functions of angles in right triangles. These functions, sine, cosine, and tangent, are known as trigonometric functions because they are used in measuring triangles. With the use of these functions we can find the missing parts of a right triangle if we know one angle and one side.

EXAMPLE 22–15 In the right triangle in Figure 22–16, $\angle\alpha = 25.7°$ and $c = 15.5$ m. Find side a, side b, and $\angle\beta$.

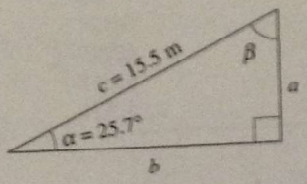


FIGURE 22–16 Right triangle for Example 22–15.

Solution Solve for side a:
Use the definition of the trigonometric functions to write equations involving the known parts and one unknown side. Which function involves an angle, the hypotenuse, and the side opposite the angle? The sine function:

$$\sin(\alpha) = a/c$$

Solve for the unknown part, a:

$$a = c \sin(\alpha)$$

Substitute:

SIN
$$a = 15.5 \sin(25.7°)$$
×
$$a = 15.5(0.434)$$
$$a = 6.72$$

$\therefore$ $a = 6.72$ m

Solve for side b:
Which function involves an angle, the hypotenuse, and the side adjacent to the angle? The cosine function:

$$\cos(\alpha) = b/c$$

Solve for the unknown b:

$$b = c \cos(\alpha)$$

Substitute:

COS
$$b = 15.5 \cos(25.7°)$$
×
$$b = 15.5(0.901)$$
$$b = 14.0$$

$\therefore$ $b = 14.0$ m

Solve for $\angle\beta$:
How do we find the missing angle β? It is the complement of $\angle\alpha$ (25.7°).

$$\therefore \quad \angle\beta = 64.3°$$

Observation Angles expressed to the nearest tenth of a degree correspond to three significant figures in the length of the sides.

EXAMPLE 22–16 In the right triangle in Figure 22–17, $\angle\alpha = 55.2°$ and $b = 74.5$ mm. Find side c, side a, and $\angle\beta$.

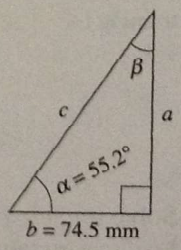


FIGURE 22–17 Right triangle for Example 22–16.

Solution Solve for side c:
Use the definitions of the trigonometric functions to write an equation involving an angle, the side adjacent, and the hypotenuse.

$$\cos(\alpha) = b/c$$

Solve for c:

$$c = b/\cos(\alpha)$$

Substitute:

COS
÷

$$c = 74.5/\cos(55.2°)$$
$$c = 74.5/0.571$$
$$c = 131$$

$$\therefore \quad c = 131 \text{ mm}$$

Solve for side a.
Write an equation involving $\angle\alpha$, b, and a:

$$\tan(\alpha) = a/b$$

Solve for a and substitute:

$$a = b \tan(\alpha)$$

TAN

$$a = 74.5 \tan(55.2°)$$

[×] $a = 74.5(1.44)$

$a = 107$

$\therefore \quad a = 107 \text{ mm}$

Solve for $\angle\beta$:
Find $\angle\beta$ as the complement of $\angle\alpha$ (55.2°).

$\therefore \quad \angle\beta = 34.8°$

EXAMPLE 22–17 In the right triangle in Figure 22–18, $\angle\beta = 47.8°$ and $b = 2.83$ m. Find side c, side a, and $\angle\alpha$.

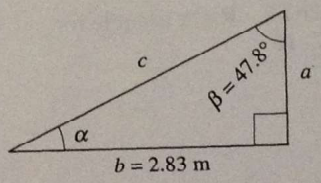


FIGURE 22–18 Right triangle for Example 22–17.

Solution Solve for side c:
Write an equation involving $\angle\beta$, b, and c:

$\sin(\beta) = b/c$

Solve for c and substitute:

$c = b/\sin(\beta)$

[SIN] $c = 2.83/\sin(47.8°)$

[÷] $c = 2.83/(0.741)$

$c = 3.82$

$\therefore \quad c = 3.82 \text{ m}$

Solve for side a.
Write an equation involving $\angle\beta$, b, and a:

$\tan(\beta) = b/a$

Solve for a and substitute:

$a = b/\tan(\beta)$

[TAN] $a = 2.83/\tan(47.8°)$

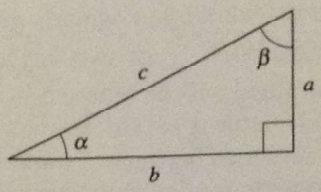
[÷] $a = 2.83/1.10$

$\therefore \quad a = 2.57 \text{ m}$

Solve for $\angle\alpha$.
Find $\angle\alpha$ as the complement of $\angle\beta = 47.8°$:

$\therefore \quad \angle\alpha = 42.2°$

FIGURE 22–19 Right triangle for Exercise 22–6.



EXERCISE 22–6

Find the missing parts of the right triangle in Figure 22–19 when you are given the following information.

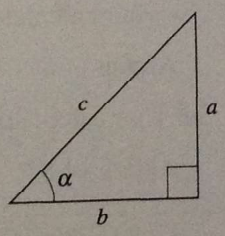
1. $\angle\alpha = 20.0°$, $c = 10$ m
2. $\angle\alpha = 59.0°$, $c = 210$ m
3. $\angle\alpha = 17.5°$, $c = 26.2$ m
4. $\angle\alpha = 83.1°$, $c = 500$ mm
5. $\angle\alpha = 28.4°$, $a = 19.5$ m
6. $\angle\alpha = 52.1°$, $a = 9.05$ m
7. $\angle\alpha = 32.3°$, $a = 2.79$ m
8. $\angle\alpha = 71.5°$, $a = 9.05$ m
9. $\angle\alpha = 14.1°$, $b = 14.1$ m
10. $\angle\alpha = 58.2°$, $b = 351$ mm
11. $\angle\alpha = 45.0°$, $b = 0.70$ km
12. $\angle\alpha = 60.0°$, $b = 1.86$ km
13. $\angle\beta = 15.3°$, $c = 300$ m
14. $\angle\beta = 79.1°$, $c = 121$ km
15. $\angle\beta = 50.8°$, $c = 875$ mm
16. $\angle\beta = 29.7°$, $c = 515$ m
17. $\angle\beta = 36.2°$, $a = 59.4$ m
18. $\angle\beta = 60.5°$, $a = 2.75$ km
19. $\angle\beta = 75.6°$, $b = 1.07$ km
20. $\angle\beta = 43.3°$, $b = 16.9$ mm

22–7 INVERSE TRIGONOMETRIC FUNCTIONS

We have seen that the three trigonometric functions give the value of certain ratios in a right triangle from the knowledge of the size of a particular angle. The inverse functions give the size of the angle from knowledge of the appropriate ratios. Definitions for three inverse trigonometric functions are given below.

In a right triangle we define the **arcsine function** of the ratio of a side to the hypotenuse as *the angle* whose sine is the ratio. There are several symbols in common use for the arcsine: $\sin^{-1}$, Sin^{-1}, arcsin, and Arcsin. For clarity and because we are working with right triangles, we will use Arcsin. This concept is discussed further in Chapter 23. Thus, in Figure 22–20, Arcsin $(a/c) = \angle\alpha$.

FIGURE 22–20 The inverse trigonometric functions: $\angle\alpha = \text{Arcsin}\ (a/c) = \text{Arccos}\ (b/c) = \text{Arctan}\ (a/b)$



Check your calculator and owner's guide for calculating this function. Look for a single keystroke, [SIN^{-1}], or two keystrokes, such as [Arc] [sin], [INV] [SIN], or [f^{-1}] [sin]. The angle will be returned in the angular unit you have selected, so check whether the calculator is set for degrees or radians! Finally, the arcsine function will work only for numbers between -1 and 1.

In a right triangle we define the **arccosine function** of the ratio of a side to the hypotenuse as *the angle* whose cosine is the ratio. We will use the symbol Arccos ( ) to indicate the arccosine function. Thus, in Figure 22–20, Arccos $(b/c) = \angle\alpha$.

Check your calculator and your owner's guide for calculating this function. The same cautions apply to this function that apply to the arcsine function.

In a right triangle, we define the **arctangent function** of the ratio of the two sides as *the angle* whose tangent is equal to the ratio. We will use the symbol Arctan ( ) to indicate the arctangent function. Thus, in Figure 22–20, Arctan $(a/b) = \angle\alpha$.

Check your calculator and your owner's guide for calculating this function. Notice that there is no limit on the size of the argument for the arctangent function.

EXAMPLE 22–18 Find the angle in degrees whose sine is 0.244. Then find the same angle in radians.

Solution Use your calculator; set it to work in degrees [DEG]:

[SIN^{-1}] Arcsin (0.244) = 14.1°

Set your calculator to work in radians [RAD]:

[SIN^{-1}] Arcsin (0.244) = 0.246^{r}

EXAMPLE 22–19 Find the angle in degrees whose cosine is 0.398. Then find the same angle in radians.

Solution Use your calculator; set it to work in degrees:

[COS^{-1}] Arccos (0.398) = 66.5°

Set your calculator to work in radians:

[COS^{-1}] Arccos (0.398) = 1.162^{r}

EXAMPLE 22–20 Find the angle in degrees whose tangent is 4.87. Then find the same angle in radians.

Solution Use your calculator; set it to work in degrees:

[TAN^{-1}] Arctan (4.87) = 78.4°

Set your calculator to work in radians:

[TAN^{-1}] Arctan (4.87) = 1.368^{r}

EXERCISE 22–7

Calculator Drill

1. Find the arcsine in degrees of the following numbers.

(a) 0.107	**(b)** 0.791	**(c)** 0.255	**(d)** 0.568
(e) 0.500	**(f)** 0.395	**(g)** 0.642	**(h)** 0.806
(i) 0.411	**(j)** 1.000	**(k)** 0.043	**(l)** 0.921

2. Find the arcsine in radians of the numbers in problem 1.
3. Find the arccosine in degrees of the numbers in problem 1.
4. Find the arccosine in radians of the numbers in problem 1.
5. Find the arctangent in degrees of the following numbers.

(a) 10.0	**(b)** 0.707	**(c)** 3.15	**(d)** 0.444
(e) 1.57	**(f)** 1.00	**(g)** 2.36	**(h)** 0.290
(i) 6.03	**(j)** 0.012	**(k)** 14.3	**(l)** 7.08

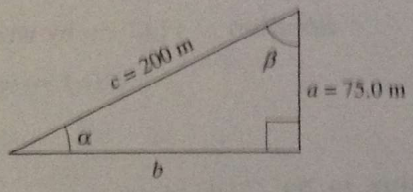
6. Find the arctangent in radians of the numbers in problem 5.

22–8 SOLVING RIGHT TRIANGLES WHEN TWO SIDES ARE KNOWN

With the aid of the inverse trigonometric functions, we can solve right triangles when two sides are known. The next three examples show how this can be done; it is important that you use your calculator.

EXAMPLE 22–21 In the right triangle in Figure 22–21, the length of the hypotenuse is 200 m and the length of side a is 75.0 m. Find $\angle\alpha$, $\angle\beta$, and side b.

FIGURE 22–21 Right triangle for Example 22–21.



Solution Find $\angle\alpha$ by writing an equation involving a, c, and $\angle\alpha$:

$$\sin(\alpha) = a/c$$

Apply the arcsine function to both sides to solve for $\angle\alpha$:

$$\text{Arcsin}[\sin(\alpha)] = \text{Arcsin}\,(a/c)$$
$$\angle\alpha = \text{Arcsin}\,(a/c)$$

Substitute:

[÷]
[SIN^{-1}] $\angle\alpha$ = Arcsin (75.0/200)
$\angle\alpha$ = Arcsin (0.375)
$\therefore$ $\angle\alpha = 22.0°$

Find $\angle\beta$ by writing an equation involving a, c, and $\angle\beta$:

$$\cos(\beta) = a/c$$

Apply the arccosine function to both sides to solve for $\angle\beta$:

Arccos [cos (β)] = Arccos (a/c)
$\angle\beta$ = Arccos (a/c)

Substitute:

[÷]
[COS^{-1}] $\angle\beta$ = Arccos (75.0/200)
$\angle\beta$ = Arccos (0.375)
$\therefore$ $\angle\beta = 68.0°$

Find side b by the Pythagorean theorem, Rule 22–3:

$$a^2 + b^2 = c^2$$

Solve for b and substitute:

$b = \sqrt{c^2 - a^2}$
[x^2] [−]
[$\sqrt{x}$] $b = \sqrt{200^2 - 75^2}$
$b = \sqrt{34\,375}$
$\therefore$ $b = 185$ m

EXAMPLE 22–22 The length of the hypotenuse is 155 mm and side b is 125 mm. Find $\angle\alpha$, $\angle\beta$, and side a in Figure 22–22.

Solution Find $\angle\alpha$ by writing an equation involving b, c, and $\angle\alpha$:

$$\cos(\alpha) = b/c$$

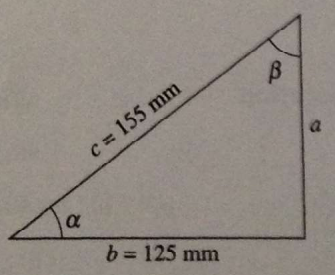


FIGURE 22–22 Right triangle for Example 22–22.

Apply the arccosine function to both sides to solve for $\angle\alpha$:

$$\angle\alpha = \text{Arccos } (b/c)$$

Substitute:

[÷] $\angle\alpha = \text{Arccos } (125/155)$

[COS^{-1}] $\angle\alpha = \text{Arccos } (0.806)$

$\therefore$ $\angle\alpha = 36.2°$

Find $\angle\beta$ as the complement of $\angle\alpha$:

$$\angle\beta = 90 - 36.2$$

$\therefore$ $\angle\beta = 53.8°$

Find side a by writing an equation involving a, b, and $\angle\alpha$:

$$\tan(\alpha) = a/b$$

Solve for a and substitute:

$$a = b \tan(\alpha)$$

[TAN] $a = 125 \tan(36.2°)$

[×] $a = 125(0.732)$

$\therefore$ $a = 91.5$ mm

EXAMPLE 22–23 In the right triangle in Figure 22–23, the length of side a is 155 km, while side b is 125 km. Find $\angle\alpha$, $\angle\beta$, and the hypotenuse c.

Solution Find $\angle\alpha$ by writing an equation involving a, b, and $\angle\alpha$:

$$\tan(\alpha) = a/b$$

Apply the arctangent function to both sides to solve for $\angle\alpha$:

$$\angle\alpha = \text{Arctan } (a/b)$$

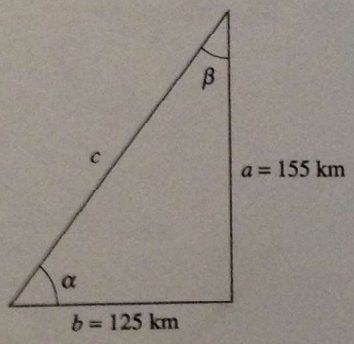


FIGURE 22–23 Right triangle for Example 22–23.

Substitute:

$\boxed{\div}$ $\quad \angle\alpha = \text{Arctan } (155/125)$

$\boxed{\text{TAN}^{-1}}$ $\quad \angle\alpha = \text{Arctan } (1.24)$

$\therefore \quad \angle\alpha = 51.1°$

Find $\angle\beta$ as the complement of $\angle\alpha$:

$$\angle\beta = 90 - 51.1$$

$\therefore \quad \angle\beta = 38.9°$

Find c from a and $\angle\alpha$:

$$\sin(\alpha) = a/c$$
$$c = a/\sin(\alpha)$$

Substitute:

$\boxed{\text{SIN}}$ $\quad c = 155/\sin(51.1°)$

$\boxed{\div}$ $\quad c = 155/0.778$

$\therefore \quad c = 199 \text{ km}$

EXERCISE 22–8

Find the missing parts in the right triangle in Figure 22–24, given the following information.

1. $a = 15.0$ m, $c = 25.0$ m
2. $a = 5.00$ m, $c = 13.0$ m
3. $a = 17.5$ mm, $c = 40.2$ mm
4. $a = 57.4$ m, $c = 77.4$ m
5. $b = 24.0$ m, $c = 25.0$ m
6. $b = 45.0$ m, $c = 100$ m
7. $b = 22.1$ km, $c = 38.3$ km
8. $b = 1.47$ km, $c = 2.00$ km
9. $a = 16.0$ mm, $b = 14.0$ mm
10. $a = 152$ m, $b = 152$ m
11. $a = 500$ m, $b = 866$ m
12. $a = 757$ mm, $b = 602$ mm

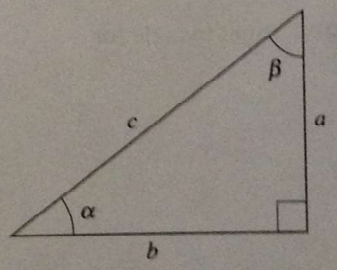


FIGURE 22–24 Right triangle for Exercise 22–8.

SELECTED TERMS

complementary angles Two angles that sum to a right angle.

hypotenuse The side opposite the right angle in a right triangle; also the longest side in a right triangle.

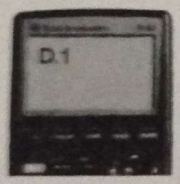
radian A unit for measuring the size of angles; a complete circle equals 2π (6.283185 . . .) radians.

revolution A unit for measuring the size of angles; a complete circle equals 1 revolution.

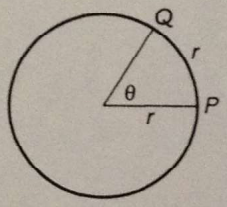
right triangle A triangle containing a right angle.

supplementary angles Two angles that sum to a straight angle.

10.3 RADIAN MEASURE



As we have already seen, many trigonometric problems may be solved in terms of the degree measure of angles. However, in many applications, as well as in our development of mathematics, another angular measurement is needed, namely, the radian. A **radian** is the measure of an angle with its vertex at the center of a circle whose intercepted arc is equal in length to the radius of the circle (see Fig. 10.22). That is, the angle θ is defined as the ratio of the length of arc PQ to the length of the radius, r. As a result, the unit *radian* has no physical dimensions since it is the ratio of two lengths.



$\theta = 1$ radian **Figure 10.22**

The circumference of any circle is given by the formula $C = 2\pi r$. The ratio of the circumference of a circle to its radius is $\frac{2\pi r}{r} = 2\pi$ (about 6.28). That is, 2π radians is the measure of the central angle of any complete circle (one complete revolution).

For comparison purposes,

$$\boxed{\begin{aligned} 2\pi \text{ radians} &= 360^\circ \\ \pi \text{ rad} &= 180^\circ \end{aligned}}$$

When we divide each side of the above relation by π, we find

$$\boxed{1 \text{ rad} = \frac{180^\circ}{\pi} = 57.2958^\circ \qquad \text{(Approximately)}}$$

Or, when we divide each side of the same relation by 180, we find

$$\boxed{1^\circ = \frac{\pi}{180} \text{ rad} = 0.01745 \text{ rad} \qquad \text{(Approximately)}}$$

Recall from Chapter 1 that to convert from one unit of measure to another, multiply the first unit by a conversion factor where the numerator given in one unit equals the denominator given in another unit. That is, multiplying by this fraction (conversion factor) equal to one does not change the quantity but changes only the units. From the relation π rad = 180°, we can form the conversion factors

$$\frac{\pi \text{ rad}}{180^\circ} \quad \text{and} \quad \frac{180^\circ}{\pi \text{ rad}}$$

EXAMPLE 1

Change each angle measure from degrees to radians.

(a) $30^\circ = 30^\circ \times \dfrac{\pi \text{ rad}}{180^\circ} = \dfrac{\pi}{6} \text{ rad} = 0.524 \text{ rad}$

(b) $45^\circ = 45^\circ \times \dfrac{\pi \text{ rad}}{180^\circ} = \dfrac{\pi}{4} \text{ rad} = 0.785 \text{ rad}$

(c) $120^\circ = 120^\circ \times \dfrac{\pi \text{ rad}}{180^\circ} = \dfrac{2\pi}{3} \text{ rad} = 2.09 \text{ rad}$

(d) $36^\circ = 36^\circ \times \dfrac{\pi \text{ rad}}{180^\circ} = \dfrac{\pi}{5} \text{ rad} = 0.628 \text{ rad}$

For some of our work, we find it convenient to express radian measure in terms of π. At other times the decimal expression is more beneficial. *Note: When no unit of angle measure is given, it is understood that the angle is expressed in radians.* Thus, $\theta = \dfrac{\pi}{6}$ is understood to be $\theta = \dfrac{\pi}{6}$ rad.

EXAMPLE 2

Change each angle measure from radians to degrees.

(a) $\dfrac{\pi}{3} \text{ rad} = \dfrac{\pi}{3} \text{ rad} \times \dfrac{180^\circ}{\pi \text{ rad}} = 60^\circ$

(b) $\dfrac{5\pi}{4} \text{ rad} = \dfrac{5\pi}{4} \text{ rad} \times \dfrac{180^\circ}{\pi \text{ rad}} = 225^\circ$

(c) $6\pi \text{ rad} = 6\pi \text{ rad} \times \dfrac{180^\circ}{\pi \text{ rad}} = 1080^\circ$

(d) $1.2 \text{ rad} = 1.2 \text{ rad} \times \dfrac{180^\circ}{\pi \text{ rad}} = 68.8^\circ$

EXAMPLE 3

Given each angle in Fig. 10.23 in radians, find the reference angle, α.

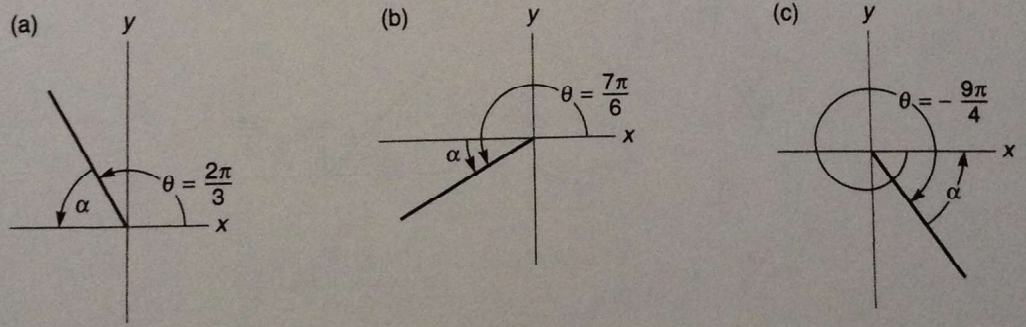


Figure 10.23

(a) $\alpha = \pi - \dfrac{2\pi}{3} = \dfrac{\pi}{3}$

(b) $\alpha = \dfrac{7\pi}{6} - \pi = \dfrac{\pi}{6}$

(c) $\alpha = \dfrac{\pi}{4}$

EXAMPLE 4

Find $\cos \frac{4\pi}{3}$.

Draw $\theta = \frac{4\pi}{3}$ as in Fig. 10.24.

$$\cos \frac{4\pi}{3} = -\cos \frac{\pi}{3} \qquad \text{(The cosine function is negative in Quadrant III.)}$$

$$= -\frac{1}{2}$$

EXAMPLE 5

Find $\tan \frac{3\pi}{4}$.

Draw $\theta = \frac{3\pi}{4}$ as in Fig. 10.25.

$$\tan \frac{3\pi}{4} = -\tan \frac{\pi}{4} \qquad \text{(The tangent function is negative in Quadrant II.)}$$

$$= -1$$

EXAMPLE 6

If $\tan \theta = -\sqrt{3}$, find θ for $0 \le \theta < 2\pi$.

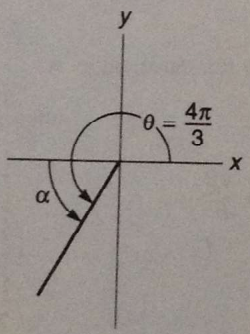
$$|-\sqrt{3}| = \sqrt{3} = \tan \frac{\pi}{3}$$

The reference angle is $\alpha = \frac{\pi}{3}$.

The tangent function is negative in Quadrants II and IV.

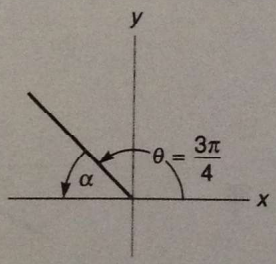
The second-quadrant angle is $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (see Fig. 10.26).

The fourth-quadrant angle is $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.



$\alpha = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$

Figure 10.24



$\alpha = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$

Figure 10.25

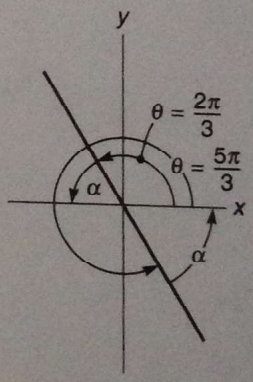


Figure 10.26

TABLE 10.4

	0 / $0°$	$\frac{\pi}{6}$ / $30°$	$\frac{\pi}{4}$ / $45°$	$\frac{\pi}{3}$ / $60°$	$\frac{\pi}{2}$ / $90°$	π / $180°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined	0
$\cot\theta$	undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	undefined
$\sec\theta$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undefined	-1
$\csc\theta$	undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	undefined

The trigonometric functions of the most used angles are tabulated in both degree and radian measures in Table 10.4.

If your calculator has a Grad button, it contains the **grad** angle system. In this system, the circle is divided into 400 equal parts, and the right angle has 100 parts (see Fig. 10.27). Each of these parts is called one **grad,** or one **new degree,** or one **grade,** written 1^g. Each grad is divided into 100 **new minutes,** written 1^c. Each new minute is divided into 100 **new seconds,** written 1^{cc}. This system is also called the **centesimal** system of measuring angles.

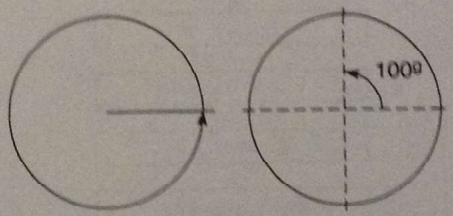


Figure 10.27 Grad angle system

A calculator may also be used to evaluate trigonometric functions in radians. Make certain that your calculator is in the radian mode.

EXAMPLE 7

Find sin 0.4, rounded to four significant digits.

Flow chart	*Buttons pushed*	*Display*
Enter sin	sin	sin
↓		
Enter 0.4	. 4	0.4
↓		
Push equals	=	0.389418

That is, $\sin 0.4 = 0.3894$ rounded to four significant digits.

EXAMPLE 8

Find cos 1.684, rounded to four significant digits.

Flow chart	*Buttons pushed*	*Display*
Enter cos	cos	cos
↓		
Enter 1.684	1 . 6 8 4	1.684
↓		
Push equals	=	−0.112962

Thus, $\cos 1.684 = -0.1130$ rounded to four significant digits.

EXAMPLE 9

Find $\tan\left(\frac{\pi}{12}\right)$, rounded to four significant digits.

Flow chart	*Buttons pushed*	*Display*
Enter tan	tan	tan
↓		
Enter π	(π	tan(π
↓		
Push divide	÷	tan(π/
↓		
Enter 12	1 2)	tan(π/12)
↓		
Push equals	=	0.267949

Thus, $\tan(\pi/12) = 0.2679$ rounded to four significant digits.

To evaluate the cotangent, secant, and cosecant functions, follow the same procedure as in Section 10.2. You may find it helpful to remember the quadrantal angles in radians as shown in Fig. 10.28.

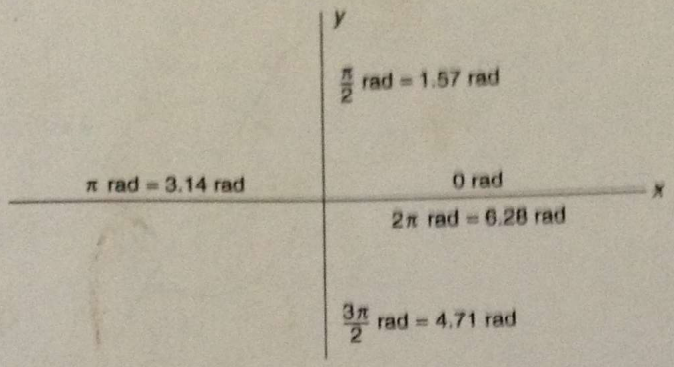


Figure 10.28

EXAMPLE 10

Given $\sin \theta = 0.7565$, find θ in radians for $0 \leq \theta < 2\pi$, rounded to four significant digits.

Flow chart	*Buttons pushed*	*Display*
Enter $\sin^{-1}$	$\sin^{-1}$	$\sin^{-1}$
↓		
Enter 0.7565	. 7 5 6 5	0.7565
↓		
Find α	=	0.857945

The reference angle is $\alpha = 0.8579$.
The sine function is positive in Quadrants I and II.
The first-quadrant angle is 0.8579 (see Fig. 10.29).
The second-quadrant angle is $\pi - 0.8579 = 2.284$ rounded to four significant digits.

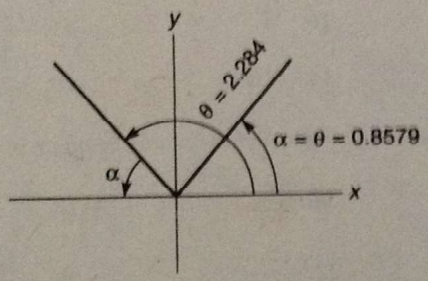


Figure 10.29

EXAMPLE 11

Given $\tan \theta = -1.632$, find θ in radians for $0 \leq \theta < 2\pi$, rounded to four significant digits.

Flow chart	*Buttons pushed*	*Display*
Enter $\tan^{-1}$	$\tan^{-1}$	$\tan^{-1}$
↓		
Enter 1.632	1 . 6 3 2	1.632
↓		
Find α	=	1.02106

The reference angle is $\alpha = 1.021$.
The tangent function is negative in Quadrants II and IV.

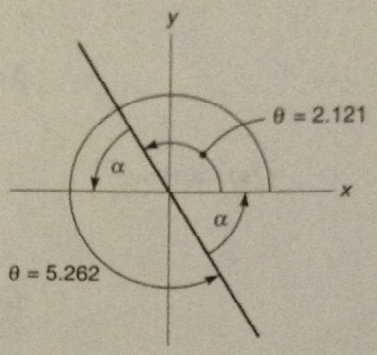


Figure 10.30

The second-quadrant angle is $\pi - 1.021 = 2.121$ (see Fig. 10.30).
The fourth-quadrant angle is $2\pi - 1.021 = 5.262$.

Exercises 10.3

Change each angle measure from degrees to radians. Express in terms of π.

1. $135°$ **2.** $210°$ **3.** $90°$ **4.** $270°$
5. $-75°$ **6.** $-230°$ **7.** $1260°$ **8.** $2490°$

Change each angle measure from radians to degrees.

9. $\frac{7\pi}{4}$ **10.** $\frac{7\pi}{6}$ **11.** $\frac{19\pi}{4}$ **12.** $\frac{27\pi}{8}$
13. 9π **14.** 12π **15.** 3.7 **16.** 1.82

Given each angle in radians, find the reference angle in radians.

17. $\frac{7\pi}{4}$ **18.** $\frac{5\pi}{6}$ **19.** $\frac{9\pi}{4}$ **20.** $\frac{4\pi}{3}$
21. $\frac{11\pi}{12}$ **22.** $\frac{17\pi}{12}$ **23.** $-\frac{8\pi}{5}$ **24.** $-\frac{15\pi}{7}$

Using Table 10.4, find the value of each trigonometric function.

25. $\sin\frac{3\pi}{4}$ **26.** $\cos\frac{7\pi}{6}$ **27.** $\tan\frac{4\pi}{3}$
28. $\sec\frac{9\pi}{4}$ **29.** $\csc\left(-\frac{3\pi}{4}\right)$ **30.** $\cot\left(-\frac{5\pi}{4}\right)$

Using Table 10.4, find θ for $0 \le \theta < 2\pi$.

31. $\cos\theta = -\frac{1}{2}$ **32.** $\sin\theta = \frac{\sqrt{3}}{2}$ **33.** $\sec\theta = 2$
34. $\tan\theta = -1$ **35.** $\cot\theta = 1$ **36.** $\csc\theta = -\frac{2\sqrt{3}}{3}$

Find the value of each trigonometric function (expressed in radians), rounded to four significant digits.

37. $\sin 0.8$ **38.** $\cos 0.4$ **39.** $\tan 1.2$

40. $\tan 0.15$ **41.** $\cos 1.0$ **42.** $\sin 0.75$

43. $\sin(-1.65)$ **44.** $\cos(-2.11)$ **45.** $\tan 18.7$

46. $\cot 23.6$ **47.** $\sec(-5.6)$ **48.** $\csc(-21.9)$

49. $\sin\left(\frac{3\pi}{4}\right)$ **50.** $\cos\left(\frac{5\pi}{8}\right)$ **51.** $\tan\left(\frac{3\pi}{5}\right)$

52. $\sin 8\pi$ **53.** $\cos\left(-\frac{\pi}{24}\right)$ **54.** $\sin\left(-\frac{5\pi}{6}\right)$

Find θ for $0 \le \theta < 2\pi$, rounded to four significant digits.

55. $\sin\theta = 0.9845$ **56.** $\cos\theta = 0.3554$ **57.** $\tan\theta = 1.685$

58. $\sin\theta = -0.6825$ **59.** $\cos\theta = -0.7540$ **60.** $\tan\theta = -0.1632$

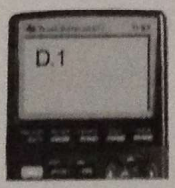
61. $\cos\theta = 0.6924$ **62.** $\sin\theta = 0.1876$ **63.** $\tan\theta = -4.672$

64. $\sec\theta = -4.006$ **65.** $\csc\theta = -2.140$ **66.** $\cot\theta = -0.1066$

67. $\cos\theta = -\frac{\sqrt{3}}{2}$ **68.** $\tan\theta = -\frac{1}{\sqrt{3}}$ **69.** $\sin\theta = \frac{1}{2}$

70. $\cot\theta = -1$ **71.** $\sec\theta = -\frac{2}{\sqrt{3}}$ **72.** $\csc\theta = -2$

10.4 USE OF RADIAN MEASURE



Radian measure has many applications in mathematics and the various technologies. We shall illustrate some of these applications by examples and exercises.

From mathematics, the length of an intercepted arc of a circle, s, equals the product of the radius, r, and the measure of the central angle, θ, in radians (see Fig. 10.31). That is,

$$\boxed{s = r\theta}$$

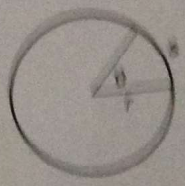


Figure 10.31

Note that when $\theta = 2\pi$, $s = 2\pi r$, where s is the circumference.

EXAMPLE 1

Find the length of the arc of a circle of radius 4.00 ft and with a central angle of 240°.

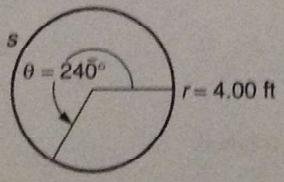


Figure 10.32

$$\theta = 240° = 240° \times \frac{\pi}{180°} = \frac{4\pi}{3}$$

$s = r\theta$ (See Fig. 10.32.)

$= (4.00 \text{ ft})\left(\frac{4\pi}{3}\right)$

$= 16.8 \text{ ft}$

Also from mathematics, the area of a sector of a circle (see Fig. 10.33) is given by the formula

$$A = \frac{1}{2} r^2 \theta$$

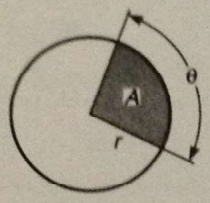


Figure 10.33

where r is the radius and θ is the measure of the central angle in radians.

EXAMPLE 2

Find the area of the sector of the circle from Example 1.

$$\begin{aligned} A &= \frac{1}{2} r^2 \theta \\ &= \frac{1}{2} (4.00 \text{ ft})^2 \left(\frac{4\pi}{3}\right) \\ &= 33.5 \text{ ft}^2 \end{aligned}$$

Other illustrations of the use of radian measure from the technologies are given in the following examples.

EXAMPLE 3

An automobile engine is rated at 200 hp at 4000 revolutions per minute (rpm). Find its torque given by the formula

$$G = \frac{P}{\omega}$$

where G is the torque, P is the power, and ω is the angular speed in radians per unit time, usually radians per s.

The horsepower measure needs to be converted to foot-pounds per second, as follows:

$$\begin{aligned} P = 200 \text{ hp} &= 200 \text{ hp} \times \frac{550 \dfrac{\text{ft-lb}}{\text{s}}}{1 \text{ hp}} \\ &= 110{,}000 \frac{\text{ft-lb}}{\text{s}} \end{aligned}$$

The angular speed needs to be converted to radians per second, as follows:

$$\omega = 4000 \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{\text{rev}} \times \frac{1 \text{ min}}{60 \text{ s}} = 419 \frac{\text{rad}}{\text{s}}$$

Therefore,

$$G = \frac{P}{\omega} = \frac{110{,}000 \dfrac{\text{ft-lb}}{\cancel{s}}}{419 \dfrac{\text{rad}}{\cancel{s}}} = 263 \text{ ft-lb}$$

Note: Recall that the *radian* unit has no physical dimensions.

EXAMPLE 4

The flywheel of a gasoline engine rotates at an angular speed of 3000 rpm. Find its angular displacement in 5.00 s.

The angular displacement of a revolving body is given by the formula

$$\theta = \omega t$$

where θ is the angular displacement in radians, ω is angular speed in radians per second, and t is the time in seconds.

$$\omega = 3000 \frac{\cancel{\text{rev}}}{\cancel{\text{min}}} \times \frac{2\pi \text{ rad}}{\cancel{\text{rev}}} \times \frac{1 \cancel{\text{min}}}{60 \text{ s}} = 314 \frac{\text{rad}}{\text{s}}$$

$$\theta = \omega t$$

$$= \left(314 \frac{\text{rad}}{\cancel{s}}\right)(5.00 \cancel{s})$$

$$= 1570 \text{ rad}$$

or

$$1570 \cancel{\text{rad}} \times \frac{1 \text{ rev}}{2\pi \cancel{\text{rad}}} = 250 \text{ revolutions}$$

EXAMPLE 5

An airplane propeller is rotating at 1800 rpm. If the blades of the propeller are 2.00 m long, find the linear speed of each point:

(a) 1.00 m from the axis of rotation.

(b) On the end of a blade.

The linear speed of a revolving body is given by the formula

$$v = \omega r$$

where v is the linear speed, ω is the angular speed, and r is the distance from the axis of rotation.

$$\omega = 1800 \frac{\cancel{\text{rev}}}{\cancel{\text{min}}} \times \frac{2\pi \text{ rad}}{\cancel{\text{rev}}} \times \frac{1 \cancel{\text{min}}}{60 \text{ s}} = 188 \frac{\text{rad}}{\text{s}}$$

(a) $v = \omega r = \left(188 \dfrac{\text{rad}}{\text{s}}\right)(1.00 \text{ m}) = 188 \text{ m/s}$

(b) $v = \omega r = \left(188 \dfrac{\text{rad}}{\text{s}}\right)(2.00 \text{ m}) = 376 \text{ m/s}$

To illustrate the difference between linear and angular velocity, consider a common ice-skating routine performed by many traveling shows. A line of skaters moves around in a circle as shown in Fig. 10.34. Who skates the fastest? As you may recall, the person

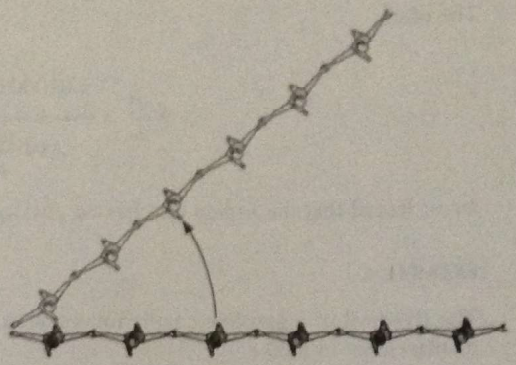


Figure 10.34

on the end usually has difficulty in keeping up and in maintaining a straight line. Thus, each successive person out from the center skates faster (has a greater linear velocity) than the person nearer the center. Note that the angular velocity of the line of skaters and of each skater remains the same as they each make the same number of rotations per unit of time.

Exercises 10.4

1. From a circle of radius 12.0 in., find the length of intercepted arc when the central angle is $\frac{2\pi}{3}$ radians.
2. From a circle of radius 8.00 in., find the length of intercepted arc when the central angle is $48°$.
3. Find the area of the sector of the circle from Exercise 1.
4. Find the area of the sector of the circle from Exercise 2.
5. Find the central angle, in degrees, of a circle of radius 6.00 in. when the intercepted arc length is 5.00 in.
6. Find the radius of a circle in which a central angle of $\frac{3\pi}{4}$ radians intercepts an arc of 24.0 in.
7. Given the two concentric circles in Fig. 10.35 where $\theta = \frac{\pi}{6}$ rad, $r_1 = 3.00$ m, and $r_2 = 5.00$ m, find the shaded area of the figure.

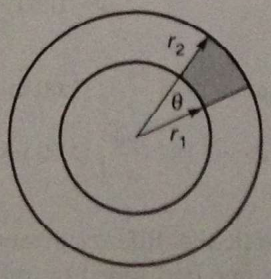


Figure 10.35

8. If a sector of a given circle has an area of 36.0 in^2 and its radius is 6.00 in., find the measure of the central angle.

9. A pendulum of length 5.00 m swings through an angle of 6°. Find the length of arc through which the pendulum swings in one complete swing.

10. An airplane travels in a circular path at 400 mi/h for 3.00 min. What is the radius of the circle when the central angle is 10°?

11. An automobile engine is rated at 275 hp at 4200 rpm. Find the torque developed.

12. A gasoline engine develops a torque of 300 ft-lb at 3600 rpm. What is its horsepower rating?

13. The flywheel of a steam engine is rotating at 420 rpm.
    (a) Express this angular speed in radians per second.
    (b) Find the angular displacement of the wheel in 10.0 s.
    (c) Find the linear speed of a point on the rim of the wheel which has a radius of 1.75 ft.

14. An airplane propeller whose blades are 6.00 ft long is rotating at 2200 rpm.
    (a) Express its angular speed in radians per second.
    (b) Find the angular displacement in 3.00 s.
    (c) Find the linear speed of a point on the end of the blade.

15. The angular velocity of an airplane propeller is increased from 1800 rpm to 2200 rpm in 10.0 s. Find its angular acceleration. The angular acceleration of a rotating body is given by the formula

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

where α is the angular acceleration, $\Delta\omega$ is the change in angular velocity, and Δt is the change in time.

16. Find the angular acceleration of the airplane propeller in Exercise 15 when its angular velocity is increased as given:
    (a) From 1800 rpm to 2200 rpm in 6.00 s.
    (b) From 1800 rpm to 2600 rpm in 10.0 s.

17. The earth rotates on its axis at 1 rev per 24 h. Assume that the average radius of the earth is 3960 mi. Find the linear speed in mi/h of a point on the equator.

18. The earth is revolving about the sun in 1 rev per 365 days. Assume an average circular orbit of radius 1.50×10^8 km. Find the linear speed in kilometres per second of the earth about the sun.

19. In the pulley system in Fig. 10.36, find the following.
    (a) The linear velocity of a point on the rim of pulley *A* in m/s.
    (b) The linear velocity of a point on the rim of pulley *B* in m/s.
    (c) The angular velocity of pulley *B* in rad/s.

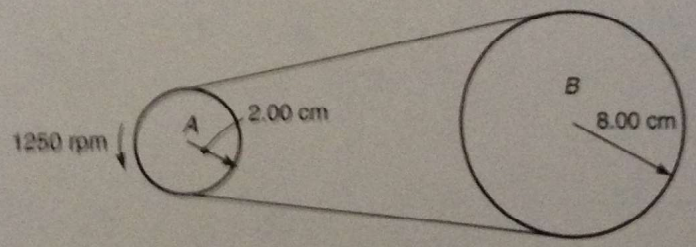


Figure 10.36

Day _____ Date _____

38. An automobile is traveling at 55.0 mi/h. Its tires have a radius of 13.0 in. Find the following.
(a) The tires' angular velocity in rad/s.
(b) The number of revolutions each tire completes in 10.0 s.
(c) The linear distance traveled in feet by a point on the tread in 10.0 s.

INTRODUCTION

A lack of rain for six weeks, a wind speed of 19 mph, and an abundance of campers had increased the likelihood of a forest fire in the Smoky Mountains. Rangers in observation towers and helicopter crews were on the alert. The ranger at tower A was the first to spot a fire in a direction 51.5° east of north. Minutes later the ranger in tower B, 5.0 miles due east of A, sighted the fire at 17.2° west of north. The ranger in station A radioed the helicopter crews and determined that there was a helicopter 4.5 miles north of station A. Because of cloud cover, the pilot could not see the fire. Did the ranger have enough information to give the pilot the location of the fire?

Solving the preceding problem requires the use of the law of sines and the law of cosines. Part of the information needed is requested in Problem 7 in Exercises 11.4. The ranger radioed the pilot to fly in a direction of 72.1° east of south for 4.2 miles to find the fire.

Objectives

- Find the sides and angles of a triangle given three parts, one of which is a side.
- Use the law of sines.
- Use the law of cosines.
- Add vectors, using trigonometric methods.
- Express vectors in terms of components.
- Use vectors to solve problems involving displacement, velocity, and force.

11.1 LAW OF SINES

An **oblique,** or **general,** triangle is a triangle that contains no right angles. We shall use the standard notation of labeling the vertices of a triangle by the capital letters A, B, and

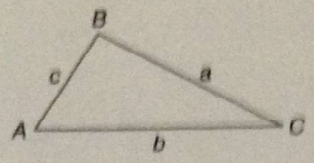


Figure 11.1

C and using the small letters a, b, and c as the labels for the sides opposite angles A, B, and C, respectively (see Fig. 11.1).

Solving a triangle means finding all those sides and angles that are not given or known. To solve a triangle, we need three parts (including at least one side). Solving any oblique triangle falls into one of four cases where the following parts of a triangle are known:

1. Two sides and an angle opposite one of them (SSA).
2. Two angles and a side opposite one of them (AAS).
3. Two sides and the included angle (SAS).
4. Three sides (SSS).

One law that we use to solve triangles is called the **law of sines**. In words, for any triangle the ratio of any side to the sine of the opposite angle is a constant. The formula for the law of sines is as follows.

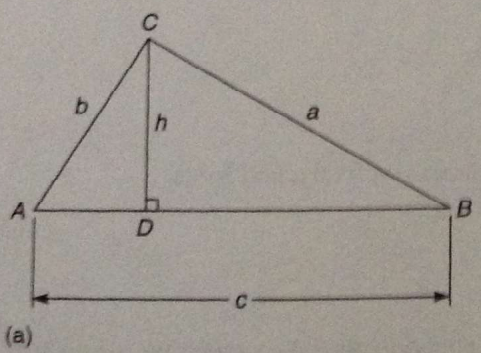
LAW OF SINES

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

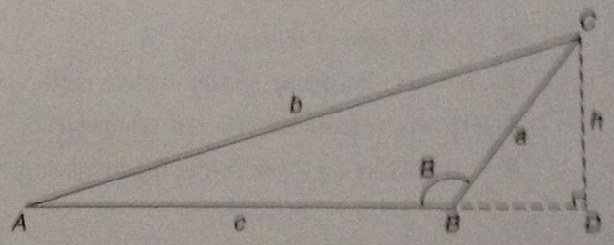
To derive the law of sines, let's take any general triangle ABC. From C draw CD perpendicular to AB. Note that every oblique triangle is either acute (all three angles are between 0° and 90°) or obtuse (one angle is between 90° and 180°). Both cases are shown in Fig. 11.2. Line CD is called the altitude, h, and forms two right triangles in each case.

In right triangle ADC in Figs. 11.2(a) and (b),

$$\sin A = \frac{h}{b} \quad \text{or} \quad h = b \sin A$$



(a)



(b)

Figure 11.2

In right triangle BCD in Fig. 11.2(a),

$$\sin B = \frac{h}{a} \quad \text{or} \quad h = a \sin B$$

In Fig. 11.2(b), note that angle DBC is the reference angle for angle ABC or angle B. Then in right triangle BDC,

$$\sin B = \sin DBC = \frac{h}{a} \quad \text{or } h = a \sin B$$

In either case we have

$$h = a \sin B = b \sin A$$

Dividing each side by $(\sin B)(\sin A)$, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

By drawing a perpendicular from A and B to the opposite side, we can show in a similar manner that $\frac{c}{\sin C} = \frac{b}{\sin B}$ and $\frac{c}{\sin C} = \frac{a}{\sin A}$. By combining these equations, we have the law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

In order to use the law of sines, we must know either of the following:

1. Two sides and an angle opposite one of them (SSA).
2. Two angles and a side opposite one of them (AAS). *Note:* Knowing two angles and any side is sufficient because knowing two angles, we can easily find the third.

You must select the proportion that contains three parts that are known and the unknown part.

As in Chapter 3, when calculations with measurements involve a trigonometric function, we shall use the following rule of thumb.

Angles expressed to the nearest:	*Lengths of sides of a triangle will contain:*
1°	Two significant digits
0.1° or 1′	Three significant digits
0.01° or 1″	Four significant digits

Day: Date:

EXAMPLE 1

If $A = 65.0°$, $a = 20.0$ m, and $b = 15.0$ m, solve the triangle.

First, draw a triangle as in Fig. 11.3 and find angle B by using the law of sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{20.0 \text{ m}}{\sin 65.0°} = \frac{15.0 \text{ m}}{\sin B}$$

$$\sin B = \frac{(15.0 \text{ m})(\sin 65.0°)}{20.0 \text{ m}} = 0.6797$$

$$B = 42.8°$$

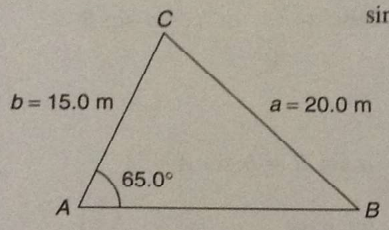


Figure 11.3

This angle may be found using a calculator as follows:

Flow chart	*Buttons pushed*	*Display*
Enter 15	15	15
↓ Push times	x	
↓ Enter sin 65°	sin 6 5	sin 65
↓ Push divide	÷	
↓ Enter 20	2 0	20
↓ Push equals	=	0.679731
↓ Find angle B	$\sin^{-1}$ → ans	$\sin^{-1}$ (ans)
↓ Push equals	=	42.8226

So $B = 42.8°$ rounded to the nearest tenth of a degree.

To find C, use the fact that the sum of the angles of any triangle is 180°. Therefore,

$$C = 180° - 65.0° - 42.8°$$

$$= 72.2°$$

Finally, find c using the law of sines.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{20.0 \text{ m}}{\sin 65.0°} = \frac{c}{\sin 72.2°}$$

$$c = \frac{(20.0 \text{ m})(\sin 72.2°)}{\sin 65.0°} = 21.0 \text{ m}$$ (Rounded to three significant digits)

This side may be found using a calculator as follows:

Flow chart	Buttons pushed	Display
Enter 20	20	20
Push times	×	
Enter sin 72.2°	sin 7 2 . 2	sin 72.2
Push divide	÷	
Enter sin 65°	sin 6 5	sin 65
Push equals	=	21.0112

That is, side $c = 21.0$ m rounded to three significant digits.
The solution is $B = 42.8°$, $C = 72.2°$, and $c = 21.0$ m.

EXAMPLE 2

If $C = 25°$, $c = 59$ ft, and $B = 108°$, solve the triangle.
First, draw a triangle as in Fig. 11.4 and find b.

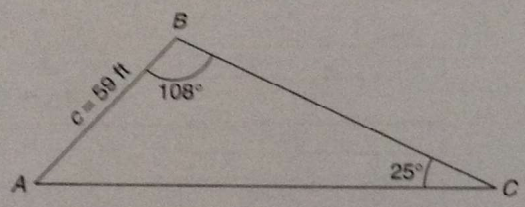


Figure 11.4

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{59 \text{ ft}}{\sin 25°} = \frac{b}{\sin 108°}$$

$$b = \frac{(59 \text{ ft})(\sin 108°)}{\sin 25°} = 130 \text{ ft} \qquad \text{(Rounded to two significant digits)}$$

$$A = 180° - 25° - 108° = 47°$$

Find a.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 47°} = \frac{59 \text{ ft}}{\sin 25°}$$

$$a = \frac{(59 \text{ ft})(\sin 47°)}{\sin 25°} = \bar{1}\bar{0}\bar{0} \text{ ft}$$

The solution is $A = 47°$, $a = \bar{1}\bar{0}\bar{0}$ ft, and $b = 130$ ft.

EXAMPLE 3

If $B = 36°21'45''$, $a = 3745$ m, and $b = 4551$ m, solve the triangle.

Draw a triangle as in Fig. 11.5 and find angle A.

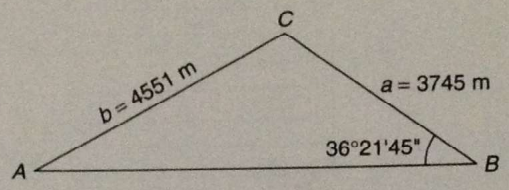


Figure 11.5

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{4551 \text{ m}}{\sin 36°21'45''} = \frac{3745 \text{ m}}{\sin A}$$

$$\sin A = \frac{(\sin 36°21'45'')(3745 \text{ m})}{4551 \text{ m}} = 0.4879$$

$$A = 29°12'7'' \qquad \text{(Rounded to the nearest second)}$$

This angle may be found using a calculator with DMS-DD buttons as follows:

Flow chart	*Buttons pushed*	*Display*
Enter sin 36°21′45″	sin → 3 6 → DMS → 2 1 → DMS → 4 5	36.2145
↓ Push times	x	
↓ Enter 3745	3 7 4 5	3745
↓ Push divide	÷	
↓ Enter 4551	4 5 5 1	4551
↓ Push equals	=	0.487888
↓ Find angle A	$\sin^{-1}$ → ans	29.2019
↓ Push DMS	DMS	29°12′06.81″

So angle $A = 29°12'7''$ rounded to the nearest second. Then

$$C = 180° - A - B$$
$$= 180° - 29°12'7'' - 36°21'45'' = 114°26'8''$$

Find side c.

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 114°26'8''} = \frac{4551 \text{ m}}{\sin 36°21'45''}$$

$$c = \frac{(\sin 114°26'8'')(4551 \text{ m})}{\sin 36°21'45''}$$

$$= 6988 \text{ m} \qquad \text{(Rounded to four significant digits)}$$

The solution is $A = 29°12'7''$, $C = 114°26'8''$, and $c = 6988$ m.

Exercises 11.1

*Solve each triangle using the labels as shown in Fig. 11.6.**

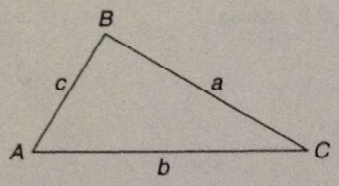


Figure 11.6

Express the lengths of sides to three significant digits and the angles to the nearest tenth of a degree.

1. $A = 69.0°$, $a = 25.0$ m, $b = 16.5$ m

2. $C = 57.5°$, $c = 166$ mi, $b = 151$ mi

3. $B = 61.4°$, $b = 124$ cm, $c = 112$ cm

4. $A = 19.5°$, $a = 487$ km, $c = 365$ km

5. $B = 75.3°$, $A = 57.1°$, $b = 257$ ft

6. $C = 59.6°$, $B = 43.9°$, $b = 4760$ m

7. $A = 115.0°$, $a = 5870$ m, $b = 4850$ m

8. $A = 16.4°$, $a = 205$ ft, $b = 187$ ft

Express the lengths of sides to four significant digits and the angles to the nearest hundredth of a degree.

9. $C = 72.58°$, $b = 28.63$ cm, $c = 42.19$ cm

10. $A = 58.95°$, $a = 3874$ m, $c = 2644$ m

11. $B = 28.76°$, $C = 19.30°$, $c = 39{,}750$ mi

12. $A = 35.09°$, $B = 48.64°$, $a = 8.362$ km

Express the lengths of sides to two significant digits and the angles to the nearest degree.

13. $A = 25°$, $a = 5\bar{0}$ cm, $b = 4\bar{0}$ cm

14. $B = 42°$, $b = 5.3$ km, $c = 4.6$ km

15. $C = 8°$, $c = 16$ m, $a = 12$ m

16. $A = 105°$, $a = 460$ mi, $c = 380$ mi

Express the lengths of sides to four significant digits and the angles to the nearest second.

17. $B = 51°17''$, $b = 1948$ ft, $c = 1525$ ft

18. $A = 49°31'50''$, $a = 37{,}560$ ft, $b = 24{,}350$ ft

19. $A = 31°14'35''$, $B = 85°45'15''$, $c = 4.575$ mi

20. $B = 75°30'6''$, $C = 70°12'18''$, $c = 93.45$ m

*Because of differences in rounding, your answers may differ slightly from the answers in the text if you choose to solve for the parts of a triangle in an order different from that chosen by the authors.

11.2 THE AMBIGUOUS CASE

The solution of a triangle when two sides and an angle opposite one of the sides (SSA) are given requires special care. There may be one, two, or no triangles formed from the given data. By construction and discussion, let's study the possibilities.

EXAMPLE 1

Construct a triangle given that $A = 35°$, $b = 10$, and $a = 7$.

As you can see from Fig. 11.7, two triangles that satisfy the given information can be drawn: triangles ACB and ACB'. Note that in one triangle angle B is acute, and in the other triangle angle B is obtuse.

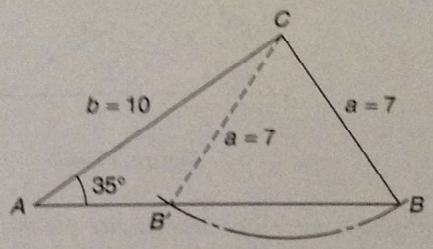


Figure 11.7

EXAMPLE 2

Construct a triangle given that $A = 45°$, $b = 10$, and $a = 5$.

As you can see from Fig. 11.8, no triangle can be drawn that satisfies the given information. Side a is simply not long enough to reach the side opposite angle C.

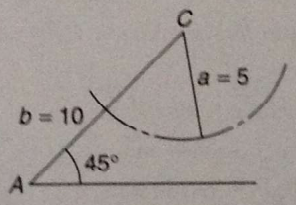


Figure 11.8

EXAMPLE 3

Construct a triangle given that $A = 60°$, $b = 6$, and $a = 10$.

As you can see from Fig. 11.9, only one triangle that satisfies the given information can be drawn. Side a is too long for two solutions.

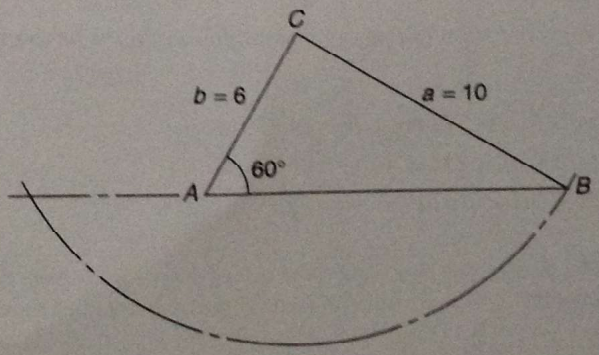
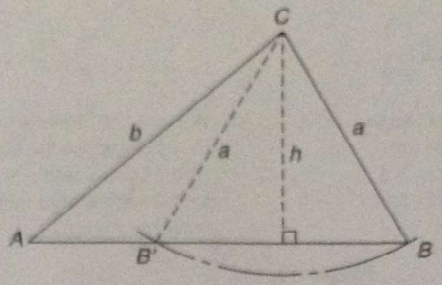
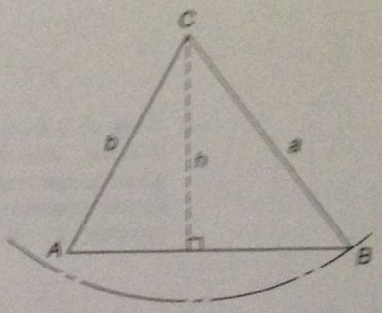


Figure 11.9

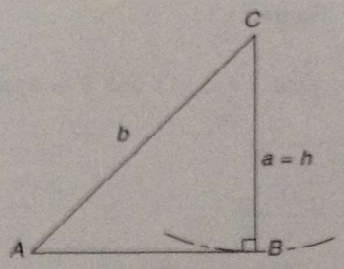
In summary, let's list the possible cases when two sides and an angle opposite one of the sides are given. Assume that *acute* angle A and adjacent side b are given. As a result of $h = b \sin A$, h is also determined. Depending on the length of the opposite side, a, we have the four cases shown in Fig. 11.10.



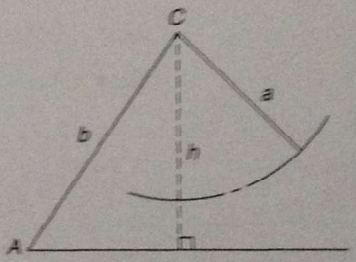
(a) When $h < a < b$, there are two possible triangles. In words, when the side opposite the given *acute* angle is less than the known adjacent side but greater than the altitude, there are two possible triangles.



(b) When $h < b < a$, there is only one possible triangle. In words, when the side opposite the given acute angle is greater than the known adjacent side, there is only one possible triangle.



(c) When $a = h$, there is one possible (right) triangle. In words, when the side opposite the given *acute* angle equals the length of the altitude, there is only one possible (right) triangle.



(d) When $a < h$, there is no possible triangle. In words, when the side opposite the given acute angle is less than the length of the altitude, there is no possible triangle.

Figure 11.10

If angle A is *obtuse*, we have two possible cases (Fig. 11.11).

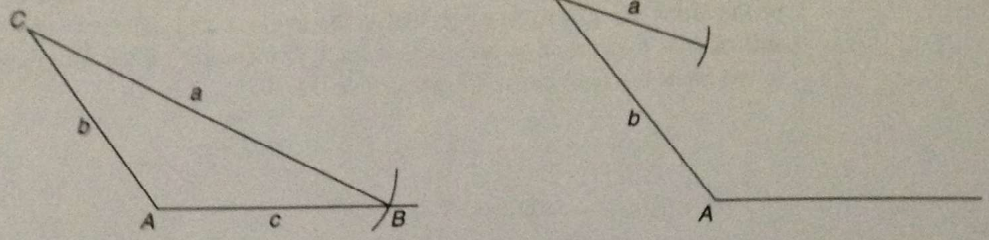
Note: If the given parts are not angle A, side opposite a, and side adjacent b as in our preceding discussions, then you must substitute the given angle and sides accordingly. This is why it is so important to understand the general word description corresponding to each case.

EXAMPLE 4

If $A = 26°$, $a = 25$ cm, and $b = 41$ cm, solve the triangle.

First, find h.

$$h = b \sin A = (41 \text{ cm})(\sin 26°) = 18 \text{ cm}$$



(a) When $a > b$, there is one possible triangle. In words, when the side opposite the given *obtuse* angle is greater than the known adjacent side, there is only one possible triangle.

(b) When $a \leq b$, there is no possible triangle. In words, when the side opposite the given *obtuse* angle is less than or equal to the known adjacent side, there is no possible triangle.

Figure 11.11

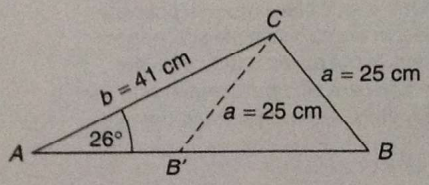


Figure 11.12

Since $h < a < b$, there are two solutions. First, let's find B in triangle ACB in Fig. 11.12.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{25 \text{ cm}}{\sin 26^\circ} = \frac{41 \text{ cm}}{\sin B}$$

$$\sin B = \frac{(41 \text{ cm})(\sin 26^\circ)}{25 \text{ cm}} = 0.7189$$

$$B = 46^\circ$$

$$C = 180^\circ - 26^\circ - 46^\circ = 108^\circ$$

Find c.

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 108^\circ} = \frac{25 \text{ cm}}{\sin 26^\circ}$$

$$c = \frac{(25 \text{ cm})(\sin 108^\circ)}{\sin 26^\circ} = 54 \text{ cm}$$

Therefore, the first solution is $B = 46^\circ$, $C = 108^\circ$, and $c = 54$ cm.

The second solution occurs when B is obtuse, as in triangle ACB'. That is, find the obtuse angle whose sine is 0.7189.

$$B' = 180^\circ - 46^\circ = 134^\circ$$

Then $C = 180^\circ - 26^\circ - 134^\circ = 20^\circ$.

For c,

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\frac{c}{\sin 20^\circ} = \frac{25 \text{ cm}}{\sin 26^\circ}$$

$$c = \frac{(25 \text{ cm})(\sin 20^\circ)}{\sin 26^\circ} = 20 \text{ cm}$$

The second solution is $B' = 134^\circ$, $C = 20^\circ$, and $c = 20$ cm.

Note: The triangles in Section 11.1 were carefully chosen so that they had only one solution.

EXAMPLE 5

If $A = 62.0^\circ$, $a = 415$ m, and $b = 855$ m, solve the triangle.

First, find h.

$$h = b \sin A$$

$$h = (855 \text{ m})(\sin 62.0^\circ)$$

$$= 755 \text{ m}$$

Since $a < h$, there is no possible solution. What would happen if you applied the law of sines anyway?

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{415 \text{ m}}{\sin 62.0^\circ} = \frac{855 \text{ m}}{\sin B}$$

$$\sin B = \frac{(855 \text{ m})(\sin 62.0^\circ)}{415 \text{ m}} = 1.819 \qquad \text{(Tilt!)}$$

Note: $\sin B = 1.819$ is impossible because $-1 \leq \sin B \leq 1$.

In summary,

1. **Given two angles and one side (AAS):** There is only one possible triangle.
2. **Given two sides and an angle opposite one of them (SSA):** There are three possibilities. If the side opposite the given angle is
   (a) greater than the known adjacent side, there is only one possible triangle.
   (b) less than the known adjacent side but greater than the altitude, there are two possible triangles.
   (c) less than the altitude, there is no possible triangle.

Since solving a general triangle requires several operations, errors are often introduced. The following points may be helpful in avoiding some of these errors:

1. Always choose a given value over a calculated value when doing calculations.
2. Always check your results to see that the largest angle is opposite the largest side, and the smallest angle is opposite the smallest side.

Exercises 11.2

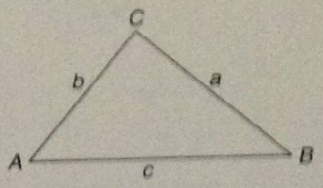


Figure 11.13

For each general triangle as in Fig. 11.13,
(a) determine the number of solutions, and
(b) solve the triangle, if possible.

Express the lengths of sides to three significant digits and the angles to the nearest tenth of a degree.

1. $A = 37.0°$, $a = 21.5$ cm, $b = 16.4$ cm
2. $B = 55.0°$, $b = 182$ m, $c = 203$ m
3. $C = 26.5°$, $c = 42.7$ km, $a = 47.2$ km
4. $B = 40.4°$, $b = 81.4$ m, $c = 144$ m
5. $A = 71.5°$, $a = 3.45$ m, $c = 3.50$ m
6. $C = 17.2°$, $c = 2.20$ m, $b = 2.00$ m
7. $B = 105.0°$, $b = 16.5$ mi, $a = 12.0$ mi
8. $A = 98.8°$, $a = 707$ ft, $b = 585$ ft

Express the lengths of sides to two significant digits and the angles to the nearest degree.

9. $C = 18°$, $c = 24$ mi, $a = 45$ mi
10. $B = 36°$, $b = 75$ cm, $a = 95$ cm
11. $C = 60°$, $c = 150$ m, $b = 180$ m
12. $A = 30°$, $a = 4800$ ft, $c = 3600$ ft
13. $B = 8°$, $b = 450$ m, $c = 850$ m
14. $B = 45°$, $c = 2.5$ m, $b = 3.2$ m

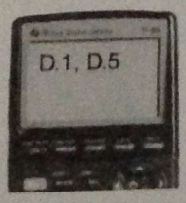
Express the lengths of sides to four significant digits and the angles to the nearest hundredth of a degree.

15. $B = 41.50°$, $b = 14.25$ km, $a = 18.50$ km
16. $A = 15.75°$, $a = 642.5$ m, $c = 592.7$ m
17. $C = 63.85°$, $c = 29.50$ cm, $b = 38.75$ cm
18. $B = 50.00°$, $b = 41{,}250$ km, $c = 45{,}650$ km
19. $C = 8.75°$, $c = 89.30$ m, $a = 61.93$ m
20. $A = 31.50°$, $a = 375.0$ mm, $b = 405.5$ mm

Express the lengths of sides to four significant digits and the angles to the nearest second.

21. $B = 29°16'37''$, $b = 215.6$ m, $c = 304.5$ m
22. $A = 61°12'30''$, $a = 3457$ ft, $c = 2535$ ft
23. $C = 25°45''$, $a = 524.5$ ft, $c = 485.6$ ft
24. $A = 21°45'$, $a = 1785$ m, $b = 2025$ m

11.3 LAW OF COSINES



When the law of sines cannot be used, we use the **law of cosines.** In words, the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of these two sides and the cosine of their included angle (see Fig. 11.14). By formula, the law is stated as follows.

LAW OF COSINES

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

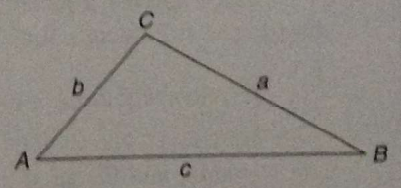


Figure 11.14

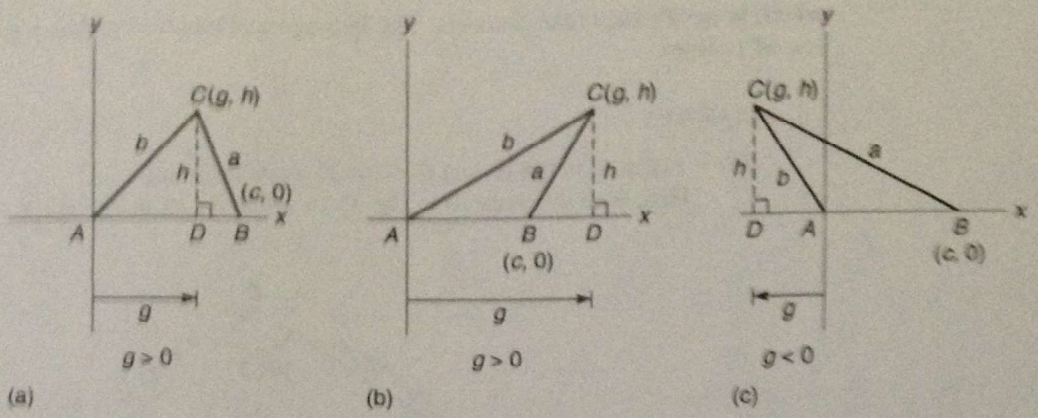


Figure 11.15

To derive the law of cosines, let triangle ABC be any general triangle. Figure 11.15 shows the three possible triangles or cases: point C between A and B, point C to the right of B, and point C to the left of A.

The length of side a in any of the three triangles in Fig. 11.15 is found using the formula for the distance between two points:

$$a = \sqrt{(g - c)^2 + (h - 0)^2}$$

$$a^2 = (g - c)^2 + (h - 0)^2 \qquad \text{(Square each side.)}$$

$$a^2 = g^2 - 2gc + c^2 + h^2 \tag{11.1}$$

From Fig. 11.15, note that $b^2 = g^2 + h^2$, and make this substitution in Equation (11.1).

$$a^2 = b^2 + c^2 - 2gc \tag{11.2}$$

Note also in each triangle that $\cos A = \dfrac{g}{b}$ or $g = b \cos A$. Now, make this substitution in Equation (11.2).

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Note: If angle A is acute, $\cos A > 0$; if angle A is obtuse, $\cos A < 0$.

The other two forms of the law of cosines may be derived in a similar manner by relabeling the vertices.

There are two cases when the law of sines does not apply and we use the law of cosines to solve triangles:

1. Two sides and the included angle are known (SAS).
2. All three sides are known (SSS).

Do you see that when the law of cosines is used, there is no possibility of an ambiguous case? If not, draw a few triangles for each of these two cases (SAS and SSS) to convince yourself intuitively.

If $A = 90°$, then $\cos A = 0$, and the law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$

reduces to

$$a^2 = b^2 + c^2$$

which is the Pythagorean theorem. The Pythagorean theorem is thus a special case of the law of cosines.

EXAMPLE 1

If $a = 112$ m, $b = 135$ m, and $C = 104.3°$, solve the triangle.

First, draw a triangle as in Fig. 11.16 and find c by using the law of cosines.

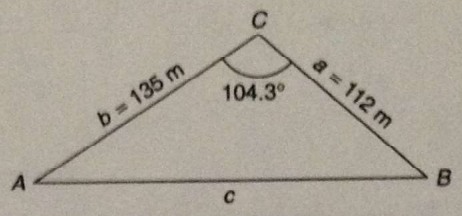


Figure 11.16

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$c^2 = (112 \text{ m})^2 + (135 \text{ m})^2 - 2(112 \text{ m})(135 \text{ m})(\cos 104.3°)$$
$$c = 196 \text{ m}$$

This side may be found using a calculator as follows:

Flow chart	*Buttons pushed*	*Display*
Enter 112^2	1 1 2 x^2	112^2
↓ Push plus	+	
↓ Enter 135^2	1 3 5 x^2	135^2
↓ Push minus	−	
↓ Enter 2	2	2
↓ Push times	×	
↓ Enter 112	1 1 2	112
↓ Push times	×	
↓ Enter 135	1 3 5	135
↓ Push times	×	
↓ Enter cos 104.3°	cos 1 0 4 . 3	cos 104.3
↓ Push equals	=	38238.3
↓ Find square root	√ ans =	195.546

So $c = 196$ m rounded to three significant digits.

To find A, use the law of sines since it requires less computation.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{112 \text{ m}}{\sin A} = \frac{196 \text{ m}}{\sin 104.3^\circ}$$

$$\sin A = \frac{(112 \text{ m})(\sin 104.3^\circ)}{196 \text{ m}} = 0.5537$$

$$A = 33.6^\circ$$

$$B = 180^\circ - 104.3^\circ - 33.6^\circ = 42.1^\circ$$

The solution is $A = 33.6^\circ$, $B = 42.1^\circ$, and $c = 196$ m.

EXAMPLE 2

If $a = 375.0$ ft, $b = 282.0$ ft, and $c = 114.0$ ft, solve the triangle.
First, draw a triangle as in Fig. 11.17 and find A by using the law of cosines.

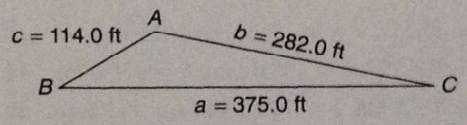


Figure 11.17

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$(375.0 \text{ ft})^2 = (282.0 \text{ ft})^2 + (114.0 \text{ ft})^2 - 2(282.0 \text{ ft})(114.0 \text{ ft}) \cos A$$

$$\cos A = \frac{(375.0 \text{ ft})^2 - (282.0 \text{ ft})^2 - (114.0 \text{ ft})^2}{-2(282.0 \text{ ft})(114.0 \text{ ft})}$$

$$\cos A = -0.7482$$

$$A = 138.43^\circ$$

(Rounded to the nearest hundredth of a degree)

Next, to find B, let's use the law of sines.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{375.0 \text{ ft}}{\sin 138.43^\circ} = \frac{282.0 \text{ ft}}{\sin B}$$

$$\sin B = \frac{(282.0 \text{ ft})(\sin 138.43^\circ)}{375.0 \text{ ft}} = 0.4990$$

$$B = 29.93^\circ$$

$$C = 180^\circ - 138.43^\circ - 29.93^\circ = 11.64^\circ$$

The solution is $A = 138.43^\circ$, $B = 29.93^\circ$, and $C = 11.64^\circ$.

Exercises 11.3

Solve each triangle using the labels shown in Fig. 11.18.

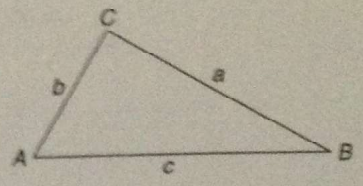


Figure 11.18

Express the lengths of sides to three significant digits and the angles to the nearest tenth of a degree.

1. $A = 60.0°$, $b = 19.5$ m, $c = 25.0$ m
2. $B = 19.5°$, $a = 21.5$ ft, $c = 12.5$ ft
3. $C = 109.0°$, $a = 140$ km, $b = 215$ km
4. $A = 94.7°$, $c = 875$ yd, $b = 185$ yd
5. $a = 19.2$ m, $b = 21.3$ m, $c = 27.2$ m
6. $a = 125$ km, $b = 195$ km, $c = 145$ km
7. $a = 4.25$ ft, $b = 7.75$ ft, $c = 5.50$ ft
8. $a = 3590$ m, $b = 7950$ m, $c = 4650$ m

Express the lengths of sides to two significant digits and the angles to the nearest degree.

9. $A = 45°$, $b = 51$ m, $c = 39$ m
10. $B = 60°$, $a = 160$ cm, $c = 230$ cm
11. $a = 7000$ m, $b = 5600$ m, $c = 4800$ m
12. $a = 5.8$ cm, $b = 5.8$ cm, $c = 9.6$ cm
13. $C = 135°$, $a = 36$ ft, $b = 48$ ft
14. $A = 5°$, $b = 19$ m, $c = 25$ m

Express the lengths of sides to four significant digits and the angles to the nearest hundredth of a degree.

15. $B = 19.25°$, $a = 4815$ m, $c = 1925$ m
16. $C = 75.00°$, $a = 37{,}550$ mi, $b = 45{,}250$ mi
17. $C = 108.75°$, $a = 405.0$ mm, $b = 325.0$ mm
18. $A = 111.05°$, $b = 1976$ ft, $c = 3250$ ft
19. $a = 207.5$ km, $b = 105.6$ km, $c = 141.5$ km
20. $a = 19.45$ m, $b = 36.50$ m, $c = 25.60$ m

Express the lengths of sides to four significant digits and the angles to the nearest second.

21. $A = 72°18'0''$, $b = 1074$ m, $c = 1375$ m
22. $C = 101°25'30''$, $a = 685.0$ ft, $b = 515.0$ ft
23. $a = 1.250$ mi, $b = 1.975$ mi, $c = 1.250$ mi
24. $a = 375.1$ m, $b = 286.0$ m, $c = 305.0$ m

11.4 APPLICATIONS OF OBLIQUE TRIANGLES

In Section 3.4, we saw several applications of right triangles. Similarly, there are many applications of oblique, or general, triangles in the world around us. The following examples and exercises illustrate a few of these applications.

EXAMPLE 1

The ground of a subdivision lot slopes upward from the street at an angle of 8°. The builder wants to build a house on a level lot that is 60 ft deep. To control erosion, the back of the lot must be cut to have a slope of 25°. How far from the street, measured along the present slope, will the excavation extend?

First, draw a diagram as in Fig. 11.19 and find AC. In $\triangle ABC$, $A = 8°$ and

$$\angle ABC = 180° - 25° = 155°$$

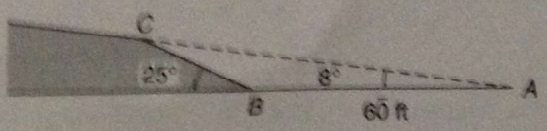


Figure 11.19

Thus, $C = 180° - 8° - 155° = 17°$.

Using the law of sines, we have

$$\frac{AC}{\sin(\angle ABC)} = \frac{AB}{\sin C}$$

$$\frac{AC}{\sin 155°} = \frac{60 \text{ ft}}{\sin 17°}$$

$$AC = \frac{(\sin 155°)(60 \text{ ft})}{\sin 17°} = 87 \text{ ft}$$

EXAMPLE 2

The Sanchez family wants a northern skylight built into the roof line of their house, as shown in Fig. 11.20. The windows for the skylight are to be 4.00 m long and make an angle of 40.0° with the horizontal.

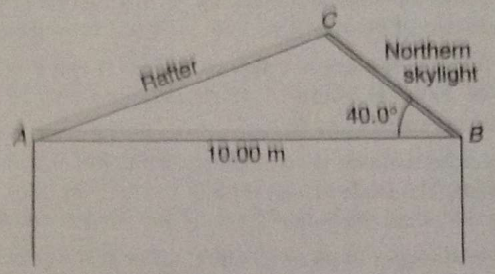


Figure 11.20

(a) How long must the rafters on the south side be?

(b) What angle does the roof make with the horizontal on the south?

(a) We need to find AC. In $\triangle ABC$, $B = 40.0°$, $AB = 10.00$ m, and $BC = 4.00$ m. Using the law of cosines, we have

$$(AC)^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos B$$

$$(AC)^2 = (10.00 \text{ m})^2 + (4.00 \text{ m})^2 - 2(10.00 \text{ m})(4.00 \text{ m})\cos 40.0°$$

$$AC = 7.40 \text{ m}$$

(b) We need to find A. Using the law of sines, we have

$$\frac{BC}{\sin A} = \frac{AC}{\sin B}$$

$$\frac{4.00 \text{ m}}{\sin A} = \frac{7.40 \text{ m}}{\sin 40.0°}$$

$$\sin A = \frac{(\sin 40.0°)(4.00 \text{ m})}{7.40 \text{ m}} = 0.3475$$

$$A = 20.3°$$

Exercises 11.4

1. Find the lengths of rafters AC and BC for the roof in Fig. 11.21.
2. The sides of a triangular metal plate measure 17.5 in., 12.3 in., and 21.3 in. Find the measure of the largest angle.

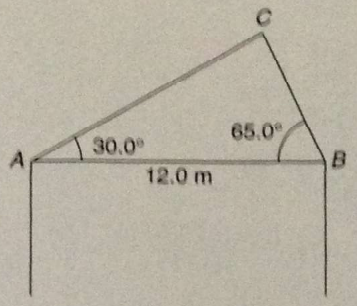


Figure 11.21

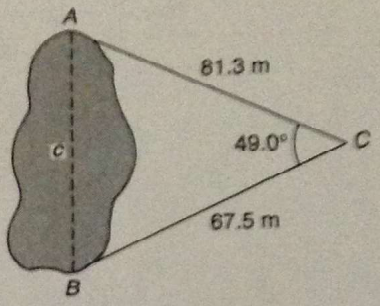


Figure 11.22

3. Find the distance c across the pond in Fig. 11.22.
4. A surveyor on the side of a hill, whose surface makes an angle of 25.0° with the horizontal, measures the angle of depression to the top of a tree at the bottom of the hill as 10.0°. Find the height of the tree if its base is 90.0 ft from the surveyor.
5. To find the distance between two points A and B lying on opposite banks of a river, a length AC of $3\overline{0}0$ m is measured. Angle BAC measures 58° and $\angle ACB$ measures 49°. Find the distance between A and B.
6. A weather balloon is sighted between points A and B, which are 3.6 mi apart on level ground. The angle of elevation of the balloon from A is 28°, and its angle of elevation from B is 49°. Find the height (in ft) of the balloon above the ground.
7. A forest ranger in an observation tower at A sights a fire in a direction 51.5° east of north. Another ranger in a tower at B 5.00 mi due east of A sights the fire at 17.2° west of north. How far is the fire from each observation tower?
8. The angle at one corner of a triangular plot of ground measures 65.5°. If the sides that meet at this corner measure 225 m and $32\overline{0}$ m, what is the length of the third side?
9. Two automobiles depart at the same time from the intersection of two straight highways, which intersect at 75°. If the automobiles' speeds are $6\overline{0}$ mph and 45 mph, how far apart will they be after 1.5 h?
10. A vertical cable television tower as in Fig. 11.23 is standing on the side of a hill, which makes an angle of 25.0° with the horizontal. Guy wires are attached $16\overline{0}$ ft up the tower. What lengths of guy wires are needed to reach points 50.0 ft uphill and 80.0 ft downhill from the base of the tower?

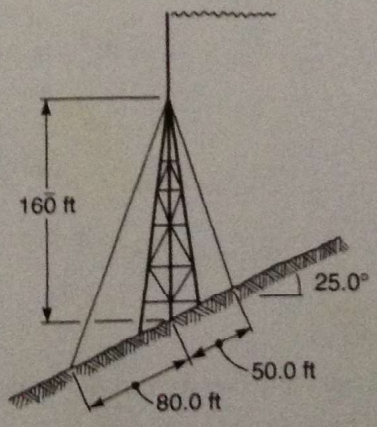


Figure 11.23

11. From point A the angle of elevation to the top of a cliff is 37° as in Fig. 11.24. On level ground the angle of elevation from point B to the top of the cliff is 24°. Point A is 270 ft closer to the cliff than point B. Find the height of the cliff.

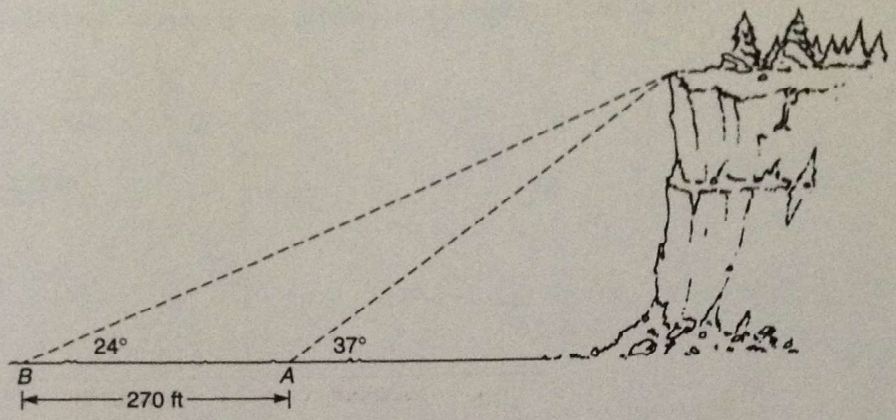


Figure 11.24

12. A lighthouse 150 ft high is on the edge of a vertical cliff overlooking the ocean. The angle of elevation from a ship to the bottom of the lighthouse measures 15°. The angle of elevation from the ship to the top of the lighthouse measures 24°.
    (a) Find the distance from the ship to the cliff.
    (b) Find the height of the cliff.
13. A tower 75 m high is on a vertical cliff on the bank of a river. From the top of the tower, the angle of depression to a piece of driftwood on the opposite bank of the river is 28°. From the bottom of the tower, the angle of depression to the same piece of driftwood is 19°.
    (a) Find the width of the river.
    (b) Find the height of the cliff.
14. Find the lengths L_1 and L_2 of the rafters in Fig. 11.25.

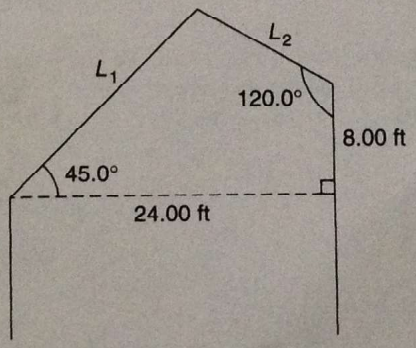


Figure 11.25

15. An owner of a triangular lot wishes to fence it in along the lot lines. Lot markers at A and B have been located, but the lot marker at C cannot be located. The owner's attorney gives the following information by phone: $AB = 245$ ft, $BC = 185$ ft, and $A = 35.0°$. What is the length of AC?
16. The average distance from the sun to Earth is 1.5×10^8 km, and from the sun to Venus it is 1.1×10^8 km. Find the distance between Earth and Venus when the angle between Earth and

the sun and Earth and Venus is 25°. (Assume that Earth and Venus have circular orbits around the sun.)

17. A farmer wants to extend the roof of a barn in Fig. 11.26 to add a storage area for machinery. If 20.0 ft of clearance is needed at the lowest point, how wide can the addition be?

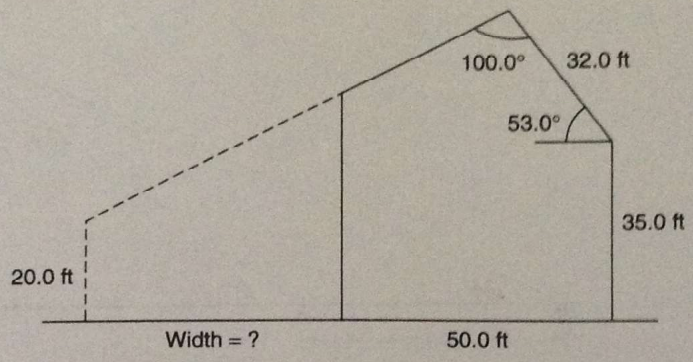


Figure 11.26

18. In the framework in Fig. 11.27, we know that $AE = CD$, $AB = BC$, $BD = BE$, and $AC \parallel ED$. Find the measure of each.

(a) $\angle AEB$ **(b)** $\angle A$ **(c)** BE **(d)** DE

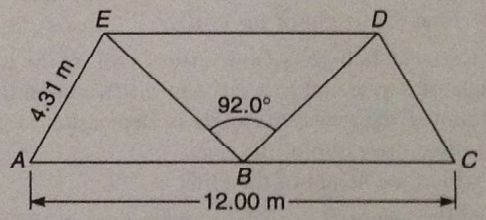


Figure 11.27

19. In the roof truss in Fig. 11.28, we know that $AB = CD$, $AG = DE$, $GF = FE$, $BG = CE$, and $BF = CF$. Find the measure of each.

(a) BG **(b)** $\angle ABG$ **(c)** BF **(d)** $\angle GFB$ **(e)** $\angle FBC$

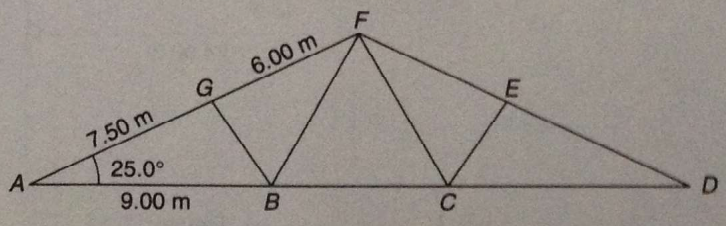


Figure 11.28

20. Find the distance between the peaks of two hills across the gorge in Fig. 11.29. Points A and B are trees on the peaks of the two hills. Point C is where you stand. Measure a length of $10\overline{0}$ m from point C to point D. Then $\angle BCD$ measures 115.0°, $\angle CDA$ measures 120.5°, $\angle BCA$ measures 86.5°, and $\angle ADB$ measures 98.1°.

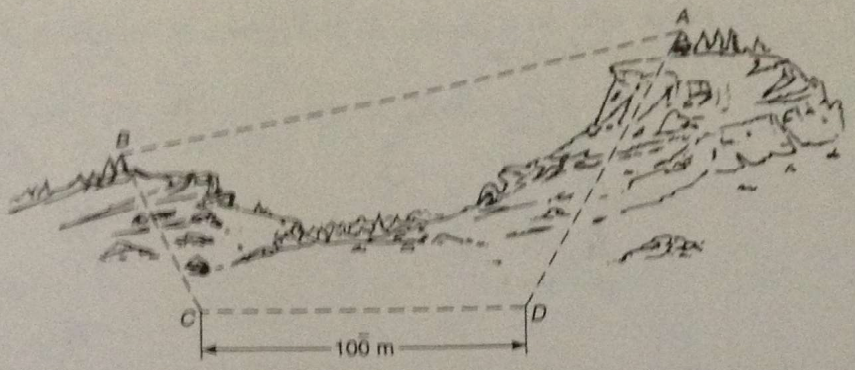


Figure 11.29

21. A deck is to be built on the house in Fig. 11.30. How long will the stairway, l, be?

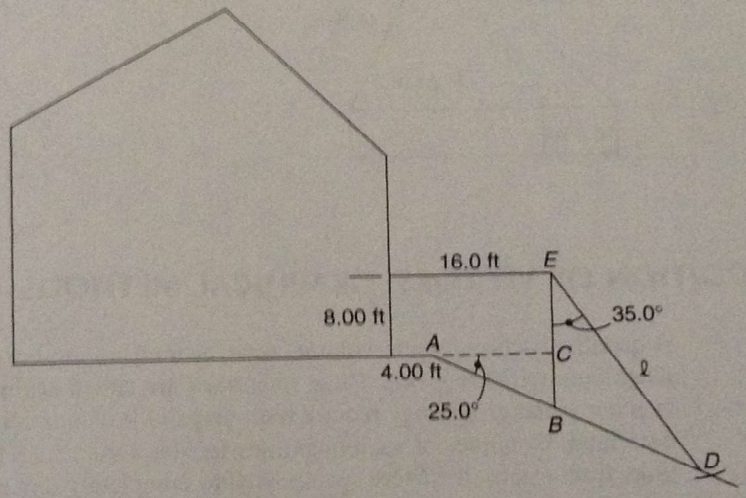


Figure 11.30

22. A dental office building in the shape of a regular pentagon is planned as in Fig. 11.31. The central records department is located at point A. The various offices receive the dental records by a conveyor system.

(a) Find the length of the conveyor from A to D.

(b) Find the length of the conveyor from A to N, the midpoint of side BC.

(c) Find the length of the conveyor from A to M, the midpoint of side CD.

Note: Each interior angle of a regular pentagon is 108°.

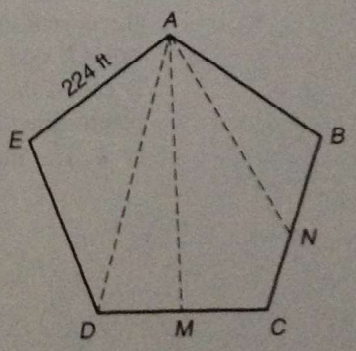
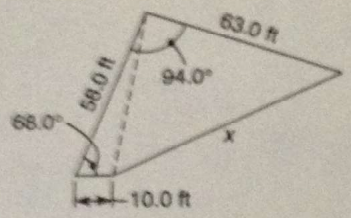


Figure 11.31

Find the value of x in each diagram. Find angles to the nearest tenth of a degree and sides to three significant digits.

23.



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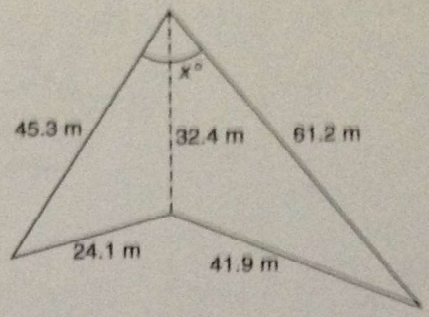

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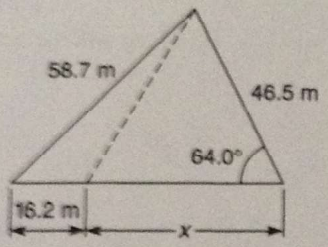
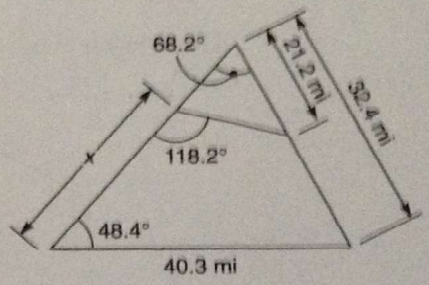


26.



11.5 ADDITION OF VECTORS: GRAPHICAL METHODS

A quantity such as length, volume, time, or temperature is completely described when its magnitude (size) is given. These quantities are called **scalars.** Other physical quantities in science and technology require both magnitude and direction in order to be completely described. Examples of such quantities include force, velocity, torque, and certain quantities from electricity. More specifically, to completely describe wind velocity, it requires not only a speed, such as 25 mph, but also the direction, 20° east of north. When a lawnmower is pushed with a given force of 20 lb, the direction of the force (angle of the handle with the ground) determines the ease of mowing. These are called **vector** quantities.

Graphically, vectors are usually represented by directed line segments. The length of a segment indicates the magnitude of the quantity. An arrowhead is used to indicate the direction (see Fig. 11.32). If A and B are the end points of a line segment, the symbol **AB** denotes the **vector from *A* to *B*;** point A is called the **initial point,** and point B is called the **terminal point.** The vector **BA** has the same length as **AB** but has the opposite direction. Vectors may also be denoted by a single lowercase letter, such as **v, u,** or **w**. The magnitude or length of a vector **v** is denoted as $|\mathbf{v}|$.

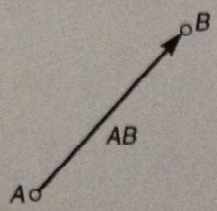


Figure 11.32

When writing vectors on paper or a chalkboard, we use a small arrow above the vector quantity, such as $\vec{v}$, $\vec{R}$, or $\overrightarrow{AB}$ as in Fig. 11.33.

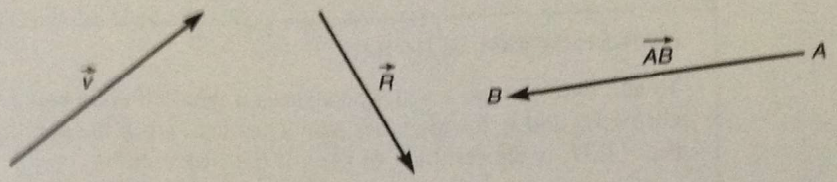


Figure 11.33

Two vectors are equal when they have the same magnitude and the same direction. Two vectors are negatives of each other when they have the same magnitude but opposite directions (see Fig. 11.34).

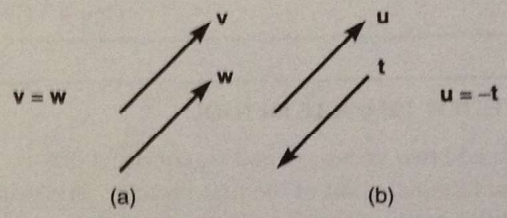


Figure 11.34

Note that a given vector **v** may be placed in any position as long as its magnitude and direction are not changed. Such vectors are called *free vectors* (see Fig. 11.35).

A vector is in **standard position** when its initial point is at the origin of the xy-coordinate system. A vector in standard position is expressed in terms of its length and angle θ, where θ is measured counterclockwise from the positive x-axis to the vector. The vector in Fig. 11.36 is expressed in standard position.

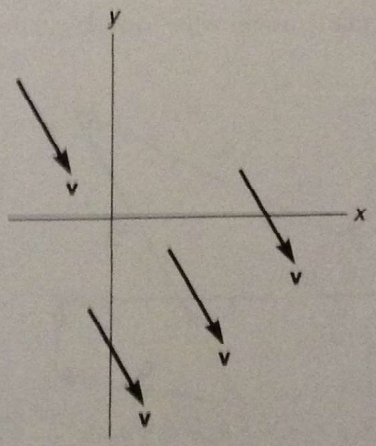


Figure 11.35

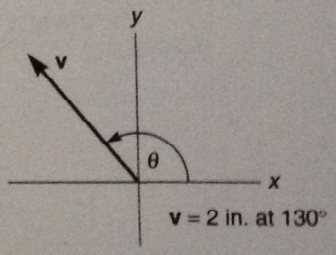


Figure 11.36

The sum of two or more vectors is called the **resultant.** This sum may be obtained graphically using one of two methods: the parallelogram method or the vector triangle method, defined as follows.

PARALLELOGRAM METHOD

To add two vectors, **v** and **w,** construct a parallelogram using **v** as one pair of parallel sides and **w** for the other pair. The diagonal of the parallelogram, as shown in Fig. 11.37, is the resultant or sum of the two vectors.

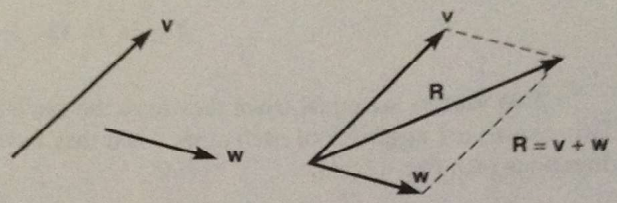


Figure 11.37

VECTOR TRIANGLE METHOD

To add two vectors, **v** and **w**, construct the second vector **w** with its initial point on the terminal point of the first vector **v**. The resultant vector is the vector joining the initial point of the first vector to the terminal point of the second vector (see Fig. 11.38).

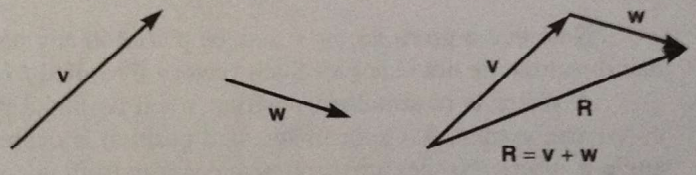
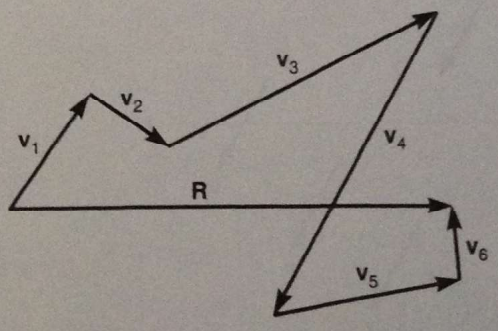


Figure 11.38

The triangle method is particularly useful when several vectors are to be added (see Fig. 11.39).



$$R = v_1 + v_2 + v_3 + v_4 + v_5 + v_6$$

Figure 11.39

A vector may be subtracted by adding its negative. That is, $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$. Construct $\mathbf{v}$ as usual, construct the negative of $\mathbf{w}$, and find the resultant as in Fig. 11.40.

Suppose a friend offers to fly you from Parkland to Kampsville and asks you how to get there. You reply, "It is 250 km." Have you given the friend enough information to find Kampsville? Obviously not! You must also tell him or her in what direction to go. If you reply, "Go 250 km due west," the friend can find Kampsville. This change in position is represented by the vector **PK** in Fig. 11.41.

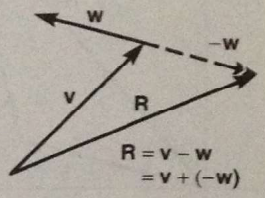


Figure 11.40

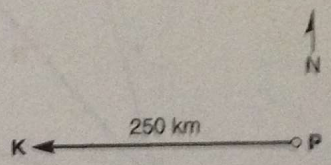


Figure 11.41

Perhaps the simplest vector is **displacement,** which is the net change in position. Displacement is a vector because it requires both a magnitude and a direction for its complete description.

Suppose your friend needs to stop at Hillsfield on the way to Kampsville, as shown in the flight plan in Fig. 11.42. Which is the displacement vector now? Since displacement is the net change in position, the displacement is the shortest distance between the beginning point and the ending point, or vector **PK**. The displacement vector is the same no matter which route is taken. This second situation may be expressed by vectors as follows:

$$\mathbf{PH} + \mathbf{HK} = \mathbf{PK}$$

This does not mean the distances are the same. In fact, we know that

$$|\mathbf{PH}| + |\mathbf{HK}| > |\mathbf{PK}|$$

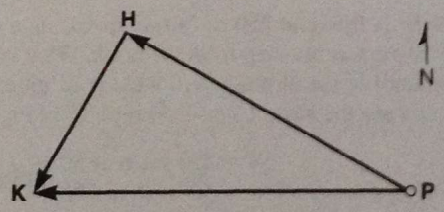


Figure 11.42

EXAMPLE 1

Find the vector sum of the following two vectors, each given in standard position, by using each graphic method:

$$\mathbf{v} = 2.4 \text{ km at } 15°$$
$$\mathbf{w} = 3.3 \text{ km at } 75°$$

Choose a suitable scale as in Fig. 11.43. Using ruler and protractor for measuring, we find that

$$\mathbf{v} + \mathbf{w} = \mathbf{R} = 5.0 \text{ km at } 50°$$

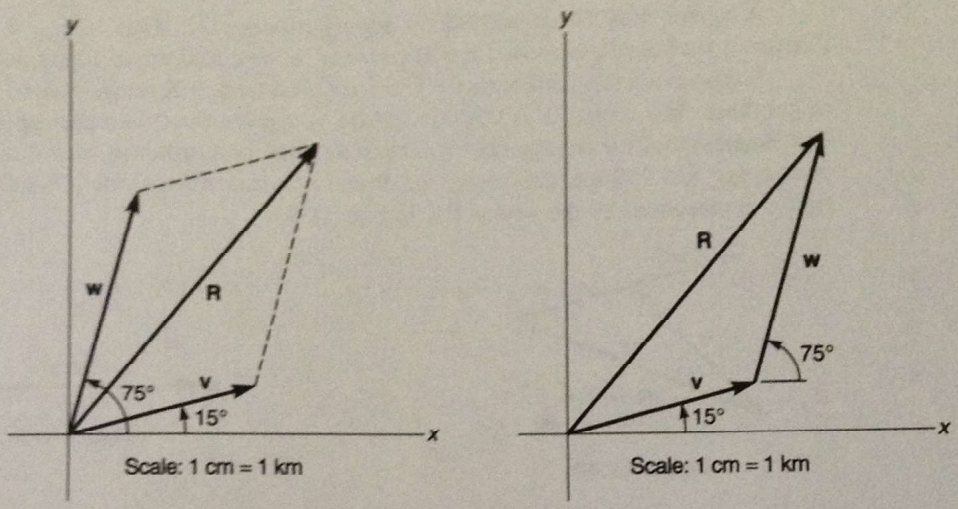


Figure 11.43

EXAMPLE 2

Given the following three vectors, each in standard position, find the vector sum graphically:

$$\mathbf{v}_1 = 38 \text{ mi at } 12\overline{0}°$$
$$\mathbf{v}_2 = 48 \text{ mi at } 35°$$
$$\mathbf{v}_3 = 82 \text{ mi at } 195°$$

Choose a suitable scale as in Fig. 11.44. Using ruler and protractor for measuring, we find that

$$\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{R} = 71 \text{ mi at } 146°$$

EXAMPLE 3

An airplane is flying at 250 mi/h (air speed) on a compass heading of $4\overline{0}°$ east of north. A wind of $5\overline{0}$ mi/h is blowing from the north. What is the true course (the heading with respect to the ground) of the airplane, and what is its ground speed?

Let's use the scale 1 cm = $5\overline{0}$ mph as in Fig. 11.45. Let

$$\left.\begin{aligned}\mathbf{v} &= 250 \text{ mi/h at } 5\overline{0}°\\ \mathbf{w} &= 5\overline{0} \text{ mi/h at } 27\overline{0}°\end{aligned}\right\} \text{(Standard position)}$$

By measuring the length of $\mathbf{R}$ with a ruler, we find

$$(4.2 \text{ cm}) \times \frac{5\overline{0} \text{ mi/h}}{\text{cm}} = 210 \text{ mi/h}$$

Now use a protractor to measure the angle that $\mathbf{R}$ makes with the north line, 51°. Therefore,

$$\mathbf{v} + \mathbf{w} = \mathbf{R} = 210 \text{ mi/h at } 51° \text{ east of north}$$
$$= 210 \text{ mi/h at } 39° \text{ (in standard position)}$$

Velocity may be defined as the time rate of change of displacement. Thus, velocity is a vector, and both its magnitude (speed) and its direction are required for its complete description.

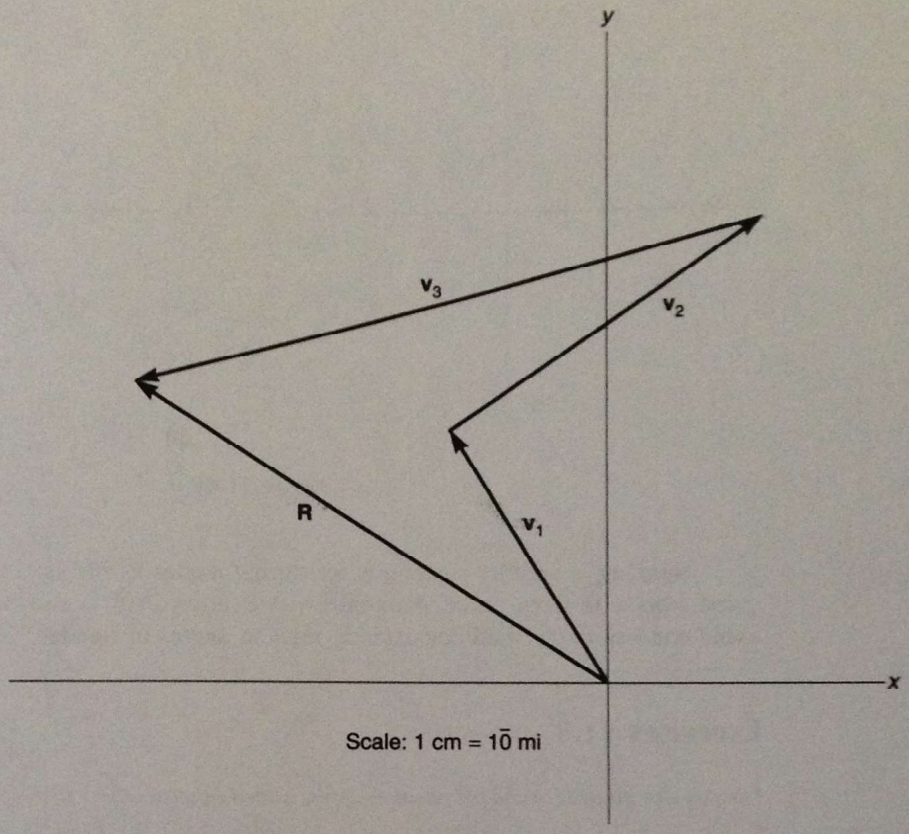


Figure 11.44

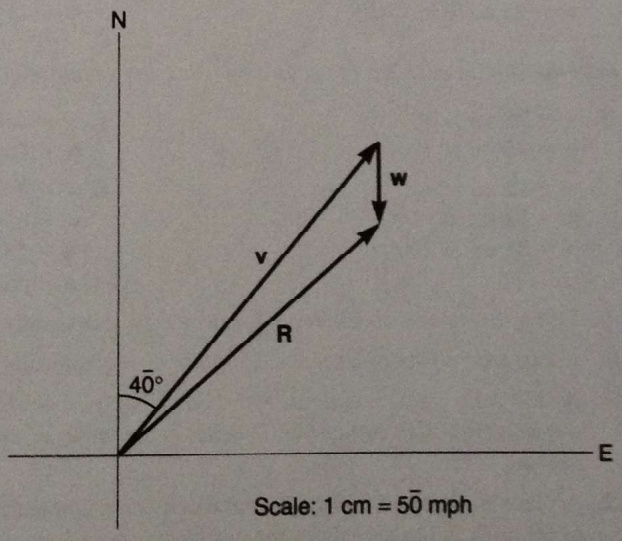


Figure 11.45

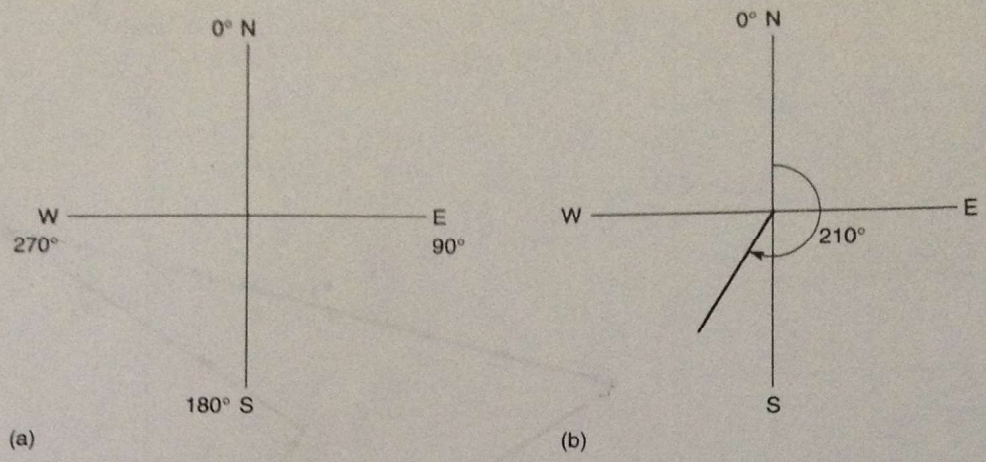


Figure 11.46

Bearing is another system of measuring angles as in Fig. 11.46. Angles are measured *clockwise* from north. An angle with bearing 210° is shown in Fig. 11.46(b). To avoid confusion, we shall consistently refer to angles in standard position.

Exercises 11.5

(Angles are given in standard position unless stated otherwise.)

Find the sum of each set of vectors by using the parallelogram method. Choose a suitable scale.

1. $\mathbf{v} = 4.5$ km at 67°
$\mathbf{w} = 6.5$ km at 105°

2. $\mathbf{v} = 150$ mi at 175°
$\mathbf{w} = 270$ mi at 215°

3. $\mathbf{v} = 75$ mi/h at 345°
$\mathbf{w} = 25$ mi/h at $27\overline{0}$°

4. $\mathbf{v} = 85$ km/h at 25°
$\mathbf{w} = 25$ km/h at 165°

Find the sum of each set of vectors by using the vector triangle method. Choose a suitable scale.

5. $\mathbf{v} = 75$ mi at $30\overline{0}$°
$\mathbf{w} = 25$ mi at 195°

6. $\mathbf{v} = 250$ km at 25°
$\mathbf{w} = 450$ km at 345°

7. $\mathbf{v} = 18$ mi at 45°
$\mathbf{w} = 32$ mi at $11\overline{0}$°
$\mathbf{t} = 28$ mi at $20\overline{0}$°

8. $\mathbf{v} = 95$ km at $35\overline{0}$°
$\mathbf{w} = 65$ km at $30\overline{0}$°
$\mathbf{t} = 25$ km at 255°
$\mathbf{u} = 75$ km at $18\overline{0}$°

9. Using the vectors in Exercise 1, find $\mathbf{v} - \mathbf{w}$ graphically.

10. Using the vectors in Exercise 2, find $\mathbf{w} - \mathbf{v}$ graphically.

11. A ship sails 55 mi due north and then sails $3\overline{0}$ mi at an angle of 45° east of north. How far is the ship from its starting point? What is the angle of the resultant path with respect to the starting point?

12. A plane is flying at $3\overline{0}0$ mi/h (air speed) on a compass heading of $4\overline{0}$° west of south. A wind of $6\overline{0}$ mi/h is blowing from the east. What are its true course and its ground speed?

13. Do Exercise 12 when the wind is blowing from the west.

Day: _____ Date: _____

11.6 ADDITION OF VECTORS: TRIGONOMETRIC METHODS

Vector sums may be found more accurately using trigonometry.

EXAMPLE 1

Using trigonometry, find the sum of the given vectors.

$\mathbf{v}$ = 19.5 km due west

$\mathbf{w}$ = 45.0 km due north

The vector triangle method gives the sketch in Fig. 11.47.

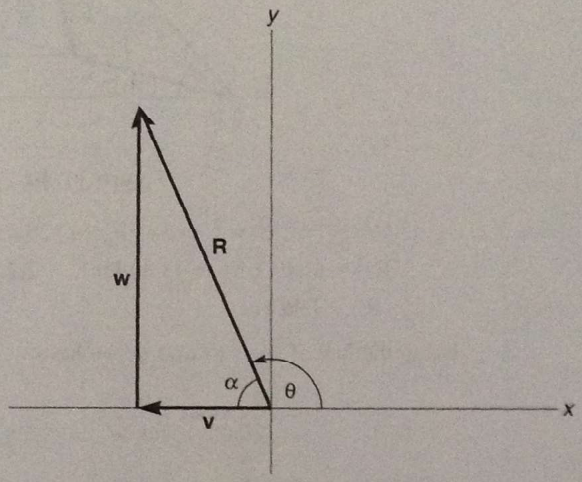


Figure 11.47

Using the Pythagorean theorem to find $|\mathbf{R}|$, we have

$$|\mathbf{R}| = \sqrt{|\mathbf{v}|^2 + |\mathbf{w}|^2}$$

$$|\mathbf{R}| = \sqrt{(19.5 \text{ km})^2 + (45.0 \text{ km})^2}$$

$$= 49.0 \text{ km}$$

Next, find α.

$$\tan \alpha = \frac{\text{side opposite } \alpha}{\text{side adjacent to } \alpha} = \frac{|\mathbf{w}|}{|\mathbf{v}|}$$

$$\tan \alpha = \frac{45.0 \text{ km}}{19.5 \text{ km}} = 2.308$$

$$\alpha = 66.6^\circ$$

$$\theta = 180^\circ - \alpha = 180^\circ - 66.6^\circ = 113.4^\circ$$

Therefore,

$$\mathbf{R} = 49.0 \text{ km at } 113.4^\circ$$

Vector sums may also be found by using the law of sines and the law of cosines.

EXAMPLE 2

Using trigonometry, find the sum of the given vectors.

$$\mathbf{v} = 2.40 \text{ km at } 15.0^\circ$$
$$\mathbf{w} = 3.30 \text{ km at } 75.0^\circ$$

First, draw the vectors as in Fig. 11.48. Use the law of cosines to find the length of $\mathbf{R}$.

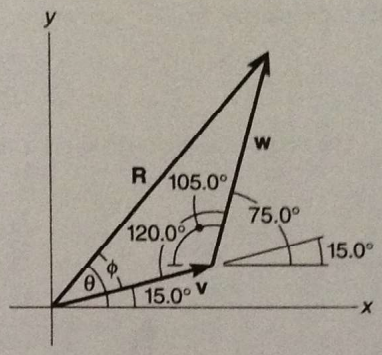


Figure 11.48

$$|\mathbf{R}|^2 = |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2|\mathbf{v}||\mathbf{w}|\cos 120.0^\circ$$
$$|\mathbf{R}|^2 = (2.40 \text{ km})^2 + (3.30 \text{ km})^2 - 2(2.40 \text{ km})(3.30 \text{ km})\cos 120.0^\circ$$
$$|\mathbf{R}| = 4.96 \text{ km}$$

Using the law of sines to find ϕ, we have

$$\frac{|\mathbf{R}|}{\sin 120.0^\circ} = \frac{|\mathbf{w}|}{\sin \phi}$$

$$\frac{4.96 \text{ km}}{\sin 120.0^\circ} = \frac{3.30 \text{ km}}{\sin \phi}$$

$$\sin \phi = \frac{(\sin 120.0^\circ)(3.30 \text{ km})}{4.96 \text{ km}} = 0.5762$$

$$\phi = 35.2^\circ$$
$$\theta = \phi + 15.0^\circ = 35.2^\circ + 15.0^\circ = 50.2^\circ$$

Therefore,

$$\mathbf{R} = 4.96 \text{ km at } 50.2^\circ$$

Exercises 11.6

(Angles are given in standard position unless stated otherwise.)

Using trigonometry, find the sum of each set of vectors.

1. $\mathbf{v} = 65.3$ km/h at 270.0°
   $\mathbf{w} = 40.5$ km/h at 180.0°
2. $\mathbf{v} = 6150$ m due south
   $\mathbf{w} = 1780$ m due east
3. $\mathbf{v} = 4.50$ km at 67.0°
   $\mathbf{w} = 6.50$ km at 105.0°
4. $\mathbf{v} = 15\bar{0}$ mi at 175.0°
   $\mathbf{w} = 27\bar{0}$ mi at 215.0°

5. $\mathbf{v} = 87.1$ mi/h at 130.5°
   $\mathbf{w} = 46.7$ mi/h at 207.0°

6. $\mathbf{v} = 60.0$ km/h at 286.0°
   $\mathbf{w} = 60.0$ km/h at 254.0°

7. $\mathbf{v} = 605$ m at 60.0°
   $\mathbf{w} = 415$ m at 120.0°
   $\mathbf{t} = 295$ m at 90.0°

8. $\mathbf{v} = 15.3$ mi at 135.0°
   $\mathbf{w} = 24.5$ mi at 75.0°
   $\mathbf{t} = 19.7$ mi at 180.0°

9. Using the vectors in Exercise 3, find $\mathbf{v} - \mathbf{w}$.

10. Using the vectors in Exercise 8, find $\mathbf{w} - \mathbf{v}$.

11. An airplane takes off and flies 175 km on a course of 15.0° west of north and then changes course and flies 105 km due north to where it lands. Find the displacement from the starting point to the landing point.

12. A ship travels 75.0 mi on a course 25.0° north of east; then it travels 45.0 mi on a course 15.0° east of north to a point where it lands. Find the displacement from the starting point to the landing point.

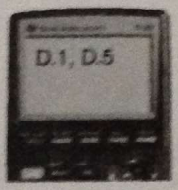
13. An automobile is driven 25.0 km due east, then 10.0 km due north, then 15.0 km due east, then 10.0 km due south. Find the displacement from the starting point to the ending point.

14. En route between two cities, a car travels $10\overline{0}$ km due west, then 125 km southwest, then $15\overline{0}$ km due south. Find the displacement from the starting point to the ending point.

15. A plane is traveling south at 175 mi/h in still air. What is its velocity with a wind of 65.0 mi/h blowing from the east?

16. A plane is traveling due north at 175 mi/h in still air. What is its velocity with a wind of 65.0 mi/h blowing from 45.0° east of south?

11.7 VECTOR COMPONENTS



We shall find it desirable to express a given vector as the sum of two vectors, especially two vectors along the coordinate axes. In Fig. 11.49,

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$
$$\mathbf{v} = \mathbf{u}_1 + \mathbf{u}_2$$
$$\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y$$

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_x$, and $\mathbf{v}_y$ are called **components** of vector $\mathbf{v}$. That is, if two or more vectors are added and their sum is the resultant vector $\mathbf{v}$, then each of these vectors is called a component of $\mathbf{v}$. We call $\mathbf{v}_x$ the **horizontal component** and $\mathbf{v}_y$ the **vertical component.**

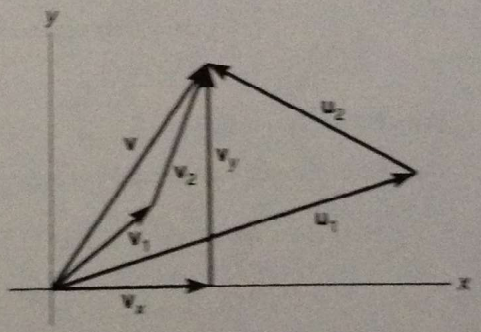


Figure 11.49

In general, the horizontal and vertical components may be found using the definitions for sine and cosine.

$$\mathbf{v}_x = |\mathbf{v}|\cos\theta$$
$$\mathbf{v}_y = |\mathbf{v}|\sin\theta$$

where θ is in standard position (see Fig. 11.50).

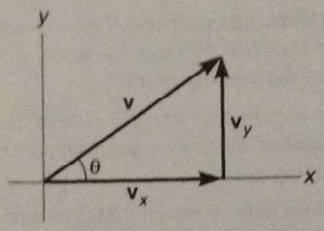


Figure 11.50

EXAMPLE 1

Find the horizontal and vertical components of the following vector in standard position: 50.0 mi/h at 60.0°.

First, draw the vector as in Fig. 11.51. Thus,

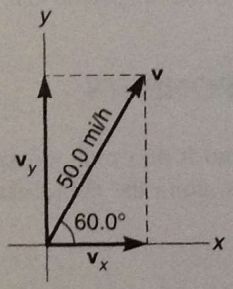


Figure 11.51

$$\begin{aligned}
\mathbf{v}_x &= |\mathbf{v}|\cos\theta \\
&= (50.0 \text{ mi/h})(\cos 60.0°) \\
&= 25.0 \text{ mi/h} \qquad \text{(At 0° is understood.)} \\
\mathbf{v}_y &= |\mathbf{v}|\sin\theta \\
&= (50.0 \text{ mi/h})(\sin 60.0°) \\
&= 43.3 \text{ mi/h} \qquad \text{(At 90° is understood.)}
\end{aligned}$$

EXAMPLE 2

Find the horizontal and vertical components of the following vector in standard position: $10\overline{0}$ m at 153°.

First, draw the vector as in Fig. 11.52. Then

$$\begin{aligned}
\mathbf{v}_x &= |\mathbf{v}|\cos\theta \\
&= (10\overline{0} \text{ m})(\cos 153°) \\
&= -8.9 \text{ m}
\end{aligned}$$

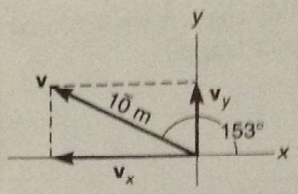


Figure 11.52

$$\begin{aligned} v_y &= |\mathbf{v}| \sin \theta \\ &= (10 \text{ m})(\sin 153°) \\ &= 4.5 \text{ m} \end{aligned}$$

EXAMPLE 3

The person in Fig. 11.53 exerts a 50-lb force on the handle of a lawnmower that is at an angle of 40° with the ground. What is the net horizontal component of the force that pushes the mower ahead? What is the net vertical component of the force that pushes the mower into the ground?

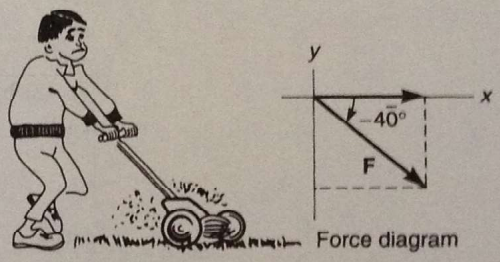


Figure 11.53

$$\begin{aligned} \mathbf{F}_x &= |\mathbf{F}| \cos \theta \\ &= (50 \text{ lb})[\cos(-40°)] \\ &= 38 \text{ lb} \\ \mathbf{F}_y &= |\mathbf{F}| \sin \theta \\ &= (50 \text{ lb})[\sin(-40°)] \\ &= -32 \text{ lb} \end{aligned}$$

That is, a 50-lb force at −40° and a 38-lb force at 0° produce the same force in keeping the lawnmower moving forward.

We must also be able to find a vector $\mathbf{v}$ when its horizontal and vertical components, $\mathbf{v}_x$ and $\mathbf{v}_y$, are given. In general, to express $\mathbf{v}$ in standard position, use these steps:

1. The length of $\mathbf{v}$ may be found by using the Pythagorean theorem.

$$|\mathbf{v}| = \sqrt{|\mathbf{v}_x|^2 + |\mathbf{v}_y|^2}$$

2. To find angle θ, we first need to find α, the reference angle, by using the definition of tangent.

$$\tan \alpha = \frac{|\mathbf{v}_y|}{|\mathbf{v}_x|}$$

EXAMPLE 4

If $\mathbf{v}_x = 36$ km/h and $\mathbf{v}_y = 52$ km/h, find $\mathbf{v}$.

First, draw the components and $\mathbf{v}$ as in Fig. 11.54. Then

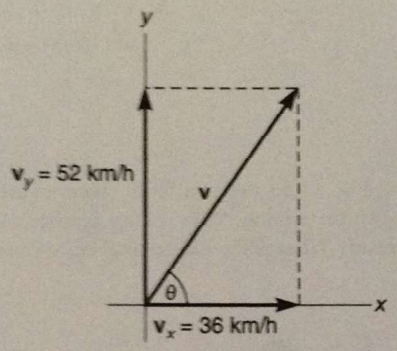


Figure 11.54

$$|\mathbf{v}| = \sqrt{|\mathbf{v}_x|^2 + |\mathbf{v}_y|^2}$$
$$= \sqrt{(36 \text{ km/h})^2 + (52 \text{ km/h})^2}$$
$$= 63 \text{ km/h}$$

$$\tan \alpha = \frac{|\mathbf{v}_y|}{|\mathbf{v}_x|}$$

$$\tan \alpha = \frac{52 \text{ km/h}}{36 \text{ km/h}} = 1.444$$

$$\alpha = 55°$$

Since θ is in the first quadrant,

$$\alpha = \theta = 55°$$

Therefore, $\mathbf{v} = 63$ km/h at 55°

EXAMPLE 5

If $\mathbf{v}_x = -26.4$ N and $\mathbf{v}_y = -15.3$ N, find $\mathbf{v}$.

First, draw the components and $\mathbf{v}$ as in Fig. 11.55.

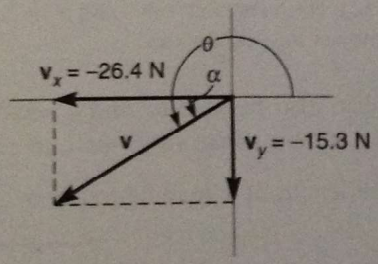


Figure 11.55

$$|\mathbf{v}| = \sqrt{|\mathbf{v}_x|^2 + |\mathbf{v}_y|^2}$$
$$= \sqrt{|-26.4\ \text{N}|^2 + |-15.3\ \text{N}|^2}$$
$$= 30.5\ \text{N}$$
$$\tan\alpha = \frac{|\mathbf{v}_y|}{|\mathbf{v}_x|}$$
$$\tan\alpha = \frac{15.3\ \text{N}}{26.4\ \text{N}} = 0.5795$$
$$\alpha = 30.1°$$

Since θ is in the third quadrant,

$$\theta = 180° + \alpha = 180° + 30.1° = 210.1°$$

Therefore,

$$\mathbf{v} = 30.5\ \text{N at } 210.1°$$

The **impedance** of a series circuit containing a **resistance** and an **inductance** can be represented by the vector diagram in Fig. 11.56, where ϕ is the phase angle that equals the amount by which the *current lags behind the voltage.* The resistance is always drawn as a vector pointing in the positive x-direction, and the inductive reactance is always drawn as a vector pointing in the positive y-direction.

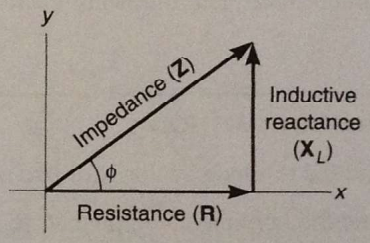


Figure 11.56

EXAMPLE 6

If the resistance is $60\bar{}$ ohms (Ω) and the inductive reactance is 36 Ω, find the impedance.

Using Fig. 11.56, we have

$$|\mathbf{Z}| = \sqrt{(60\ \Omega)^2 + (36\ \Omega)^2} = 70\bar{}\ \Omega$$
$$\tan\phi = \frac{36\ \Omega}{60\bar{}\ \Omega} = 0.6000$$
$$\phi = 31°$$

Therefore,

$$\mathbf{Z} = 70\bar{}\ \Omega \text{ at } 31°$$

The **impedance** of a series circuit containing a **resistance** and a **capacitance** can be represented by the vector diagram in Fig. 11.57, where ϕ is the phase angle that equals the amount by which the *voltage lags behind the current.* The resistance is always drawn

as a vector pointing in the positive x-direction, and the capacitive reactance is drawn as a vector pointing in the negative y-direction.

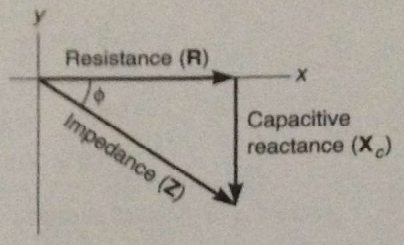


Figure 11.57

EXAMPLE 7

If the impedance is 120 Ω and $\phi = 40°$, find the resistance and the capacitive reactance.

Using Fig. 11.57, we have

$$|\mathbf{R}| = |\mathbf{Z}|\cos\phi = (120\ \Omega)(\cos 40°) = 92\ \Omega$$
$$|\mathbf{X}_c| = |\mathbf{Z}|\sin\phi = (120\ \Omega)(\sin 40°) = 77\ \Omega$$

Addition of vectors using graphic methods has a limited degree of accuracy. Also, addition of vectors using the law of sines and the law of cosines, while accurate, is sometimes cumbersome. The following method of adding vectors is accurate and rather efficient.

COMPONENT METHOD

To find the resultant vector **R** of two or more vectors using the component method:

1. Find the horizontal component, $\mathbf{R}_x$, of vector **R** by finding the algebraic sum of the horizontal components of each of the vectors being added.
2. Find the vertical component, $\mathbf{R}_y$, of vector **R** by finding the algebraic sum of the vertical components of each of the vectors being added.
3. Find the length of **R**:

$$|\mathbf{R}| = \sqrt{|\mathbf{R}_x|^2 + |\mathbf{R}_y|^2}$$

4. To find angle θ, first find α, the reference angle.

$$\tan\alpha = \frac{|\mathbf{R}_y|}{|\mathbf{R}_x|}$$

EXAMPLE 8

Using the component method, find the sum of the given vectors.

$$\mathbf{v} = 25.0\ \text{km at } 121.0°$$
$$\mathbf{w} = 66.0\ \text{km at } 245.0°$$

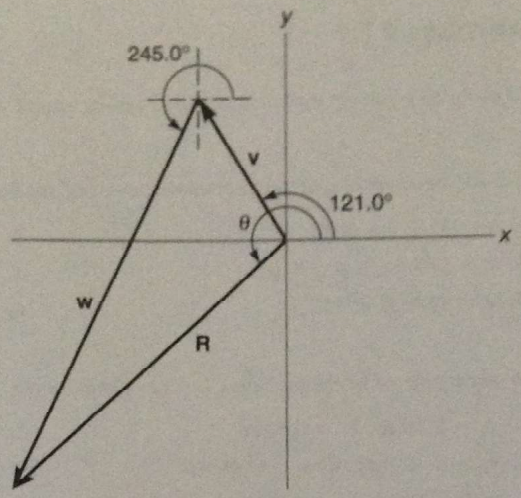


Figure 11.58

First, draw the components and **R** as in Fig. 11.58.

$$\begin{aligned} \mathbf{v}_x &= |\mathbf{v}|\cos\theta \\ &= (25.0\text{ km})(\cos 121.0^\circ) \\ &= -12.9\text{ km} \\ \mathbf{w}_x &= |\mathbf{w}|\cos\theta \\ &= (66.0\text{ km})(\cos 245.0^\circ) \\ &= -27.9\text{ km} \\ \mathbf{R}_x\text{: Sum of } x\text{-components} &= -40.8\text{ km} \end{aligned}$$

$$\begin{aligned} \mathbf{v}_y &= |\mathbf{v}|\sin\theta \\ &= (25.0\text{ km})(\sin 121.0^\circ) \\ &= 21.4\text{ km} \\ \mathbf{w}_y &= |\mathbf{w}|\sin\theta \\ &= (66.0\text{ km})(\sin 245.0^\circ) \\ &= -59.8\text{ km} \\ \mathbf{R}_y\text{: Sum of } y\text{-components} &= -38.4\text{ km} \end{aligned}$$

Then

$$\begin{aligned} \mathbf{R} &= \sqrt{|\mathbf{R}_x|^2 + |\mathbf{R}_y|^2} \\ &= \sqrt{|-40.8\text{ km}|^2 + |-38.4\text{ km}|^2} \\ &= 56.0\text{ km} \end{aligned}$$

$$\tan\alpha = \frac{|\mathbf{R}_y|}{|\mathbf{R}_x|}$$

$$\tan\alpha = \frac{38.4\text{ km}}{40.8\text{ km}} = 0.9412$$

$$\alpha = 43.3^\circ$$

Since $\mathbf{R}_x < 0$ and $\mathbf{R}_y < 0$, **R** is in the third quadrant. So

$$\theta = 180^\circ + \alpha = 180^\circ + 43.3^\circ = 223.3^\circ$$

Therefore,

$$\mathbf{R} = 56.0\text{ km at } 223.3^\circ$$

Exercises 11.7

(Angles are given in standard position unless stated otherwise.)

Find the horizontal and vertical components of each vector given in standard position.

1. $\mathbf{v} = 18.2$ km at $85.0°$
2. $\mathbf{v} = 27.9$ mi at $138.0°$
3. $\mathbf{v} = 135$ mi/h at $270.0°$
4. $\mathbf{v} = 448$ m at $319.0°$
5. $\mathbf{v} = 2680$ ft at $152.5°$
6. $\mathbf{v} = 3620$ ft at $187.3°$

For each pair of horizontal and vertical components, find the vector.

7. $\mathbf{v}_x = 8.70$ m, $\mathbf{v}_y = 6.40$ m
8. $\mathbf{v}_x = -2.10$ ft, $\mathbf{v}_y = 3.20$ ft
9. $\mathbf{v}_x = 4.70$ m/s, $\mathbf{v}_y = -6.60$ m/s
10. $\mathbf{v}_x = -925$ m, $\mathbf{v}_y = 125$ m
11. $\mathbf{v}_x = -14.7$ km, $\mathbf{v}_y = 0$
12. $\mathbf{v}_x = 427$ mi/h, $\mathbf{v}_y = 381$ mi/h
13. A series circuit containing a resistance and an inductance has an impedance of $90\ \Omega$, and the phase angle is $20°$. Find the resistance and the inductive reactance.
14. A series circuit contains a resistance of $85\ \Omega$ and has an inductive reactance of $45\ \Omega$. Find the impedance and the phase angle.
15. A series circuit contains a resistance of $240\ \Omega$ and a capacitive reactance of $140\ \Omega$. Find the impedance and the phase angle.
16. A series circuit containing a resistance and a capacitance has an impedance of $110\ \Omega$, and $\phi = 49°$. Find the resistance and the capacitive reactance.

Using the component method, find the sum of each set of vectors. Find the magnitudes to three significant digits and the angles to the nearest tenth of a degree.

17. $\mathbf{v} = 324$ ft at $0°$
    $\mathbf{w} = 576$ ft at $90.0°$
18. $\mathbf{v} = 91.2$ km/h at $180.0°$
    $\mathbf{w} = 84.7$ km/h at $270.0°$
19. $\mathbf{v} = 28.9$ mi/h at $52.0°$
    $\mathbf{w} = 16.2$ mi/h at $310.0°$
20. $\mathbf{v} = 59.7$ km at $125.0°$
    $\mathbf{w} = 86.4$ km at $298.0°$
21. $\mathbf{v} = 655$ km at $108.0°$
    $\mathbf{w} = 655$ km at $27.0°$
    $\mathbf{u} = 655$ km at $270.0°$
22. $\mathbf{v} = 29.7$ mi at $237.0°$
    $\mathbf{w} = 16.4$ mi at $180.0°$
    $\mathbf{u} = 18.5$ mi at $15.0°$
23. $\mathbf{v} = 5020$ m at $0°$
    $\mathbf{w} = 3130$ m at $148.0°$
    $\mathbf{u} = 6250$ m at $65.0°$
    $\mathbf{t} = 4620$ m at $335.0°$
24. $\mathbf{v} = 5760$ ft at $90.0°$
    $\mathbf{w} = 3940$ ft at $205.0°$
    $\mathbf{u} = 6140$ ft at $150.0°$
    $\mathbf{t} = 1230$ ft at $330.0°$
25. Using the vectors in Exercise 19, find $\mathbf{v} - \mathbf{w}$.
26. Using the vectors in Exercise 20, find $\mathbf{w} - \mathbf{v}$.
27. An airplane flies 165 mi on a course of $25.0°$ west of south; then it changes course and flies 125 mi on a course of $15.0°$ north of west to where it lands. Find the displacement from the starting point to the landing point.
28. A ship travels 75.0 mi on a course of $35.0°$ west of north; then it travels 45.0 mi on a course of $60.0°$ south of west to where it docks. Find the displacement from the starting point to the docking point.

11.8 VECTOR APPLICATIONS

Many applications involve vector quantities. Displacement, velocity, and force have been chosen here to illustrate some basic applications.

Displacement

Recall that **displacement** is a change of position; that is, the difference between the initial position of a body and any later position.

EXAMPLE 1

A ship travels 175 mi at 52.0°, then 295 mi at 141.0°, and then 225 mi due west. Find the displacement; that is, find the net distance between the starting point and the end point and the angle.

Using the component method, let

$$\mathbf{u} = 175 \text{ mi at } 52.0°$$
$$\mathbf{v} = 295 \text{ mi at } 141.0°$$
$$\mathbf{w} = 225 \text{ mi at } 180.0°$$

Draw the components and $\mathbf{R}$ as in Fig. 11.59. Then we have the following:

$$\begin{aligned} u_x &= |\mathbf{u}|\cos\theta \\ &= (175 \text{ mi})(\cos 52.0°) \\ &= 108 \text{ mi} \\ v_x &= |\mathbf{v}|\cos\theta \\ &= (295 \text{ mi})(\cos 141.0°) \\ &= -229 \text{ mi} \\ w_x &= |\mathbf{w}|\cos\theta \\ &= (225 \text{ mi})(\cos 180.0°) \\ &= -225 \text{ mi} \\ R_x\text{: Sum of } x\text{-components} &= -346 \text{ mi} \end{aligned}$$

$$\begin{aligned} u_y &= |\mathbf{u}|\sin\theta \\ &= (175 \text{ mi})(\sin 52.0°) \\ &= 138 \text{ mi} \\ v_y &= |\mathbf{v}|\sin\theta \\ &= (295 \text{ mi})(\sin 141.0°) \\ &= 186 \text{ mi} \\ w_y &= |\mathbf{w}|\sin\theta \\ &= (225 \text{ mi})(\sin 180.0°) \\ &= 0 \text{ mi} \\ R_y\text{: Sum of } y\text{-components} &= 324 \text{ mi} \end{aligned}$$

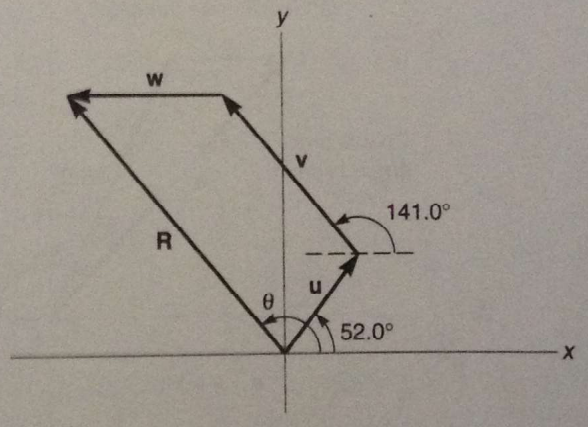


Figure 11.59

Then

$$\begin{aligned} R &= \sqrt{|R_x|^2 + |R_y|^2} \\ &= \sqrt{|-346 \text{ mi}|^2 + |324 \text{ mi}|^2} \\ &= 474 \text{ mi} \end{aligned}$$

$$\tan \alpha = \frac{|R_y|}{|R_x|}$$

$$\tan \alpha = \frac{324 \text{ mi}}{346 \text{ mi}} = 0.9364$$

$$\alpha = 43.1°$$

Since $R_x < 0$ and $R_y > 0$, $\mathbf{R}$ is in the second quadrant. So

$$\theta = 180° - \alpha = 180° - 43.1° = 136.9°$$

Therefore, the displacement is

$$\mathbf{R} = 474 \text{ mi at } 136.9°$$

Velocity

Velocity is a vector quantity that is described in terms of both speed and direction. For example, wind typically has an effect on both the speed and the direction of a plane in flight. The *heading* of an airplane is the direction the plane is pointed with respect to the compass. The *course* of the airplane is the direction the plane is actually flying with respect to the ground. The *air speed* is the speed of the plane with respect to the air; the *ground speed* is the speed of the plane with respect to the ground.

EXAMPLE 2

An airplane is flying on a heading of 25.0° west of north at an air speed of 95.0 mi/h. The wind is from the east at 30.0 mi/h. Find the plane's course and ground speed.

First, draw a vector diagram as in Fig. 11.60. You often have a choice of vector methods to use to solve a particular problem. Here, let's use trigonometry.

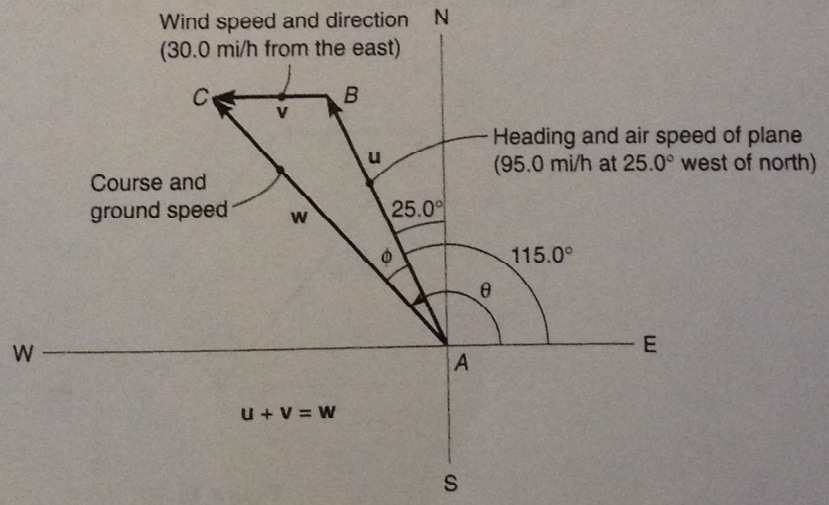


Figure 11.60

Since **BC** is parallel to the x-axis, angle $B = 115.0°$. Then use the law of cosines in triangle ABC.

$$|\mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos B$$

$$|\mathbf{w}| = \sqrt{(95.0 \text{ mi/h})^2 + (30.0 \text{ mi/h})^2 - 2(95.0 \text{ mi/h})(30.0 \text{ mi/h})(\cos 115.0°)}$$

$$= 111 \text{ mi/h}$$

Next, find angle ϕ using the law of sines.

$$\frac{|\mathbf{w}|}{\sin B} = \frac{|\mathbf{v}|}{\sin \phi}$$

$$\frac{111 \text{ mi/h}}{\sin 115.0°} = \frac{30.0 \text{ mi/h}}{\sin \phi}$$

$$\sin \phi = 0.2449$$

$$\phi = 14.2°$$

Then

$$\theta = 115.0° + 14.2° = 129.2°$$

The ground speed is 111 mi/h, and the course is 129.2° in standard position or 39.2° west of north.

Force

In physics a **force** is defined as a push or a pull that tends to cause or prevent motion. When a force is applied, it must be applied in some direction. Thus, force is a vector quantity.

EXAMPLE 3

Two forces act on a point at an angle of 70.0°. One force is 1850 newtons (N). The resultant force is 2250 N. Find the second force and the angle it makes with the resultant.

First, draw a force diagram as in Fig. 11.61.

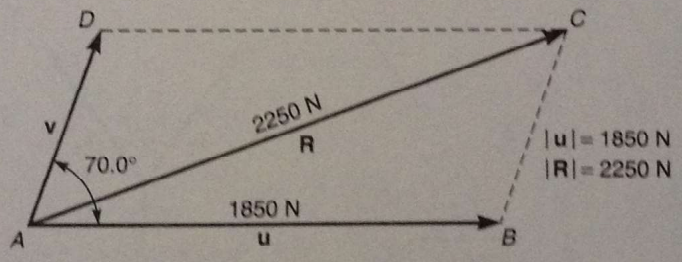


Figure 11.61

The consecutive angles of a parallelogram are supplementary; angle $B = 110.0°$. Let's first find angle ACB using the law of sines.

$$\frac{\overline{AC}}{\sin B} = \frac{|\mathbf{u}|}{\sin ACB}$$

$$\frac{2250 \text{ N}}{\sin 110.0°} = \frac{1850 \text{ N}}{\sin ACB}$$

$$\sin ACB = \frac{(\sin 110.0°)(1850 \text{ N})}{2250 \text{ N}} = 0.7726$$

$$\text{angle } ACB = 50.6°$$

Then angle $CAB = 180° - 110.0° - 50.6° = 19.4°$.

Now find $|\mathbf{v}|$ using the law of sines and noting that $\overline{BC} = |\mathbf{v}|$ in triangle ABC.

$$\frac{\overline{BC}}{\sin CAB} = \frac{\overline{AC}}{\sin B}$$

$$\frac{|\mathbf{v}|}{\sin 19.4°} = \frac{2250 \text{ N}}{\sin 110.0°} \qquad (\textit{Note: } \overline{BC} = |\mathbf{v}|.)$$

$$|\mathbf{v}| = \frac{(\sin 19.4°)(2250 \text{ N})}{\sin 110.0°} = 795 \text{ N}$$

Thus, $\mathbf{v} = 795$ N at 50.6° from $\mathbf{R}$.

When two or more forces act at a point, the force applied at the same point that produces equilibrium is called the *equilibrant force*. Thus, the equilibrant force is *equal* in magnitude but *opposite* in direction to the resultant force.

EXAMPLE 4

A 1250-lb weight hangs from two ropes of equal lengths suspended from a support as shown in Fig. 11.62. Find the tension in each rope.

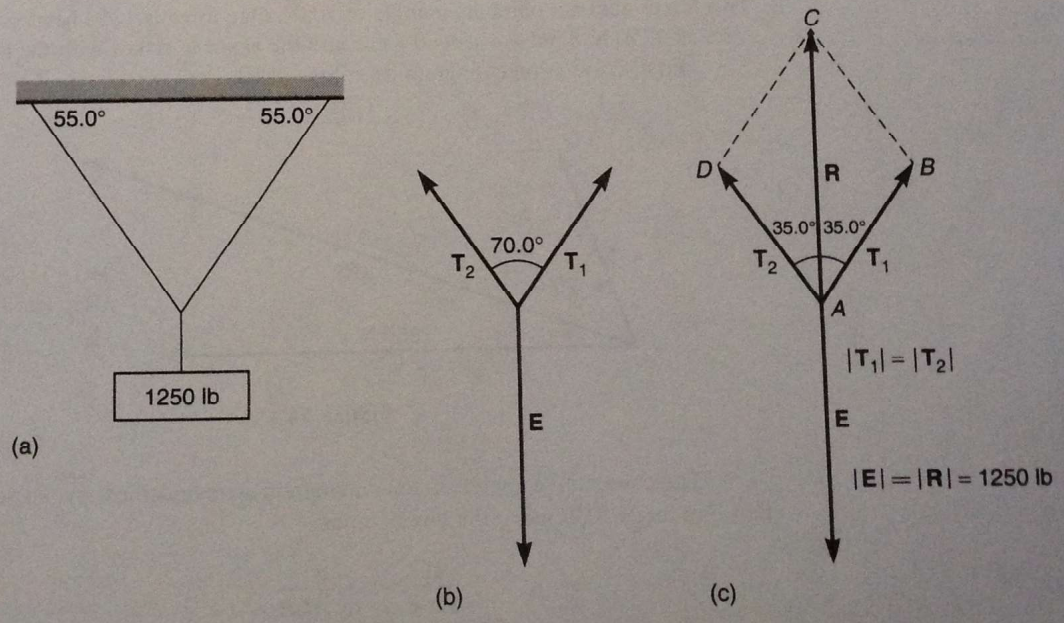


Figure 11.62

First, draw the force diagram as in Fig. 11.62(b). Note that the equilibrant force is 1250 lb; the resultant force is shown in Fig. 11.62(c). Find $|\mathbf{T}_1|$ using the law of sines in triangle ABC as follows:

$$\frac{|\mathbf{T}_1|}{\sin ACB} = \frac{|\mathbf{R}|}{\sin B}$$

$$\frac{|\mathbf{T}_1|}{\sin 35.0^\circ} = \frac{1250 \text{ lb}}{\sin 110.0^\circ} \qquad \textit{(Note: Angle } B = 110.0^\circ\textit{.)}$$

$$|\mathbf{T}_1| = \frac{(\sin 35.0^\circ)(1250 \text{ lb})}{\sin 110.0^\circ} = 763 \text{ lb} = |\mathbf{T}_2|$$

The tension in each rope is 763 lb.

Exercises 11.8

1. An airplane takes off and flies 125 mi on a course of 61.5° south of west; then it changes course and flies 185 mi due south, where it lands. Find the displacement from the starting point to the landing point.
2. A ship travels 18.5 km on a course of 31.2° south of east, then 12.7 km due south, then 21.5 km on a course of 61.3° west of south, where it lands. Find the displacement from the starting point to the landing point.
3. En route between two cities, a person travels by automobile 115 km due south, then 195 km at 45.0° west of south by plane, and then 45.0 km due west by boat. Find the displacement from the starting point to the ending point.
4. Natasha jogs 2.00 km due west, then 1.00 km due north, then 3.00 km due east, and then 2.00 km due south. Find the displacement from her starting point to her ending point.

In Exercises 5–8, given displacements with magnitudes of 10.0 mi and 15.0 mi:

5. Find the magnitude of the maximum resultant displacement.
6. Find the magnitude of the minimum resultant displacement.
7. Find the angle between the two original displacements if the magnitude of the resultant displacement is 12.0 mi.
8. Find the angle between the two original displacements if the magnitude of the resultant displacements is 7.50 mi.

In Exercises 9–10, given three displacements of magnitudes 10.0 km, 15.0 km, and 20.0 km:

9. Find the magnitude of the maximum resultant displacement.
10. Find the magnitude of the minimum resultant displacement.
11. A boat travels 12 mi/h in still water. The flow of the current is 3 mi/h.
    (a) What is the boat's speed going downstream?
    (b) What is its speed going upstream?
12. A plane's air speed is 125 mi/h. The wind speed is 21 mi/h.
    (a) What is the plane's ground speed when it is flying into the wind?
    (b) What is the plane's ground speed when it is flying with the wind?
13. A plane is flying due west at 175 mi/h. Suddenly a wind of 25.0 mi/h from the south develops. Find the plane's new course and ground speed.

14. A plane is flying north at 145 mi/h in still air. Find the velocity of the plane when the wind is blowing 55.0 mi/h from the east.

15. A plane is flying on a heading of 35.0° west of north at an air speed of 315 km/h. The wind is blowing from the west at 50.0 km/h. Find the plane's course and ground speed.

16. A plane is flying on a heading of 60.0° east of south at an air speed of 165 mi/h. The wind is blowing at 45.0 mi/h from 25.0° south of west. Find the plane's course and ground speed.

17. A pilot is flying on a course of 25.5° north of west at 215 mi/h. The wind is blowing from the east at 45.6 mi/h. Find the plane's heading and air speed.

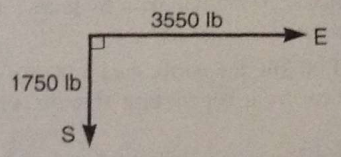
18. During a model boating competition, the following circumstances are presented to the contestants: width of stream, 125 ft; speed of current, 4.00 ft/s; speed of boat in still water, 12.0 ft/s. At what angle must the boat be steered to reach a point on the opposite side of the stream **(a)** that is directly across the stream, **(b)** that is 45.0 ft upstream, and **(c)** that is 45.0 ft downstream?

19. DeCarlos applies a 25.0-lb force to the handle of a lawnmower, which makes a 35.0° angle with the ground. Find the horizontal and vertical components of this force.

20. Cindy pulls a sled with a rope with a force of 50.0 lb. The rope makes an angle of 30.0° with the ground. Find the horizontal and vertical components of this force.

In Exercises 21–24, find the vector sum of each set of forces.

21.



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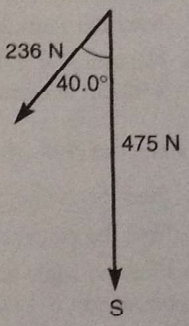

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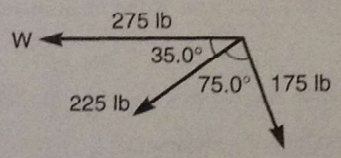

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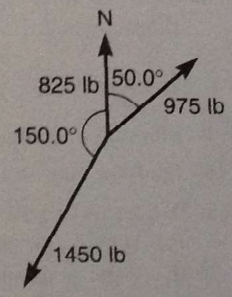
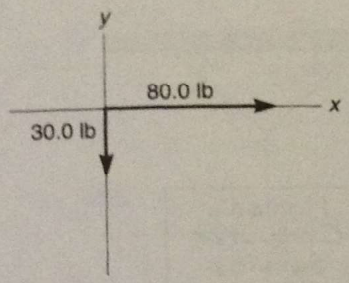


25. Two forces act on a point at an angle of 125.0°. One force is 225 lb. The resultant force is 195 lb. Find the second force and the angle it makes with the resultant.

26. Two forces act on a point at an angle of 65.0°. One force is 175 lb. The resultant force is 215 lb. Find the second force and the angle it makes with the resultant.

In Exercises 27–30, find each equilibrant force in standard position.

27.



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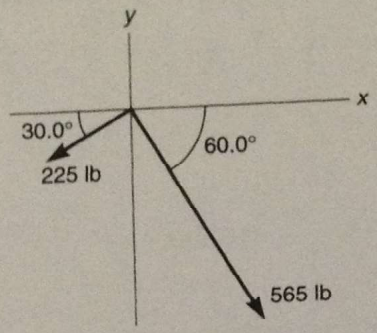

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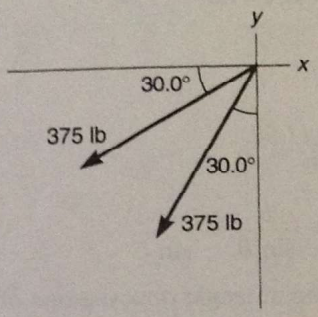
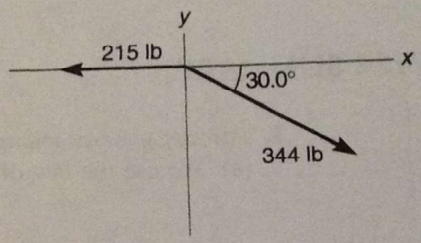


30.



31. A sign weighing 615 lb is supported as shown in Fig. 11.63. Find the tension in the cable.

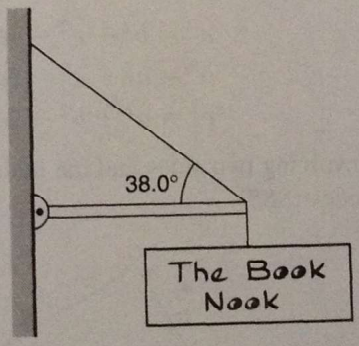


Figure 11.63

32. A sign weighing 275 lb is suspended by two cables attached between two buildings at equal heights above the ground. Each cable makes an angle of 65.0° with its building. Find the tension in each cable.

33. A 475-lb sign hangs from two cables as shown in Fig. 11.64. Find the tension in each cable.

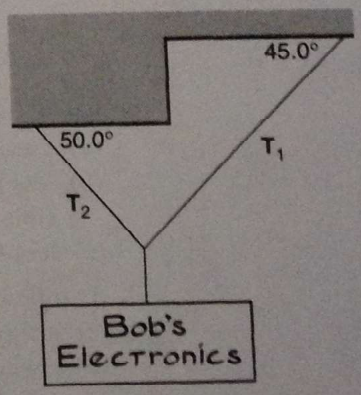


Figure 11.64

34. A 1250-lb weight hangs from two cables of equal length as shown in Fig. 11.65. If each cable can withstand a maximum tension of 825 lb, at what angle A are the cables unsafe?

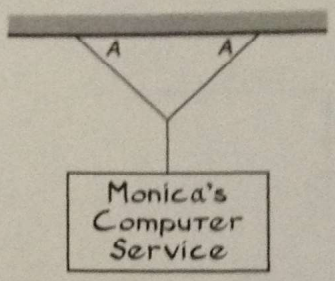


Figure 11.65

CHAPTER SUMMARY

1. *For any general triangle as in Fig. 11.66:*

(a) We use the law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

for cases involving two sides and an angle opposite one of them (SSA) and for cases involving two angles and a side opposite one of them (AAS).

(b) We use the law of cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

for cases involving two sides and the included angle (SAS) and for cases involving three sides (SSS).

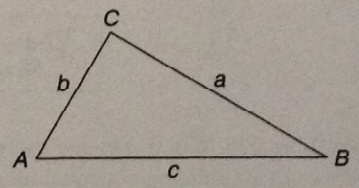


Figure 11.66

2. The SSA case requires special consideration. If the side opposite the given angle is:

(a) greater than the known adjacent side, there is only one possible triangle.

(b) less than the known adjacent side but greater than the altitude, there are two possible triangles.

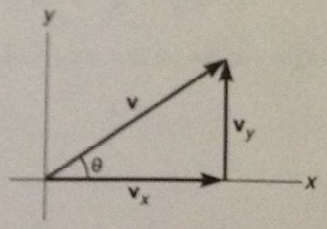
(c) less than the altitude, there is no possible triangle.

3. *As a final check:*

(a) always choose a given value over a calculated value for doing calculations; and

(b) always check your results to see that the largest angle is opposite the largest side and that the smallest angle is opposite the smallest side.

4. Vector problems may be solved:
   (a) graphically using
      (i) the parallelogram method or
      (ii) the vector triangle method;
   (b) algebraically using the law of sines and/or the law of cosines; or
   (c) by the component method.
5. Given $|\mathbf{v}|$ and angle θ, the horizontal and vertical components are found as follows (see Fig. 11.67):



$$\mathbf{v}_x = |\mathbf{v}|\cos\theta$$
$$\mathbf{v}_y = |\mathbf{v}|\sin\theta$$

Figure 11.67

6. Given the horizontal and vertical components $\mathbf{v}_x$ and $\mathbf{v}_y$, the magnitude of $\mathbf{v}$ may be found as follows:

$$|\mathbf{v}| = \sqrt{|\mathbf{v}_x|^2 + |\mathbf{v}_y|^2}$$

The reference angle α may be found as follows:

$$\tan\alpha = \frac{|\mathbf{v}_y|}{|\mathbf{v}_x|}$$

Then angle θ in standard position is determined from angle α and the quadrant in which $\mathbf{v}$ lies.

7. *Component method of adding vectors:* To find the resultant vector $\mathbf{R}$ of two or more vectors using the component method:
   (a) Find the horizontal component, $\mathbf{R}_x$, of vector $\mathbf{R}$ by finding the algebraic sum of the horizontal components of each of the vectors being added.
   (b) Find the vertical component, $\mathbf{R}_y$, of vector $\mathbf{R}$ by finding the algebraic sum of the vertical components of each of the vectors being added.
   (c) Find the length of $\mathbf{R}$: $|\mathbf{R}| = \sqrt{|\mathbf{R}_x|^2 + |\mathbf{R}_y|^2}$.
   (d) To find angle θ, first find α, the reference angle.

$$\tan\alpha = \frac{|\mathbf{R}_y|}{|\mathbf{R}_x|}$$

CHAPTER 11 REVIEW

Solve each triangle using the labels as in Fig. 11.68.

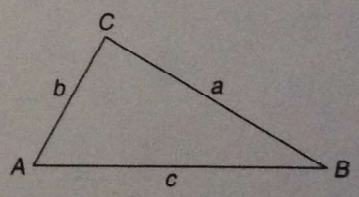


Figure 11.68

Express the lengths of sides to three significant digits and the angles to the nearest tenth of a degree.

1. $B = 71.4°$, $b = 409$ ft, $c = 327$ ft

2. $A = 25.1°$, $C = 37.7°$, $a = 15.7$ m

3. $A = 15.5°$, $b = 236$ cm, $c = 209$ cm

4. $a = 25.6$ m, $b = 42.2$ m, $c = 35.2$ m

Express the lengths of sides to two significant digits and the angles to the nearest degree.

5. $B = 44°$, $b = 150$ mi, $c = 240$ mi

6. $A = 29°$, $a = 41$ cm, $b = 49$ cm

7. $C = 36°$, $a = 2100$ ft, $b = 3600$ ft

8. $C = 58°$, $a = 450$ m, $c = 410$ m

Express the lengths of sides to four significant digits and the angles to the nearest hundredth of a degree.

9. $B = 105.15°$, $a = 231.1$ m, $c = 190.7$ m

10. $A = 74.75°$, $a = 22.19$ cm, $c = 15.28$ cm

11. $B = 18.25°$, $a = 1675$ ft, $b = 1525$ ft

12. $C = 40.16°$, $b = 25{,}870$ ft, $c = 10{,}250$ ft

Express the lengths of sides to four significant digits and the angles to the nearest second.

13. $A = 48°15'35''$, $B = 68°7'18''$, $a = 2755$ ft

14. $C = 29°25'16''$, $a = 13{,}560$ ft, $b = 24{,}140$ ft

15. Find the total length of fence needed to enclose the triangular field shown in Fig. 11.69.

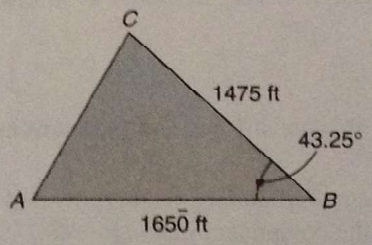


Figure 11.69

16. In surveying a tract of timber, it is necessary to find the distance to an inaccessible, but visible, point C. A distance between two points A and B of 300.0 ft is measured. Then $\angle ABC$ is measured as 85.00° and $\angle BAC$ is measured as 79.55°.

(a) Find AC.

(b) Find the perpendicular distance from C to line AB.

17. A toolmaker needs to lay out three holes in a plate, as shown in Fig. 11.70. Find the distance between holes A and C.

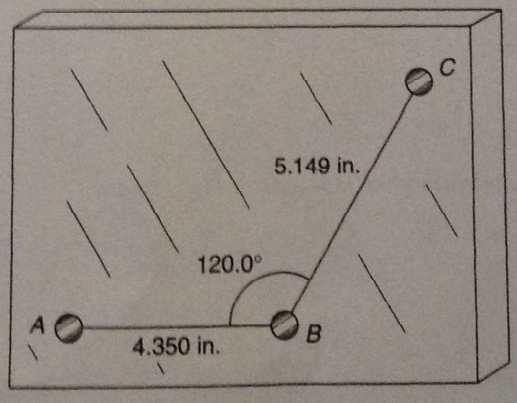


Figure 11.70

18. The centers of five holes are equally spaced around a 5.000-in.-diameter circle, as shown in Fig. 11.71.
(a) Find the distance between the centers of holes A and B.
(b) Find the distance between the centers of holes A and C.

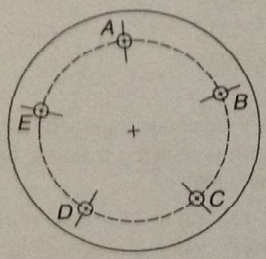


Figure 11.71

19. In the roof truss in Fig. 11.72., we know that $AB = DE$, $BC = CD$, and $AE = 20.0$ m. Find each length.

(a) AF **(b)** BF **(c)** CF **(d)** BC

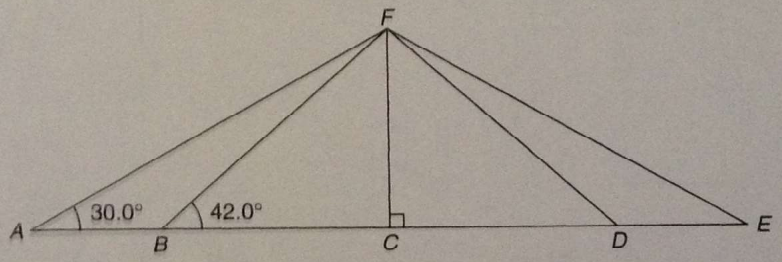


Figure 11.72

(Angles are given in standard position unless stated otherwise.)

20. Find the sum of the given vectors using a graphical method. Choose a suitable scale.

$\mathbf{v} = 75$ km/h at 87°
$\mathbf{w} = 25$ km/h at 142°

Using trigonometry, find the sum of each set of vectors.

21. $\mathbf{v} = 126$ mi at 35.0°
$\mathbf{w} = 306$ mi at 180.0°

22. $\mathbf{v} = 89.4$ mi/h at 142.0°
$\mathbf{w} = 44.7$ mi/h at 322.0°

Find the horizontal and vertical components of each vector.

23. 258 km at 135.0°

24. 42.2 mi/h at 303.0°

25. A man pushes with 160 N of force on the handle of a pushcart. If the angle between the handle and the ground is 45°, find the horizontal and vertical components of this force.

26. A woman pulls a loaded cart with a pulling force of 110 lb when the handle makes an angle of 23° with the ground. Find the horizontal and vertical components of this force.

27. If $\mathbf{v}_x = 18.5$ N and $\mathbf{v}_y = -31.0$ N, find $\mathbf{v}$.

Using the component method, find the sum of each set of vectors. Find magnitudes to three significant digits and angles to the nearest tenth of a degree.

28. $\mathbf{v} = 87.1$ mi/h at $120.0°$
$\mathbf{w} = 25.6$ mi/h at $247.0°$

29. $\mathbf{v} = 2560$ N at $237.1°$
$\mathbf{w} = 3890$ N at $346.7°$

30. $\mathbf{u} = 325$ N at $90.0°$
$\mathbf{v} = 325$ N at $162.0°$
$\mathbf{w} = 325$ N at $270.0°$

31. $\mathbf{v} = 19.7$ km at $144.5°$
$\mathbf{w} = 28.5$ km at $180.0°$
$\mathbf{u} = 10.3$ km at $225.5°$
$\mathbf{t} = 31.7$ km at $90.0°$

32. An airplane is to maintain a velocity of 550 km/h on a true course of $60°$ east of north. A wind of 90 km/h is blowing from the east. What should the air speed and compass reading be to offset the wind?

33. A ship travels 16.5 mi at $13.5°$ east of north, then 24.7 mi at $34.5°$ west of north, then 30.5 mi due north, where it lands. Find the displacement from the starting point to the landing point.

34. A weight of 850 lb is suspended by two ropes attached to opposite walls at equal heights above the floor. The first rope makes an angle of $25°$ with the first wall, and the second rope makes an angle of $45°$ with the second wall. Find the tension in each rope.

INTRODUCTION

Harmonic motion can be described using the equation $y = r\sin(\omega t + \theta)$. Voltage or current can be described using the equation $y = a\cos(\omega t + \theta)$. In this chapter we will graph these trigonometric functions and identify their amplitude, period, and phase shift.

Objectives

- Sketch the graphs of the trigonometric functions.
- Identify the amplitude, period, and phase shift for functions of the form

$$y = a\cos(\omega t + \theta) \quad \text{and} \quad y = a\sin(\omega t + \theta)$$

- Graph composite curves.

12.1 GRAPHING THE SINE AND COSINE FUNCTIONS

We shall first consider the graphs of the trigonometric equations $y = \sin x$ and $y = \cos x$. We find it convenient to express x in radian measure. (Be careful not to confuse the variables x and y in the equation $y = \sin x$ with the coordinates of a point on the terminal side of an angle. That is, in the equation of each trigonometric function, x is an angle in radian measure, and y is the trigonometric value for that angle.)

To graph $y = \sin x$, find a large number of values of x and y that satisfy the equation, and plot them in the xy-plane. It is convenient to scale the x-axis in multiples of π radians. A table of ordered pairs is as follows:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
y	0	0.5	0.71	0.87	1	0.87	0.71	0.5	0	−0.5	−0.71	−0.87	−1	−0.87	−0.71	−0.5	0

The graph is shown in Fig. 12.1. Note that the graph is a smooth, continuous curve. The domain (set of x-replacements or input) is the set of all real numbers (angles measured in radians). The range (set of output) is $-1 \le y \le 1$.

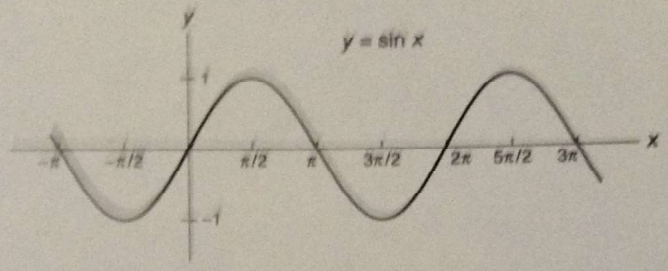


Figure 12.1

The graph of $y = \cos x$ is found in the same manner and is shown in Fig. 12.2.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
y	1	0.87	0.71	0.5	0	−0.5	−0.71	−0.87	−1	−0.87	−0.71	−0.5	0	0.5	0.71	0.87	1

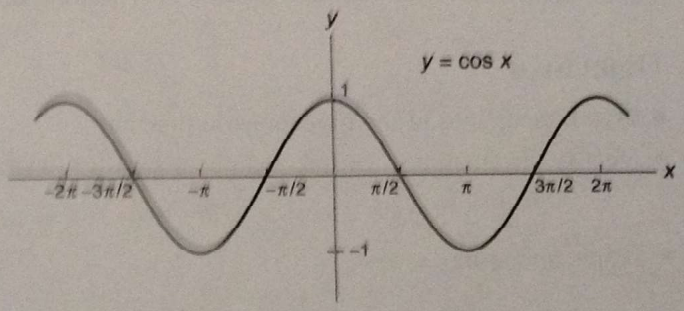


Figure 12.2

The domain and the range for $y = \cos x$ are the same as for $y = \sin x$. Note how the two graphs differ. Could we shift the cosine curve so that it coincides with the sine curve? Show $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ for all values of x by choosing various values of x and checking them in each side of the equation.

It is important to note the points at which these two curves cross the axes. We shall "sketch" similar curves based on the fact that the basic shape of the sine and cosine curves always remains the same. This will save time and work in that it will be unnecessary to plot a large number of points each time we need to sketch a curve.

The basic sine curve is an equation of the form $y = a \sin x$, where a is a real number. We have already graphed this equation for $a = 1$. For $y = 2 \sin x$, each ordered pair

satisfying this relationship has a y-value that is two times the corresponding y value for $y = \sin x$. Likewise, for $y = \frac{1}{2} \sin x$, each ordered pair satisfying this relationship has a y-value that is one-half the corresponding y-value for $y = \sin x$.

EXAMPLE 1

Graph $y = \sin x$, $y = 2 \sin x$, $y = \frac{1}{2} \sin x$, and $y = 4 \sin x$ on the same set of coordinate axes.

The graph is drawn in Fig. 12.3.

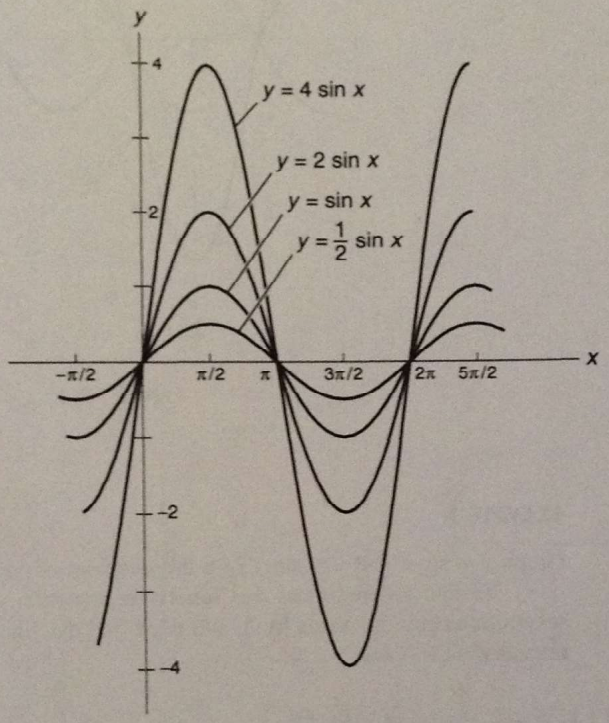


Figure 12.3

The coefficient, a, of $a \sin x$ or $a \cos x$ in the equations $y = a \sin x$ and $y = a \cos x$ determines the amplitude of the curve. The **amplitude** is the maximum y-value of the curve. The amplitudes for the curves in Example 1 are 1, 2, $\frac{1}{2}$, and 4, respectively. The effect of a negative value of a in $y = a \sin x$ or $y = a \cos x$ is to "flip," or invert, the curve about the x-axis. That is, for $y = -\cos x$, each ordered pair satisfying this relationship has a y-value that is opposite the corresponding y-value for $y = \cos x$.

EXAMPLE 2

Graph $y = \cos x$, $y = -\cos x$, and $y = -3 \cos x$ on the same set of coordinate axes.

The graph is shown in Fig. 12.4. The amplitudes for the curves in this example are 1, 1, and 3, respectively.

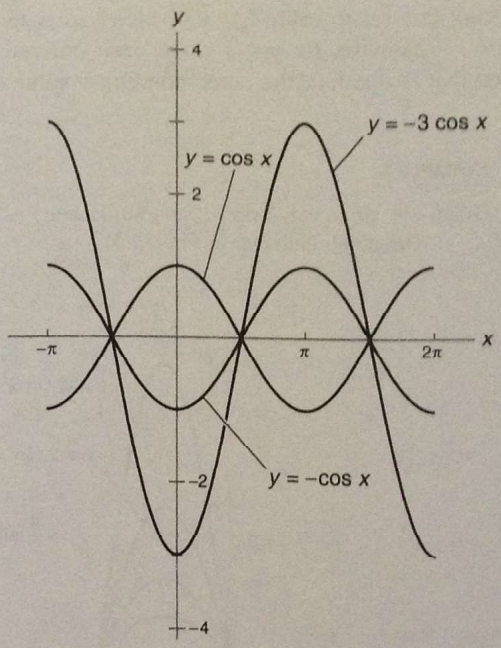


Figure 12.4

EXAMPLE 3

Graph $y = \sin x$ and $y = \sin 2x$ on the same set of coordinate axes.

To find ordered pairs that satisfy the equation $y = \sin 2x$, first choose a value for x, second multiply this value by 2, and third find the sine of this result. Some of the values are tabulated as follows:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin x$	0	0.5	0.71	0.87	1	0.87	0.71	0.5	0	−0.5	−0.71	−0.87	−1	−0.87	−0.71	−0.5	0
$2x$	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$	$\frac{8\pi}{3}$	3π	$\frac{10\pi}{3}$	$\frac{7\pi}{2}$	$\frac{11\pi}{3}$	4π
$\sin 2x$	0	0.87	1	0.87	0	−0.87	−1	−0.87	0	0.87	1	0.87	0	−0.87	−1	−0.87	0

Plot these points as in Fig. 12.5.

Note that the two curves in Fig. 12.5 have the same basic shape, but $y = \sin 2x$ goes through a complete cycle twice as $y = \sin x$ completes one cycle. The length of each of these cycles is called a *period.* The **period** is the x-distance between any point on the curve and the next corresponding point in the next cycle where the graph starts repeating itself (see Fig. 12.6). The period for $y = \sin x$ is 2π; for $y = \cos x$, 2π; and for $y = \sin 2x$, π.

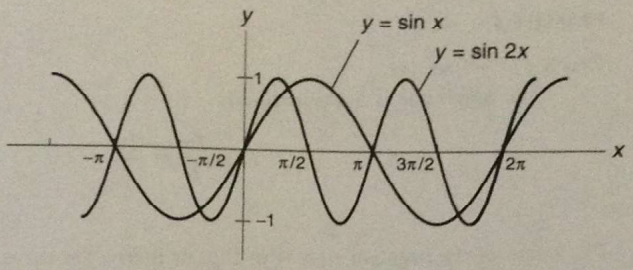


Figure 12.5

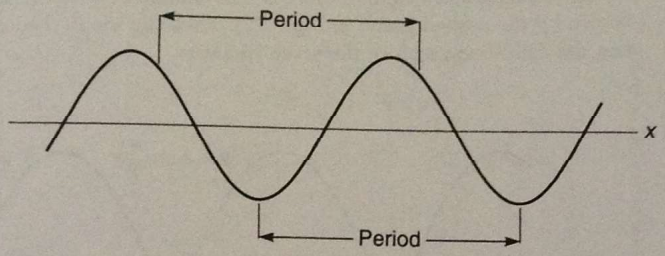


Figure 12.6

In general, the period for $y = a \sin bx$ or $y = a \cos bx$ may be found by the following formula.

$$P = \frac{2\pi}{b}$$

EXAMPLE 4

Graph $y = 2 \cos 3x$.

The amplitude is 2 and the period is

$$P = \frac{2\pi}{b} = \frac{2\pi}{3}$$

Sketch a cosine graph with amplitude 2 that completes one complete cycle each $\frac{2\pi}{3}$ radians as in Fig. 12.7.

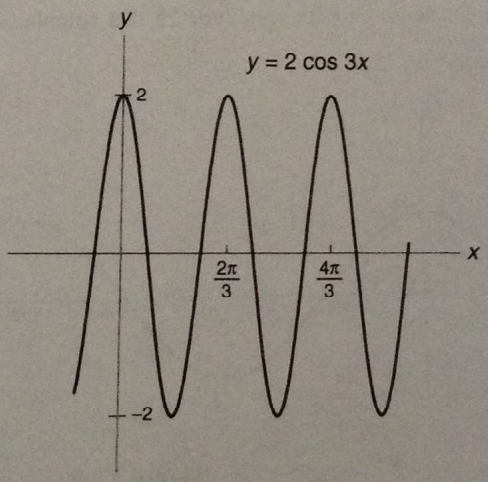


Figure 12.7

EXAMPLE 5

Graph $y = -3 \sin \frac{1}{2}x$.

The amplitude is 3; the period is

$$P = \frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$$

The effect of the negative sign is to flip, or invert, the curve $y = 3 \sin \frac{1}{2}x$ about the x-axis. Sketch a sine graph with amplitude 3 that completes one complete cycle each 4π radians as shown by the dashed graph in Fig. 12.8. Then flip the dashed curve about the x-axis to obtain the solid-line graph of the given function.

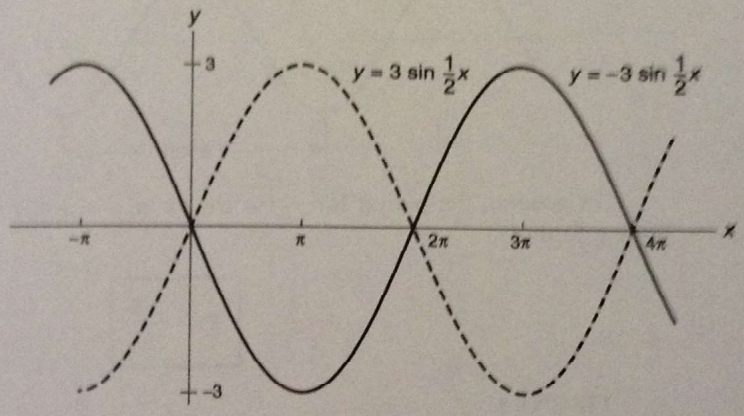


Figure 12.8

EXAMPLE 6

Graph $y = 25 \cos 120\pi x$.

The amplitude is 25; the period is

$$P = \frac{2\pi}{b} = \frac{2\pi}{120\pi} = \frac{1}{60}$$

Sketch a cosine graph with amplitude 25 that completes one complete cycle each $\frac{1}{60}$ radian as shown in Fig. 12.9.

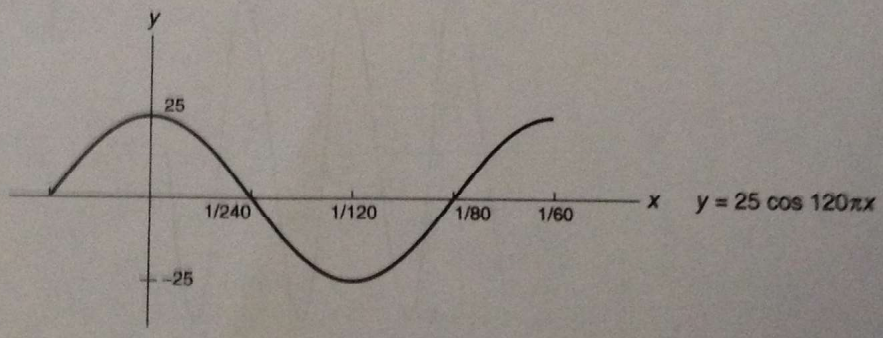


Figure 12.9

Exercises 12.1

Find the amplitude and the period of each function, and sketch its graph through at least two cycles.

1. $y = 2 \cos x$
2. $y = 5 \sin x$
3. $y = -3 \sin x$
4. $y = -4 \cos x$
5. $y = \sin 3x$
6. $y = \sin 4x$
7. $y = \cos 2x$
8. $y = \cos 3x$
9. $y = 2 \sin 4x$
10. $y = 3 \cos 6x$
11. $y = \frac{5}{2} \cos \frac{1}{2}x$
12. $y = \frac{3}{2} \sin \frac{3}{2}x$
13. $y = -\frac{1}{2} \sin \frac{2}{3}x$
14. $y = -\frac{5}{2} \cos \frac{3}{4}x$
15. $y = 2 \sin 3\pi x$
16. $y = 6 \cos \frac{4\pi x}{3}$
17. $y = -3 \cos \pi x$
18. $y = -\sin \frac{\pi x}{2}$
19. $y = 6.5 \sin 120\pi x$
20. $y = 12 \cos 160\pi x$
21. $y = 40 \cos 60x$
22. $y = 60 \sin 40x$
23. $y = -60 \sin 80\pi x$
24. $y = -240 \cos 120x$

12.2 PHASE SHIFT

If the graph of a sine curve does not pass through the origin (0, 0), or if the graph of a cosine curve does not pass through the point $(0, a)$ where a is the amplitude, the curve is **out of phase.** If a curve is out of phase, the **phase shift** is the directed distance between two successive corresponding x-intercepts of the curve $y = a \sin bx$ or $y = a \cos bx$ and the out-of-phase curve (see Fig. 12.10).

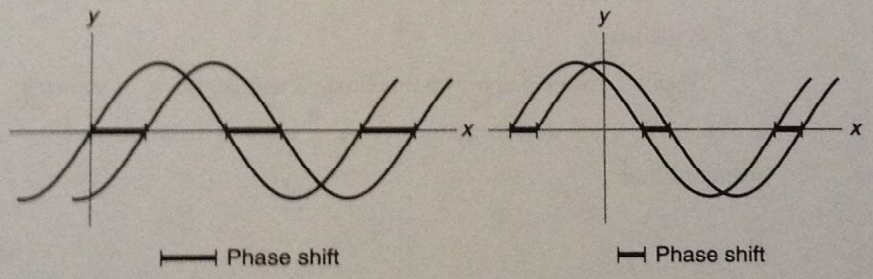


Figure 12.10

EXAMPLE 1

Graph $y = \sin x$ and $y = \sin\left(x - \frac{\pi}{4}\right)$ on the same set of coordinate axes.

For $y = \sin\left(x - \frac{\pi}{4}\right)$, complete the following table:

x	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$
y	-1	-0.71	0	0.71	1	0.71	0	-0.71	-1	-0.71	0	0.71	1

Plot the points as in Fig. 12.11.

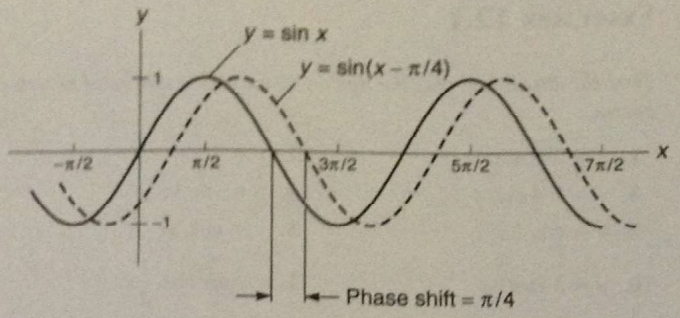


Figure 12.11

A consideration of phase shifts is helpful when graphing equations in the form $y = a\sin(bx + c)$ or $y = a\cos(bx + c)$. The effect of c in these equations is to shift the curve $y = a\sin bx$ or $y = a\cos bx$ as follows:

1. To the *left* $\frac{c}{b}$ units if $\frac{c}{b} > 0$.
2. To the *right* $\frac{c}{b}$ units if $\frac{c}{b} < 0$.

EXAMPLE 2

Graph $y = 3\cos\left(2x + \frac{\pi}{4}\right)$.

The amplitude is 3. The period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$. The phase shift is $\frac{c}{b} = \frac{\pi/4}{2} = \frac{\pi}{8}$, or $\frac{\pi}{8}$ to the left.

Sketch a cosine graph with amplitude 3 and period π, as shown by the dashed graph in Fig. 12.12. Then shift the dashed curve $\frac{\pi}{8}$ units to the left, which gives the graph of the given function.

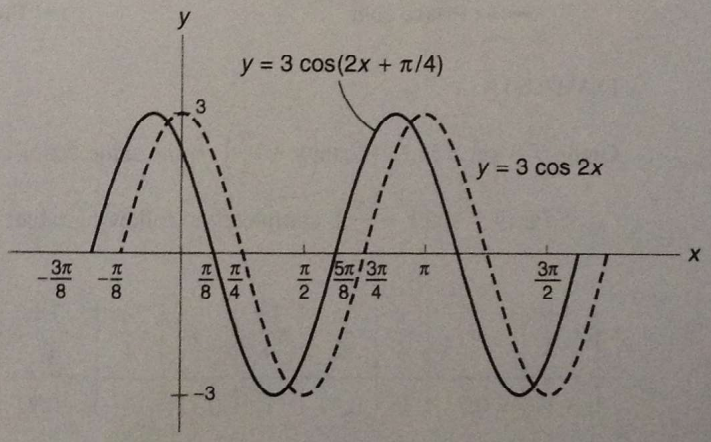


Figure 12.12

EXAMPLE 3

Graph $y = 4 \sin\left(3x + \frac{\pi}{2}\right)$.

The amplitude is 4. The period is $\frac{2\pi}{b} = \frac{2\pi}{3}$. The phase shift is $\frac{c}{b} = \frac{\pi/2}{3} = \frac{\pi}{6}$, or $\frac{\pi}{6}$ to the left.

Sketch a sine graph with amplitude 4 and period $\frac{2\pi}{3}$ as shown by the dashed graph in Fig. 12.13. Then shift the dashed curve $\frac{\pi}{6}$ units to the left, which gives the graph of the given function.

EXAMPLE 4

Graph $y = -3 \sin(\frac{1}{2}x - \pi)$.

The amplitude is 3. The period is $\frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi$. The phase shift is $\frac{c}{b} = \frac{-\pi}{\frac{1}{2}} = -2\pi$, or 2π to the right.

First, graph $y = -3 \sin \frac{1}{2}x$, the dashed curve in Fig. 12.14. Then shift the dashed curve 2π units to the right, which gives the graph of the given function.

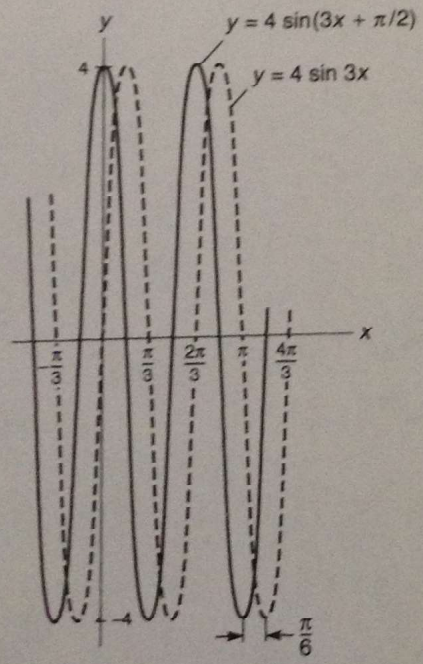


Figure 12.13

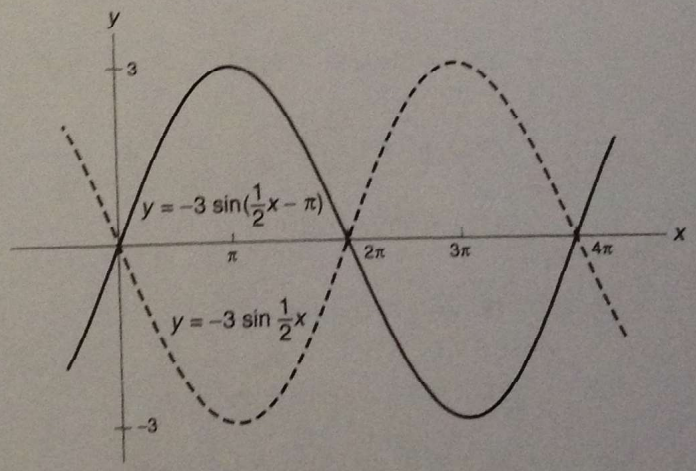


Figure 12.14

AMPLITUDE, PERIOD, AND PHASE SHIFT

In summary, the values of a, b, and c in the following equations determine amplitude, period, and phase shift as follows:

$$y = a\sin(bx + c)$$
$$y = a\cos(bx + c)$$

$$\text{Amplitude} = |a| \qquad \text{Phase shift} = \frac{c}{b} \qquad \left(\text{To the } \textit{left} \text{ if } \frac{c}{b} > 0\right)$$

$$\text{Period} = \frac{2\pi}{b} \qquad \left(\text{To the } \textit{right} \text{ if } \frac{c}{b} < 0\right)$$

Exercises 12.2

Find the amplitude, period, and phase shift of each function, and sketch its graph.

1. $y = \sin\left(x + \frac{\pi}{3}\right)$

2. $y = \cos\left(x + \frac{\pi}{4}\right)$

3. $y = 2\cos\left(x - \frac{\pi}{6}\right)$

4. $y = 3\sin\left(x - \frac{\pi}{3}\right)$

5. $y = \sin(3x - \pi)$

6. $y = \cos\left(2x + \frac{\pi}{3}\right)$

7. $y = -\cos(4x + \pi)$

8. $y = -\sin\left(4x - \frac{2\pi}{3}\right)$

9. $y = 3\sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

10. $y = 4\sin\left(\frac{1}{2}x - \frac{4\pi}{3}\right)$

11. $y = 2\sin\left(\frac{4}{3}x + \frac{\pi}{3}\right)$

12. $y = 5\cos\left(\frac{2}{3}x + \pi\right)$

13. $y = 3\sin(\pi x + \pi)$

14. $y = 3\cos\left(\frac{\pi x}{4} + \frac{\pi}{8}\right)$

15. $y = 2\cos\left(\frac{\pi x}{2} - \frac{\pi}{2}\right)$

16. $y = 4\sin\left(\frac{\pi x}{6} - \frac{\pi}{3}\right)$

17. $y = 40\cos\left(60x - \frac{\pi}{3}\right)$

18. $y = 80\sin\left(40x - \frac{\pi}{4}\right)$

19. $y = 120\cos\left(40\pi x - \frac{\pi}{2}\right)$

20. $y = 40\sin\left(20\pi x + \frac{\pi}{3}\right)$

21. $y = 7.5\sin(220x + \pi)$

22. $y = 4.5\cos\left(180x - \frac{\pi}{3}\right)$

23. $y = 20\sin(120\pi x + 4\pi)$

24. $y = 30\cos(160\pi x + 8\pi)$

12.3 GRAPHING THE OTHER TRIGONOMETRIC FUNCTIONS

The remaining trigonometric functions have interesting graphs, but they do not have as many technical applications. We shall graph these remaining functions and discuss them briefly.

To graph $y = \tan x$, we shall find a large number of values of x and y that satisfy the equation and plot them in the xy-plane as in Fig. 12.15(a).

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$
y	—	−1.7	−1	−0.58	0	0.58	1	1.7	—	−1.7	−1	−0.58	0	0.58	1	1.7	—

Note that $y = \tan x$ is a cyclic curve but not continuous. At the points

$$x = \cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$

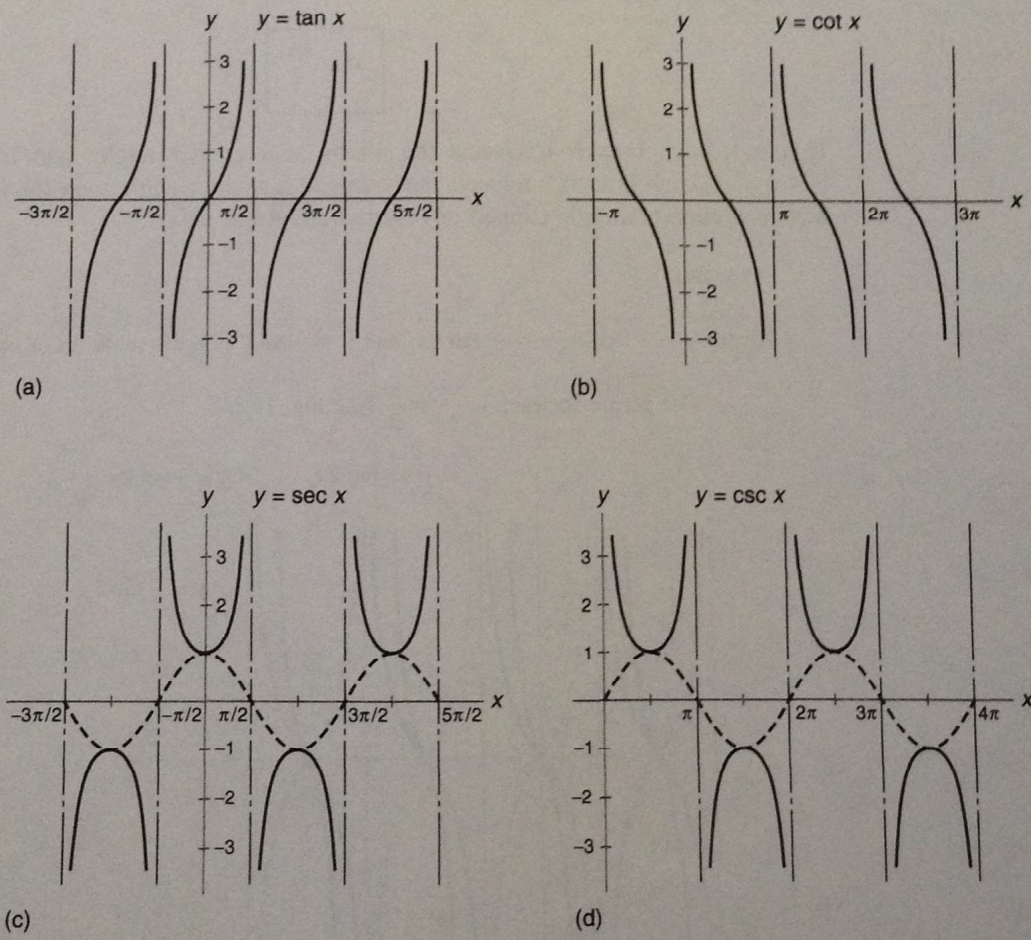


Figure 12.15

there are vertical lines called **asymptotes,** which form guidelines to the curve. That is, as x gets larger and closer to $\frac{\pi}{2}$, y becomes larger and larger and the graph approaches the asymptote. The graph never crosses the asymptote because at $x = \frac{\pi}{2}$, the tangent function is undefined.

Likewise, the three other trigonometric functions may be graphed. These graphs are shown in Figs. 12.15(b)–(d).

Since the secant and cosecant functions are reciprocals of the cosine and sine, respectively, each may be sketched by plotting reciprocals of the cosine and sine functions, which are designated by the dotted graphs in Figs. 12.15(c)–(d). Note that they, too, are cyclic but not continuous. They also have vertical asymptotes, as noted by the vertical dashed lines.

The period for both $y = \tan x$ and $y = \cot x$ is π. The period for both $y = \sec x$ and $y = \csc x$ is 2π. The amplitude for each of the four functions is undefined, since each has no maximum y-value.

Given equations in the form $y = a \tan bx$ and $y = a \cot bx$, the period may be found by the following formula.

$$P = \frac{\pi}{b}$$

If $a > 1$, each branch intersects the x-axis at a greater angle than for $y = \tan x$. If $0 < a < 1$, each branch intersects the x-axis at a smaller angle than for $y = \tan x$. If a is negative, each branch is flipped or inverted about the x-axis.

EXAMPLE 1

Graph $y = 3 \tan 2x$, $y = \frac{1}{2} \tan 2x$, and $y = -\tan 2x$ on the same set of coordinate axes.

The period for each is $\frac{\pi}{b} = \frac{\pi}{2}$ (see Fig. 12.16).

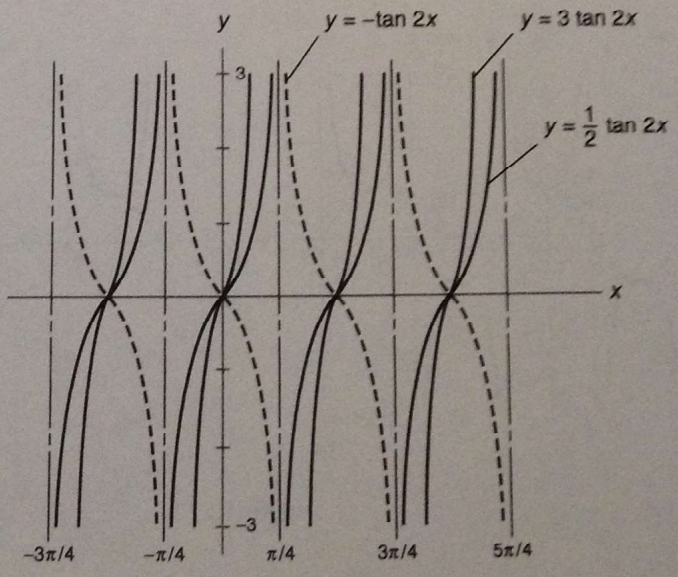


Figure 12.16

Given equations in the form $y = a \sec bx$ and $y = a \csc bx$, the period may be found by the following formula.

$$P = \frac{2\pi}{b}$$

The quantity $|a|$ is the vertical distance between the x-axis and the low point of each of the branches above the x-axis. If a is negative, each branch is flipped or inverted about the x-axis.

EXAMPLE 2

Graph $y = 2 \sec 4x$.

The period is $\frac{2\pi}{4} = \frac{\pi}{2}$ and $a = 2$. As a graphical aid, you may first graph $y = 2 \cos 4x$ as in Fig. 12.17.

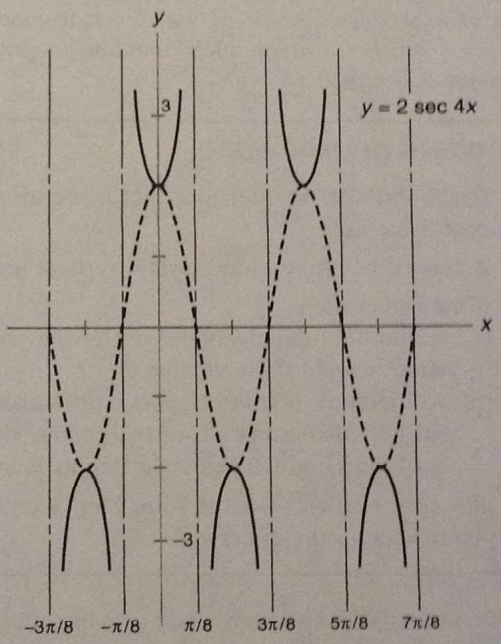


Figure 12.17

Exercises 12.3

Find the period of each function and sketch its graph.

1. $y = \tan 3x$

2. $y = \tan 4x$

3. $y = -2 \tan \frac{1}{2}x$

4. $y = -\tan \frac{3}{2}x$

5. $y = \cot 6x$

6. $y = 2 \cot \frac{1}{3}x$

7. $y = 3 \sec x$

8. $y = 4 \csc x$

9. $y = -2 \csc x$

Day: Date:

10. $y = -5 \sec x$ **11.** $y = 2 \sec 6x$ **12.** $y = 4 \sec 2x$

13. $y = -3 \sec \frac{1}{2}x$ **14.** $y = -5 \sec \frac{3}{2}x$ **15.** $y = \csc 2x$

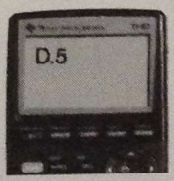
16. $y = 2 \csc 3x$

Sketch the graph of each function.

17. $y = \tan\left(x + \frac{\pi}{2}\right)$ **18.** $y = \cot\left(2x - \frac{\pi}{2}\right)$

19. $y = 3 \sec\left(2x - \frac{\pi}{2}\right)$ **20.** $y = 2 \csc\left(x + \frac{\pi}{3}\right)$

12.4 GRAPHING COMPOSITE CURVES



In applications it is common to find functions that are composites of sums or differences of expressions, some of which are trigonometric, such as $y = \cos x + \sin 2x$ or $y = x - \sin x$. To graph such functions, a graphical technique called **addition of ordinates** is useful.

ADDITION OF ORDINATES

1. Graph each of the functions that make up the composite on the same set of coordinate axes.
2. It may be helpful to draw several vertical lines perpendicular to the x-axis.
3. If the composite is
   (a) a sum, find the algebraic sum of the y-values where each vertical line intersects each of the graphs; or
   (b) a difference, find the algebraic difference of the y-values where each vertical line intersects each of the graphs, or graph the negative (or "flipped") curve and find the algebraic sum as in (a).
4. Plot each resultant y-value from Step 3 on each vertical line and connect the points with a smooth curve.

Or, instead of Step 3 above, add or subtract the y-values on the graph itself by means of a compass or ruler. Remember, the y-values of points above the x-axis are positive, and the y-values of points below the x-axis are negative.

EXAMPLE 1

Graph $y = \cos x + \sin 2x$ using the method of addition of ordinates.

The graph is shown in Fig. 12.18. *Note:* One period of $y = \sin 2x$ is not sufficient since $\cos x$ has a longer period.

EXAMPLE 2

Graph $y = x - \sin x$ using the method of addition of ordinates.

The graph is shown in Fig. 12.19.

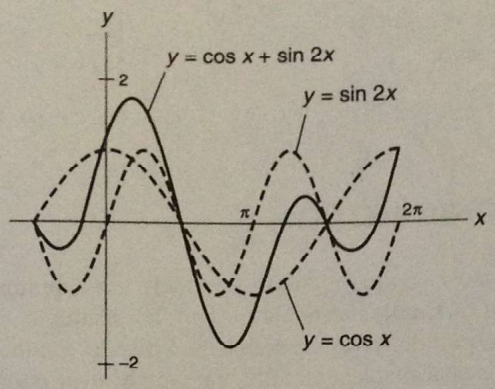


Figure 12.18

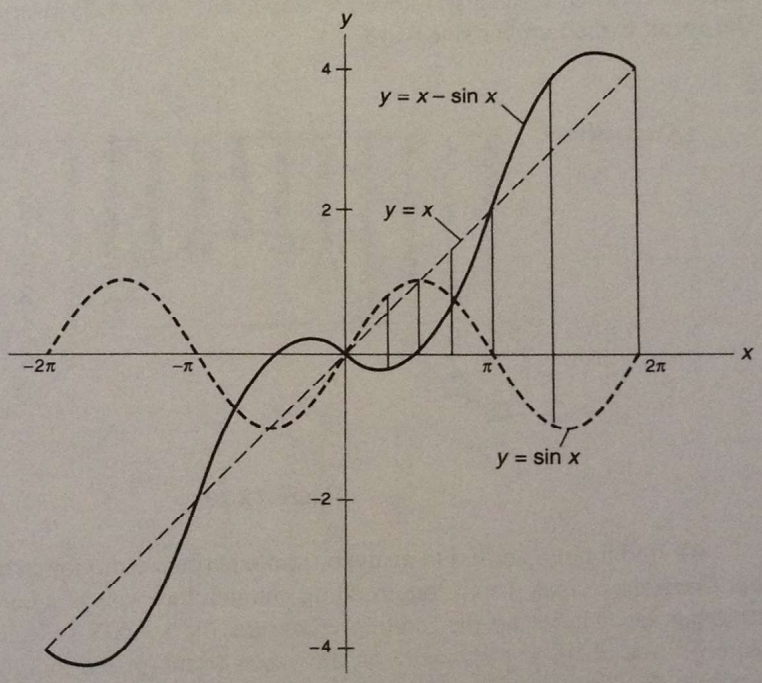


Figure 12.19

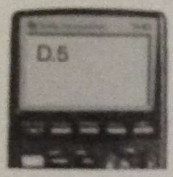
Exercises 12.4

Using the method of addition of ordinates, graph each equation for one complete period of the composite curve.

1. $y = \sin x + \cos x$
2. $y = 2\cos x + \sin x$
3. $y = 2\cos x + 2\sin x$
4. $y = \sin x + 3\cos x$
5. $y = 2\sin x + \sin \frac{x}{2}$
6. $y = 2\cos \frac{x}{2} + \sin x$
7. $y = 3\sin x + 2\cos x$
8. $y = 2\sin 2x + 2\cos x$
9. $y = 2\sin x - \cos x$
10. $y = \cos x - 2\sin x$

11. $y = 2 \sin 2x - \cos 2x$
12. $y = \cos 2x - \cos x$
13. $y = x + \sin x$
14. $y = x - 2 \cos x$
15. $y = \sin 4x + \cos\left(2x + \frac{\pi}{3}\right)$
16. $y = \sin 2x + \cos\left(2x - \frac{\pi}{4}\right)$

12.5 SIMPLE HARMONIC MOTION



One of the most important and most widely used applications of the sine function and its graph is that of simple harmonic motion. For example, consider a weight suspended on a spring. Now, pull down on the weight and then let go; the weight bobs up and down. Next, graph the vertical displacement of the weight over equal units of time as in Fig. 12.20, where successive vertical positions are displaced horizontally over equal time intervals. Our result is the familiar sine wave.

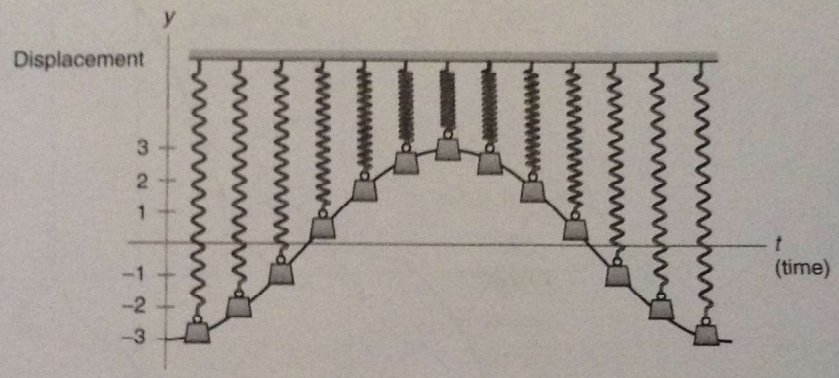


Figure 12.20

We find it quite helpful to analyze simple harmonic motion in terms of circular motion. Consider a crank 1 unit long rotating counterclockwise at a constant rate of 1 revolution per second. Set up the rotating crank and the xy-axes, as follows. The motion of the projection of the crank handle on the y-axis (think of the motion of its shadow projected on the y-axis as a function of time when a light is placed far out on the negative x-axis) is also an example of **simple harmonic motion** (see Fig. 12.21).

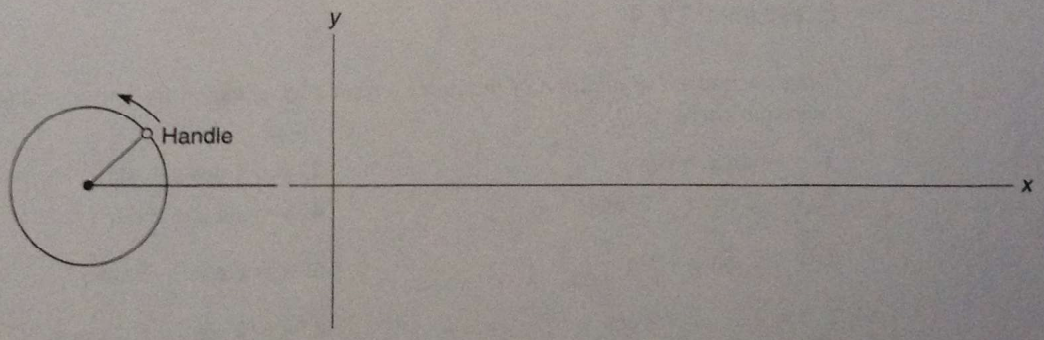


Figure 12.21

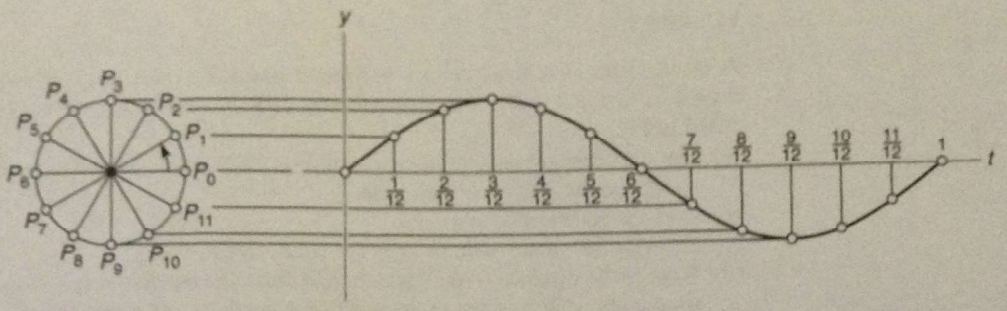


Figure 12.22

Let's assume that the crank starts from 0°. At the end of $\frac{1}{12}$ s, it will have rotated 30°, or $\frac{\pi}{6}$ rad, to position P_1 as in Fig. 12.22; at the end of $\frac{2}{12}$ s, it will have rotated 60°, or $\frac{\pi}{3}$ rad, to position P_2; and so forth; until after 1 s, or 1 rev, it will have rotated back to its original position P_0.

The projection of the crank handle on the y-axis (that is, its y-value per unit time) can be plotted as a curve. We do so by letting the x-axis be the time axis and dividing it into as many intervals as there are angles to be plotted—in this case 12, every $\frac{\pi}{6}$ rad from 0 to 2π rad. Through the points of division on the time axis (x-axis), construct vertical lines. Through the position points P_0, P_1, P_2, . . . , construct horizontal lines. Draw a small circle around the intersection of the corresponding vertical and horizontal lines; that is, P_1 and $\frac{1}{12}$, P_2 and $\frac{2}{12}$, and so on. Draw a smooth curve through these points of intersection which, as you can see, results in a sine curve. If we were to continue rotating the crank, successive rotations would generate successive periods of the sine curve. The equation is $y = \sin x$. Since the crank rotates through 2π rad in 1 s, $x = 2\pi$ in 1 s, $x = 4\pi$ in 2 s, and $x = 2\pi t$ in t s. Therefore, the y-value at any time t is given by $y = \sin 2\pi t$.

More generally, we let r be the length of a vector **B** rotating at a uniform angular velocity of ω rad per unit time. If we further assume a horizontal starting position at $t = 0$, the projections of vector **B** onto the y-axis, which are the vertical components of **B**, will be in simple harmonic motion. Their motion may be represented by the equation

$$y = r \sin \omega t$$

This rotating position vector is sometimes called a **phasor.**

This equation is of fundamental importance in that it describes the motion of any object or quantity that is in simple harmonic motion. Familiar examples of motion, that is simple harmonic motion (or very nearly so), are (1) the motion of a satellite in a circular orbit; (2) the motion of a pendulum bob, if the displacement is small; (3) the motion of the prongs of a vibrating tuning fork (sound waves); (4) the motion of a rotating wire through a magnetic field—alternating current; and (5) the motion of a mass vibrating up and down on a spring.

If the rotating vector is not in position P_0 at $t = 0$, the simple harmonic motion equation becomes

$$y = r \sin(\omega t + \theta)$$

where θ is the initial position of the vector at $t = 0$. The graph results in a phase shift of $\dfrac{\theta}{\omega}$.

EXAMPLE 1

A crank 15 in. long starts from a horizontal position (0 rad) and rotates in a counterclockwise direction at a uniform angular velocity of 3 revolutions per second (6π rad/s), as illustrated in Fig. 12.23.

(a) Plot the curve (as shown previously in Fig. 12.22) that shows the projection of the crank handle on the y-axis per unit time.
(b) Find the equation of this curve.
(c) Through how many radians does the crank turn in 0.15 s?
(d) What is the distance of the crank handle from the horizontal axis (t-axis) at $\frac{1}{9}$ s? That is, what is the length of the projection of the crank on the y-axis at $\frac{1}{9}$ s?

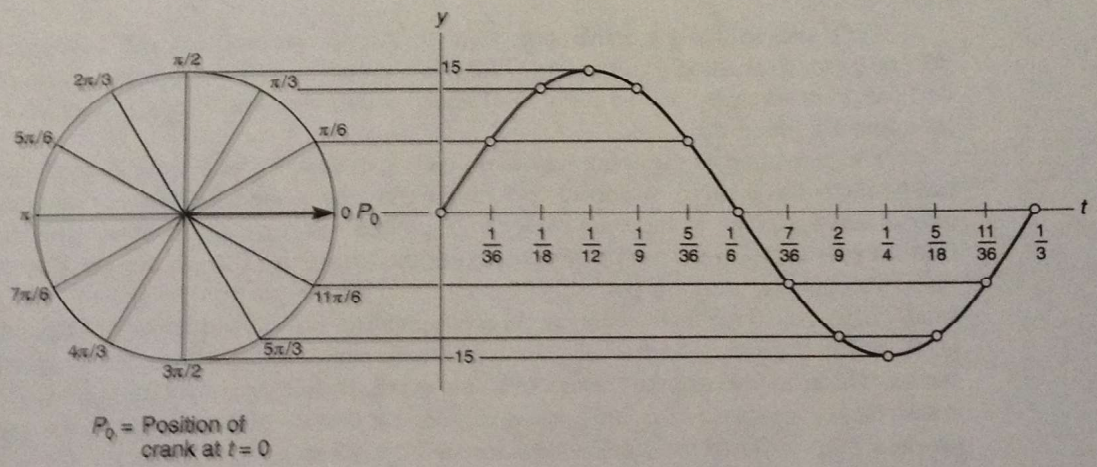


Figure 12.23

(a) The amplitude, r, is 15. The period is $\dfrac{2\pi}{6\pi} = \dfrac{1}{3}$. There is no phase shift, and $\omega = 6\pi$ rad/s.

(b) $y = r \sin \omega t$
$y = 15 \sin 6\pi t$

(c) $\theta = \omega t$
$\theta = (6\pi \text{ rad/s})(0.15 \text{ s}) = 2.8 \text{ rad}$

(d) Length of projection $= (15 \text{ in.}) \sin \dfrac{2\pi}{3} = 13 \text{ in.}$

EXAMPLE 2

A wire rotating through a magnetic field generates an alternating current. The instantaneous current, i, is given by the equation

$$i = I_{\max} \sin(\omega t + \theta)$$

where $I_{\max}$ is the maximum instantaneous current, ω is the angular velocity of the rotating wire, t is the time, and θ is the phase angle. Given $I_{\max} = 12$ A, $\omega = 120\pi$ rad/s (60 cycles/s), and $\theta = \dfrac{\pi}{3}$ (see Fig. 12.24).

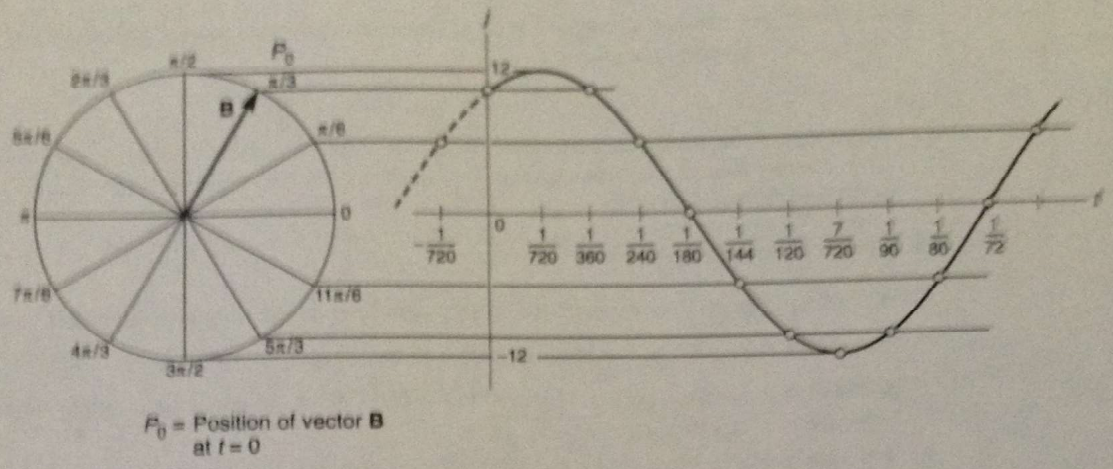


Figure 12.24

(a) Graph the resulting equation $i = 12 \sin\left(120\pi t + \frac{\pi}{3}\right)$.

(b) From the graph find the time at which the current is $+10$ A, -6 A, and 0 A.

(c) From the graph find the current at $t = \frac{1}{240}$ s and $\frac{1}{120}$ s.

(a) The amplitude is 12. The period is $\frac{2\pi}{120\pi} = \frac{1}{60}$. The phase shift is $\frac{\pi/3}{120\pi} = \frac{1}{360}$ to the left.

(b) At $i = +10$ A, $t = 0, \frac{1}{360}, \frac{1}{60}, \frac{7}{360}, \ldots$

At $i = -6$ A, $t = \frac{1}{144}, \frac{1}{80}, \frac{17}{720}, \ldots$

At $i = 0$, $t = \frac{1}{180}, \frac{1}{72}, \frac{1}{45}, \ldots$

(c) At $t = \frac{1}{240}$ s, $i = +6$ A.

At $t = \frac{1}{120}$ s, $i = -10$ A.

Exercises 12.5

1. A crank 8 in. long starts from a horizontal position (0°) and rotates in a counterclockwise direction at a uniform angular velocity of 5 revolutions per second (10π rad/s). Plot the curve that shows the simple harmonic motion of the crank handle, and find the equation of this curve.
2. Complete Exercise 1 for when the crank starts from a vertical position (90°).
3. If the rotating wire from Example 2 produces a maximum current of 15 A at 60 cycles/s, graph the resulting equation when $\theta = \frac{\pi}{4}$.
4. Complete Exercise 3 for $\theta = 0$.

5. A rotating wire through a magnetic field generates an alternating current whose instantaneous voltage, e, is given by the equation

$$e = E_{\max} \cos(\omega t + \theta)$$

where $E_{\max}$ is the maximum instantaneous voltage, ω is the angular velocity of the rotating wire, t is the time, and θ is the phase angle. Graph the resulting equation when $E_{\max} = 110$ V, $\omega = 60$ cycles/s, and $\theta = 0°$.

6. Complete Exercise 5 for $\theta = -\frac{\pi}{3}$.

7. A mass hanging on a spring is pulled down 10 cm from its equilibrium position and then released $\left(\theta = -\frac{\pi}{2}\right)$. The resulting oscillation of the mass is simple harmonic motion. The period of one complete cycle is observed to be 0.80 s. Find and graph the resulting simple harmonic motion equation.

8. A weight is attached to a spring and vibrates at a frequency of 4 vibrations per second and an observed amplitude of 3 in. Find and graph the resulting equation beginning with $t = 0$ at $\theta = 0$.

9. A thin reed fixed at one end vibrates at 200 vibrations per second with an amplitude of 0.25 cm. Find and graph the resulting equation beginning with $t = 0$ at $\theta = 0$.

10. The equation of motion for a particle in simple harmonic motion is given by

$$y = 5.0 \cos 8.0t$$

where y is in metres and t is in seconds. Graph the equation. Find the distance from the horizontal at $t = \frac{\pi}{4}$.

CHAPTER SUMMARY

1. The values of a, b, and c determine amplitude, period, and phase shift for the sine and cosine functions as follows:

$$y = a \sin(bx + c)$$
$$y = a \cos(bx + c)$$

$$\text{Amplitude} = |a| \qquad \text{Phase shift} = \frac{c}{b} \qquad \left(\text{To the } \textit{left} \text{ if } \frac{c}{b} > 0\right)$$

$$\text{Period} = \frac{2\pi}{b} \qquad \left(\text{To the } \textit{right} \text{ if } \frac{c}{b} < 0\right)$$

If a is negative, each branch is flipped or inverted about the x-axis.

2. For $y = a \tan bx$ and $y = a \cot bx$, the period may be found by

$$P = \frac{\pi}{b}$$

If $a > 1$, each branch intersects the x-axis at a greater angle than for $y = \tan x$. If $0 < a < 1$, each branch intersects the x-axis at a smaller angle than for $y = \tan x$. If a is negative, each branch is flipped or inverted about the x-axis.

3. For $y = a \sec bx$ and $y = a \csc bx$, the period may be found by

$$P = \frac{2\pi}{b}$$

The quantity $|a|$ is the vertical distance between the x-axis and the low point of each of the branches above the x-axis. If a is negative, each branch is flipped or inverted about the x-axis.

4. To add or subtract two functions graphically, use the method of addition of ordinates as follows:
   (a) Graph each of the functions that make up the composite on the same set of coordinate axes.
   (b) It may be helpful to draw several vertical lines perpendicular to the x-axis.
   (c) If the composite is
      (i) a sum, find the algebraic sum of the y-values where each vertical line intersects each of the graphs; or
      (ii) a difference, find the algebraic difference of the y-values where each vertical line intersects each of the graphs, or graph the negative (or "flipped") curve and find the algebraic sum as in (i).
   (d) Plot each resultant y-value from Step (c) on each vertical line and connect with a smooth curve.

5. Simple harmonic motion is described by the equations

$$y = r \sin(\omega t + \theta) \quad \text{or} \quad y = r \cos(\omega t + \theta)$$

CHAPTER 12 REVIEW

Find the amplitude and period of each function and sketch its graph.

1. $y = 4 \cos 6x$
2. $y = -2 \sin \frac{1}{3}x$
3. $y = 3 \cos 2\pi x$

Find the amplitude, period, and phase shift of each function and sketch its graph.

4. $y = 3 \sin\left(x - \frac{\pi}{4}\right)$
5. $y = \cos\left(2x + \frac{2\pi}{3}\right)$
6. $y = 4 \sin\left(\pi x + \frac{\pi}{2}\right)$

Find the period of each function and sketch its graph.

7. $y = \tan 5x$
8. $y = -\cot 3x$
9. $y = 2 \sec 4x$
10. Graph $y = 3 \sin x + 2 \cos 2x$ using the method of addition of ordinates for one complete period of the composite curve.
11. A pendulum has a period of 4 s and an observed amplitude of 3 ft. The motion of the projection of the pendulum bob on the floor is in simple harmonic motion. Find and graph the resulting equation. (Assume that the phase angle is 0.)

Application

Improving Color Television

The technology of gas-plasma television allows a TV set to be lightweight and thin—so much so that the TV could be hung on a wall. Currently, most color TV sets use a cathode-ray tube, which requires an electron-gun assembly to be located behind the inside screen. The bigger the screen, the further back the gun assembly must be. Gas-plasma or thin TVs use charged gas rather than electrons to light up the color pixels, thereby eliminating the distance requirement.

Both systems use red, green, and blue phosphors placed in some pattern on the inside of the screen. In the cathode-ray tube a variety of electrodes in the gun assembly and conductive coatings around the funnel tube accelerate the electrons and keep them focused. A magnetic deflection yoke guides the electrons horizontally and vertically.

A perforated sheet of metal called a *shadow mask* is the last thing to guide the electrons before they hit the screen. The screen is curved so that the electrons travel about the same distance no matter where they strike it. It helps to think of the path of the electrons as radii of a sphere. Flattening the faceplate would mean that the distance between the gun and points on the faceplate would differ, and the beam would not be focused at all points. However, we really don't want the front of a TV screen to be spherical; imagine how the picture would look to the viewer. We would like the faceplate to be as flat as possible without distorting the picture.

Engineers and technologists are continuing to find ways to improve the picture. New focusing technology converts the focus on the edges and corners to permit a flatter faceplate. For example, the Sony Trinitron™ uses a cylindrical faceplate which is flat from top to bottom and curved in the horizontal direction. This construction prevents vertical lines from bowing at the edges. Other industry improvements involve using shadow masks of invar (a steel alloy) instead of iron. The invar expands less at high temperatures and thus does not bulge. A bulge causes colors to bleed into one another. Rare-earth elements have been added to the phosphors to increase their brightness. The faceplate has been made darker which gives more detail and depth. Also, the contrast is increased by reducing the amount of light the faceplate reflects. Finally, antiglare coating is added on the outside of the faceplate.

For a television to be thin, a totally different approach is needed. Two methods for applying the gas-plasma technology are being developed by competing companies. One combines flat-panel display technology with plasma-addressed liquid-crystal technology (PALC) to activate each pixel. The liquid crystals are opened and closed when the gas plasma operates as an electronic on/off switch. The other uses two glass panels to enclose the gas plasma. Then a voltage is applied between electrodes which creates ultraviolet ra-

diation that in turn produces visible light by activating the phosphors. Both methods result in a system that weighs approximately 12 pounds, is about 4 inches thick, and can have a viewing screen from 20 to 60 inches.

Flat TVs are available to the public, but the cost can be as high as $15,000 depending on the size. When mass production becomes more efficient, the cost is expected to be around $2500.

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Chapter 14

Trigonometric Formulas and Identities

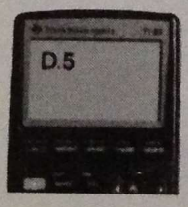
INTRODUCTION

Thus far, we have solved algebraic, exponential, and logarithmic equations. Trigonometric equations are more complicated than any we have done. They may involve more than one trigonometric function or contain functions with different (but related) arguments. In this chapter we develop many trigonometric identities that will be used to simplify expressions and convert equations into a form that we can readily solve.

Objectives

- Develop basic trigonometric identities.
- Simplify trigonometric expressions.
- Develop a strategy for proving trigonometric identities.
- Use trigonometric identities to solve equations.
- Work with inverse trigonometric functions.
- Identify the domain of an inverse trigonometric function.

13.1 BASIC TRIGONOMETRIC IDENTITIES



An equation that is true for all values of the variables for which both sides are defined is called an **identity.** The following are examples of algebraic identities:

1. $2(x - 3) = 2x - 6$ (True for all x)
2. $4(x + y) = 4x + 4y$ (True for all x and all y)
3. $\dfrac{4x + y}{x} = 4 + \dfrac{y}{x}$ (True for all y and all x except 0)

The following are examples of trigonometric identities. (Some have been shown previously; the rest will be shown later.)

4. $\sin\theta = \dfrac{1}{\csc\theta}$ (True for all θ except where $\csc\theta = 0$)

5. $\sin^2\theta + \cos^2\theta = 1$ (True for all θ)

6. $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$ (True for all θ and all ϕ)

Identities are used to prove other identities or to simplify a given trigonometric expression. They are also used to change a given trigonometric expression into a different but equivalent expression which is more useful for a particular problem, either technical or mathematical.

A rather long list of trigonometric identities exists. We shall first prove (show the validity of) the basic identities. Before you use an identity, you must be sure that it comes from a list of valid identities or be able to show that it is valid. At times, just a change in the form of an identity will greatly simplify the solution of a given problem.

A given identity may be proven by any of a number of ways, as long as all the steps are valid. We shall illustrate five basic ways to prove trigonometric identities. When a number of valid ways may be used to prove a given identity, each is *correct;* the one most preferred is the one that is shortest and most efficient. Generally, a helpful suggestion is to try to simplify the more complicated side to the less complicated side of the identity.

Method 1: Use of Trigonometric Definitions

The trigonometric definitions from Chapter 10 are shown in the following box with Fig. 13.1; they may be used to prove identities.

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\cot\theta = \frac{x}{y} \quad (y \neq 0)$$

$$\sec\theta = \frac{r}{x} \quad (x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (y \neq 0)$$

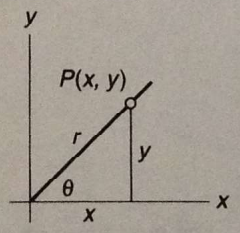


Figure 13.1

EXAMPLE 1

Prove $\cos\theta = \dfrac{1}{\sec\theta}$.

$$\frac{1}{\sec\theta} = \frac{1}{\frac{r}{x}} = \frac{x}{r} = \cos\theta$$

Each of the reciprocal identities may be proven in a similar way and is listed in the following box. All basic identities are listed and numbered for easy reference.

$$\sin\theta = \frac{1}{\csc\theta} \quad (1)$$

$$\cos\theta = \frac{1}{\sec\theta} \quad (2)$$

$$\tan\theta = \frac{1}{\cot\theta} \quad (3)$$

$$\cot\theta = \frac{1}{\tan\theta} \quad (4)$$

$$\sec\theta = \frac{1}{\cos\theta} \quad (5)$$

$$\csc\theta = \frac{1}{\sin\theta} \quad (6)$$

EXAMPLE 2

Prove $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad (7)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (8)$$

The next three identities are called the **Pythagorean identities** because they use the Pythagorean relationship, $x^2 + y^2 = r^2$.

EXAMPLE 3

Prove $\sin^2\theta + \cos^2\theta = 1$.

Given $x^2 + y^2 = r^2$, divide each side by r^2.

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

From the trigonometric definitions, $\dfrac{x}{r} = \cos\theta$ and $\dfrac{y}{r} = \sin\theta$. Therefore,

$$\cos^2\theta + \sin^2\theta = 1$$

Note: $\cos^2\theta$ may also be written $(\cos\theta)^2$.

EXAMPLE 4

Prove $1 + \tan^2 \theta = \sec^2 \theta$.

Given $x^2 + y^2 = r^2$, divide each side by x^2.

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

Then

$$1 + \tan^2 \theta = \sec^2 \theta$$

The three Pythagorean identities are as follows.

$$\sin^2 \theta + \cos^2 \theta = 1 \quad (9)$$

$$1 + \tan^2 \theta = \sec^2 \theta \quad (10)$$

$$\cot^2 \theta + 1 = \csc^2 \theta \quad (11)$$

Method 1 is typically used only to develop or prove the first 11 identities. Other methods are then needed to prove the more complex identities because method 1 commonly results in cumbersome fractional expressions. We suggest that you use method 1 only in Exercises 1–4 at the end of this section.

Method 2: Substitution of Known Identities

We may also show that an identity is valid by changing one of the sides of the given expression or any of its parts by substitution of a known identity or any of its forms one or more times.

EXAMPLE 5

Prove $\tan x = \sin x \sec x$.

Let's start with the right-hand side and show that it is equal to the left-hand side.

$$\sin x \sec x = \sin x \left(\frac{1}{\cos x}\right) \quad \text{(Identity 5)}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x \quad \text{(Identity 7)}$$

Therefore, $\tan x = \sin x \sec x$.

EXAMPLE 6

Prove $(1 - \sin^2 \theta)\tan^2 \theta = \sin^2 \theta$.

Let's start with the left-hand side and show that it is equal to the right-hand side.

$$(1 - \sin^2 \theta)\tan^2 \theta = \cos^2 \theta \tan^2 \theta \quad \text{(Form of Identity 9)}$$

$$= \cos^2 \theta \left(\frac{\sin^2 \theta}{\cos^2 \theta}\right) \quad \text{(Identity 7)}$$

$$= \sin^2 \theta$$

Therefore, $(1 - \sin^2 \theta)\tan^2 \theta = \sin^2 \theta$.

EXAMPLE 7

Prove $\cos^2 x(1 + \tan^2 x) = 1$.

Let's start with the left-hand side and show that it is equal to the right-hand side.

$$\begin{aligned}
\cos^2 x(1 + \tan^2 x) &= \cos^2 x \sec^2 x && \text{(Identity 10)} \\
&= \cos^2 x \left(\frac{1}{\cos^2 x}\right) && \text{(Identity 5)} \\
&= 1
\end{aligned}$$

Therefore, $\cos^2 x(1 + \tan^2 x) = 1$.

EXAMPLE 8

Prove $\sin \theta + \cot \theta \cos \theta = \csc \theta$.

Again, let's start with the left-hand side and show that it is equal to the right-hand side.

$$\begin{aligned}
\sin \theta + \cot \theta \cos \theta &= \sin \theta + \frac{\cos \theta}{\sin \theta} \cdot \cos \theta && \text{(Identity 8)} \\
&= \sin \theta + \frac{\cos^2 \theta}{\sin \theta} \\
&= \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta} \\
&= \frac{1}{\sin \theta} && \text{(Identity 9)} \\
&= \csc \theta && \text{(Identity 6)}
\end{aligned}$$

Therefore, $\sin \theta + \cot \theta \cos \theta = \csc \theta$.

Method 3: Factoring One Side of the Identity

EXAMPLE 9

Prove $\sin^2 \theta + \sin^2 \theta \tan^2 \theta = \tan^2 \theta$.

$$\begin{aligned}
\sin^2 \theta + \sin^2 \theta \tan^2 \theta &= \sin^2 \theta (1 + \tan^2 \theta) && \text{(Factor.)} \\
&= \sin^2 \theta \sec^2 \theta && \text{(Identity 10)} \\
&= \sin^2 \theta \left(\frac{1}{\cos^2 \theta}\right) && \text{(Identity 5)} \\
&= \tan^2 \theta && \text{(Identity 7)}
\end{aligned}$$

Therefore, $\sin^2 \theta + \sin^2 \theta \tan^2 \theta = \tan^2 \theta$.

EXAMPLE 10

Prove $\sin^4 \theta - \cos^4 \theta = 2 \sin^2 \theta - 1$.

$$\begin{aligned}
\sin^4 \theta - \cos^4 \theta &= (\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta) && \text{(Factor.)} \\
&= (1)(\sin^2 \theta - \cos^2 \theta) && \text{(Identity 9)} \\
&= \sin^2 \theta - (1 - \sin^2 \theta) && \text{(Identity 9)} \\
&= 2 \sin^2 \theta - 1
\end{aligned}$$

Therefore, $\sin^4 \theta - \cos^4 \theta = 2 \sin^2 \theta - 1$.

Method 4: Multiplication of the Numerator and Denominator of a Fractional Expression

Multiplication of the numerator and denominator by some trigonometric quantity may change the form of one side of the identity so that you may then use one of the other methods to complete the proof.

EXAMPLE 11

Prove $\dfrac{\cos x}{1 + \sin x} = \dfrac{1 - \sin x}{\cos x}$.

Multiply the numerator and the denominator of the right-hand side by $1 + \sin x$.

$$\frac{1 - \sin x}{\cos x} \cdot \frac{1 + \sin x}{1 + \sin x} = \frac{1 - \sin^2 x}{\cos x(1 + \sin x)}$$

$$= \frac{\cos^2 x}{\cos x(1 + \sin x)} \qquad \text{(Identity 9)}$$

$$= \frac{\cos x}{1 + \sin x}$$

Therefore, $\dfrac{\cos x}{1 + \sin x} = \dfrac{1 - \sin x}{\cos x}$.

Method 5: Expressing All Functions on One Side in Terms of Sines and Cosines and Using Any of Methods 1–4

EXAMPLE 12

Prove $\tan x + \cot x = \sec x \csc x$.

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \qquad \text{(Identities 7 and 8)}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\cos x \sin x} \qquad \text{(Identity 9)}$$

$$= \sec x \csc x \qquad \text{(Identities 5 and 6)}$$

Therefore, $\tan x + \cot x = \sec x \csc x$.

Note: In any method, introduce radicals only as a last resort!

Proving identities and using identities to simplify a trigonometric expression are quite different from solving an equation. You must work from only one side of the identity to show that it is equivalent to the other. When you use identities to simplify a trigonometric expression, you begin with the expression and use identities to change from one equivalent expression to another until the expression is simplified or in the form that is most useful to you. That is, you may *not* use equation-solving principles such as adding the same quantity to each side or multiplying each side by the same quantity.

Exercises 13.1

Use the trigonometric definitions opposite Fig. 13.1 to prove each identity.

1. $\dfrac{1}{\csc\theta} = \sin\theta$
2. $\dfrac{1}{\tan\theta} = \cot\theta$
3. $\dfrac{\cos\theta}{\sin\theta} = \cot\theta$
4. $\cot^2\theta + 1 = \csc^2\theta$

Prove each identity without using the trigonometric definitions.

5. $\cos\theta\sec\theta = 1$
6. $\tan\theta\cot\theta = 1$
7. $\cos x\tan x = \sin x$
8. $\csc x\tan x = \sec x$
9. $\dfrac{\csc\theta}{\sec\theta} = \cot\theta$
10. $\dfrac{\tan\theta}{\cot\theta} = \tan^2\theta$
11. $\dfrac{\tan\theta}{\sin\theta} = \sec\theta$
12. $\dfrac{\cot\theta}{\cos\theta} = \csc\theta$
13. $\sec\theta\cot\theta = \csc\theta$
14. $\sin\theta\cot\theta = \cos\theta$
15. $(1 - \cos^2 x)\csc^2 x = 1$
16. $(\cot^2 x + 1)\tan^2 x = \sec^2 x$
17. $(1 - \sin^2\theta)\cos^2\theta = \cos^4\theta$
18. $(1 + \tan^2 x)\sin^2 x = \tan^2 x$
19. $\dfrac{\cos^2 x - 1}{\sin x} = -\sin x$
20. $\dfrac{\sec^2 x - 1}{\sin^2 x} = \sec^2 x$
21. $\cos\theta(\csc\theta - \sec\theta) = \cot\theta - 1$
22. $\cot\theta - \cos\theta = \cot\theta(1 - \sin\theta)$
23. $\tan^2\theta - \tan^2\theta\sin^2\theta = \sin^2\theta$
24. $\tan\theta\sin\theta + \tan\theta\cot\theta\cos\theta = \sec\theta$
25. $\dfrac{\sec x - \cos x}{\sin x} = \tan x$
26. $\dfrac{\csc x - \sin x}{\cot x} = \cos x$
27. $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$
28. $\sin^2\theta - \cos^2\theta = 1 - 2\cos^2\theta$
29. $\dfrac{\csc^2 x}{1 + \tan^2 x} = \cot^2 x$
30. $\dfrac{1 + \tan^2 x}{\tan^2 x} = \csc^2 x$
31. $\dfrac{(1 + \sin\theta)(1 - \sin\theta)}{\sin^2\theta} = \cot^2\theta$
32. $\dfrac{\sin x\cos x}{(1 + \cos x)(1 - \cos x)} = \cot x$
33. $\dfrac{\sin^2 x}{1 + \cos x} = 1 - \cos x$
34. $\dfrac{\cos^2 x}{1 + \sin x} = 1 - \sin x$
35. $\cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$
36. $\sec^4 x - \tan^4 x = 2\tan^2 x + 1$
37. $\dfrac{1 + \sec x}{\csc x} = \sin x + \tan x$
38. $\dfrac{\cos x\tan x + \sin x}{\tan x} = 2\cos x$
39. $(\sec x - \tan x)(\csc x + 1) = \cot x$
40. $\sec x + \tan x + \sin x = \dfrac{1 + \sin x + \sin x\cos x}{\cos x}$
41. $\dfrac{1 - \sin^2 x}{1 - \cos^2 x} = \cot^2 x$
42. $\dfrac{1 - \tan x}{1 + \tan x} = \dfrac{\cot x - 1}{\cot x + 1}$
43. $\dfrac{\sin x}{1 + \cos x} = \dfrac{1 - \cos x}{\sin x}$
44. $\dfrac{\sin\theta}{1 + \cos\theta} = \dfrac{1 - \cos\theta}{\cos\theta\tan\theta}$

45. $\dfrac{\cos^2 x}{1 - \sin x} = 1 + \sin x$

46. $\dfrac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} = \dfrac{\cot\theta + 1}{\cot\theta - 1}$

47. $\dfrac{1}{\sec x - 1} - \dfrac{1}{\sec x + 1} = 2\cot^2 x$

48. $\dfrac{1}{\sin x + 1} - \dfrac{1}{\sin x - 1} = 2\sec^2 x$

49. $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$

50. $\sec^4 x - 1 = \tan^2 x(\sec^2 x + 1)$

51. $\dfrac{\tan^2 x - 1}{1 - \cot^2 x} = \tan^2 x$

52. $\sec\theta - \tan\theta = \dfrac{1}{\sec\theta + \tan\theta}$

53. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$

54. $\dfrac{\tan\theta}{\sec\theta - \cos\theta} = \csc\theta$

13.2 FORMULAS FOR THE SUM AND THE DIFFERENCE OF TWO ANGLES

The sine, cosine, and tangent of the sum and the difference of two angles have practical applications. These formulas are also used to develop the double- and half-angle formulas in the next section.

First, we want to show

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

In Fig. 13.2, angles θ and $\theta + \phi$ are constructed so that they are in standard position. The initial side of ϕ coincides with the terminal side of θ; its terminal side coincides with the terminal side of $\theta + \phi$. Now choose any point P on the terminal side of $\theta + \phi$. Then drop perpendicular lines to the terminal side of θ at Q and to the initial side of θ at S. Next, draw perpendiculars QR and QT. Because RQ and OT are parallel, $\theta = \angle 1$. (They are alternate interior angles.)

$\angle 1 + \angle 2 = 90°$	($PQ \perp OQ$)
$\angle 3 + \angle 2 = 90°$	(The sum of the two acute angles of a right triangle equals 90°.)
$\angle 1 - \angle 3 = 0$	(Subtract.)

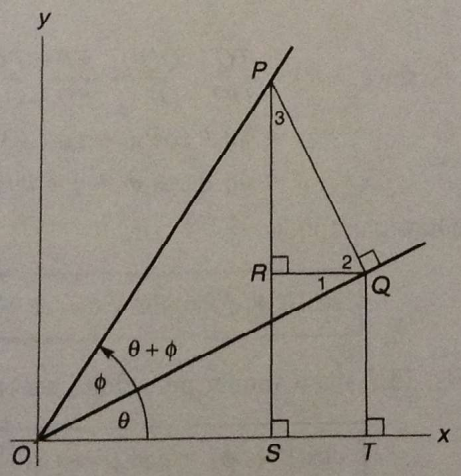


Figure 13.2

or

$$\angle 1 = \angle 3$$

We have already shown that

$$\angle 1 = \theta$$

Therefore, $\theta = \angle 3$. That is, $\angle SPQ$ and θ are equal. In right triangle OSP of Fig. 13.2,

$$\begin{aligned}\sin(\theta + \phi) = \frac{SP}{OP} &= \frac{SR + RP}{OP} \\ &= \frac{SR}{OP} + \frac{RP}{OP} \\ &= \frac{TQ}{OP} + \frac{RP}{OP} \quad \text{(In rectangle } SRQT, SR = TQ.\text{)}\end{aligned}$$

Next, multiply numerator and denominator of the first term by OQ and rearrange the terms; then multiply numerator and denominator of the second term by PQ and rearrange the terms.

$$\begin{aligned}\frac{TQ}{OP} + \frac{RP}{OP} &= \frac{TQ}{OP} \cdot \frac{OQ}{OQ} + \frac{RP}{OP} \cdot \frac{PQ}{PQ} \\ &= \frac{TQ}{OQ} \cdot \frac{OQ}{OP} + \frac{RP}{PQ} \cdot \frac{PQ}{OP}\end{aligned}$$

From Fig. 13.2 and each of the following right triangles, note that

$$\text{Right } \triangle OTQ: \qquad \sin\theta = \frac{TQ}{OQ}$$

$$\text{Right } \triangle OPQ: \qquad \cos\phi = \frac{OQ}{OP}$$

$$\text{Right } \triangle PRQ: \qquad \cos\angle 3 = \frac{RP}{PQ}$$

$$\text{Right } \triangle OPQ: \qquad \sin\phi = \frac{PQ}{OP}$$

Then

$$\begin{aligned}\sin(\theta + \phi) &= \frac{TQ}{OQ} \cdot \frac{OQ}{OP} + \frac{RP}{PQ} \cdot \frac{PQ}{OP} \\ &= \sin\theta\cos\phi + \cos\angle 3 \sin\phi \\ &= \sin\theta\cos\phi + \cos\theta\sin\phi \qquad (\angle 3 = \theta)\end{aligned}$$

Therefore, we have the following.

$$\boxed{\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi} \qquad (12)$$

Using Fig. 13.2 and a similar procedure, we can show the following.

$$\boxed{\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi} \qquad (13)$$

To illustrate these formulas let's use the following examples.

EXAMPLE 1

Simplify $\cos(\theta + 90°)$ using Formula (13).

$$\begin{aligned}\cos(\theta + 90°) &= \cos\theta \cos 90° - \sin\theta \sin 90° \\ &= \cos\theta \quad (0) \quad - \sin\theta \quad (1) \\ &= -\sin\theta\end{aligned}$$

EXAMPLE 2

Simplify $\sin(\theta + 90°)$ using Formula (12).

$$\begin{aligned}\sin(\theta + 90°) &= \sin\theta \cos 90° + \cos\theta \sin 90° \\ &= \sin\theta \quad (0) \quad + \cos\theta \quad (1) \\ &= \cos\theta\end{aligned}$$

Before proceeding, we need to show the following identities.

$$\boxed{\sin(-\theta) = -\sin\theta} \qquad (14)$$

$$\boxed{\cos(-\theta) = \cos\theta} \qquad (15)$$

First, construct any angle θ and its negative, $-\theta$, as in Fig. 13.3. Label any point P_1 on the terminal side of θ with coordinates (x_1, y_1). Let r be the distance from P_1 to the origin. On the terminal side of $-\theta$, label a point at a distance of r from the origin as P_2. Note that its coordinates are $(x_1, -y_1)$.

Then

$$\sin\theta = \frac{y_1}{r} \quad \text{and} \quad \sin(-\theta) = \frac{-y_1}{r} \qquad \text{Thus, } \sin(-\theta) = -\sin\theta$$

$$\cos\theta = \frac{x_1}{r} \quad \text{and} \quad \cos(-\theta) = \frac{x_1}{r} \qquad \text{Thus, } \cos(-\theta) = \cos\theta$$

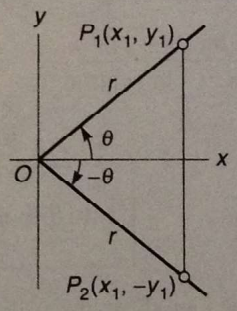


Figure 13.3

Similarly, the following may also be shown.

$$\boxed{\begin{aligned}\tan(-\theta) &= -\tan\theta \quad (16)\\ \cot(-\theta) &= -\cot\theta \quad (17)\\ \sec(-\theta) &= \sec\theta \quad (18)\\ \csc(-\theta) &= -\csc\theta \quad (19)\end{aligned}}$$

To find the formula for $\sin(\theta - \phi)$, use Formula (12), and write $\theta - \phi$ as $\theta + (-\phi)$.

$$\begin{aligned}\sin[\theta + (-\phi)] &= \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)\\ &= \sin\theta\cos\phi - \cos\theta\sin\phi\end{aligned}$$

Therefore:

$$\boxed{\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi} \quad (20)$$

Likewise, to find the formula for $\cos(\theta - \phi)$, use Formula (13), and again write $\theta - \phi$ as $\theta + (-\phi)$.

$$\begin{aligned}\cos[\theta + (-\phi)] &= \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi)\\ &= \cos\theta\cos\phi + \sin\theta\sin\phi\end{aligned}$$

Therefore:

$$\boxed{\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi} \quad (21)$$

To find a formula for $\tan(\theta + \phi)$, observe the following:

$$\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} \qquad \text{(Identity 7)}$$

$$= \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi} \qquad \text{(Identities 12 and 13)}$$

$$= \frac{\dfrac{\sin\theta\cos\phi}{\cos\theta\cos\phi} + \dfrac{\cos\theta\sin\phi}{\cos\theta\cos\phi}}{\dfrac{\cos\theta\cos\phi}{\cos\theta\cos\phi} - \dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi}} \qquad \text{(Divide numerator and denominator by } \cos\theta\cos\phi.\text{)}$$

$$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \qquad \text{(Identity 7)}$$

Therefore:

$$\boxed{\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}} \quad (22)$$

Similarly, we can show the following.

$$\boxed{\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}} \quad (23)$$

EXAMPLE 2

Find $\cos 120°$ using Formula (25a).

$$\cos 120° = \cos 2(60°) = \cos^2 60° - \sin^2 60°$$
$$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$
$$= \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

EXAMPLE 3

Simplify $2 \sin 3x \cos 3x$.

Using Formula (24), we obtain

$$2 \sin 3x \cos 3x = \sin 2(3x) = \sin 6x$$

EXAMPLE 4

Simplify $1 - 2\cos^2 5x$.

Using Formula (25b), we have

$$1 - 2\cos^2 5x = -\cos 2(5x) = -\cos 10x$$

EXAMPLE 5

Find $\sin 2\theta$ when $\cos \theta = \frac{4}{5}$ (θ in the fourth quadrant).

If $\cos \theta = \frac{4}{5}$ in the fourth quadrant (see Fig. 13.4), then $\sin \theta = -\frac{3}{5}$.

$$\sin 2\theta = 2 \sin \theta \cos \theta \qquad \text{(Identity 24)}$$
$$= 2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)$$
$$= -\frac{24}{25}$$

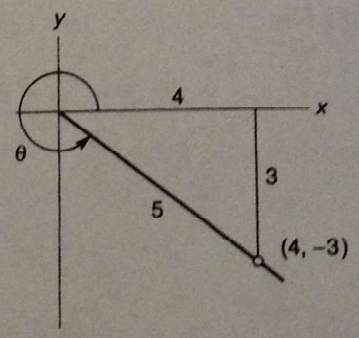


Figure 13.4

EXAMPLE 6

Find $\cos 2\theta$ when $\sin \theta = \frac{1}{2}$.

Using Formula (25c), we have

$$\cos 2\theta = 1 - 2\sin^2 \theta$$
$$= 1 - 2\left(\frac{1}{2}\right)^2 = 1 - \frac{1}{2} = \frac{1}{2}$$

In an identity the choice of specific variables does not matter. For example, the identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

may also be written in the form

$$\sin^2 m + \cos^2 m = 1$$

To find a formula for $\sin \frac{\theta}{2}$, let's begin with Formula (25c) in the form

$$\cos 2m = 1 - 2\sin^2 m$$

Since this identity is true for all values of m, let $m = \frac{\theta}{2}$ and then solve for $\sin \frac{\theta}{2}$.

$$\cos 2\left(\frac{\theta}{2}\right) = 1 - 2\sin^2 \frac{\theta}{2}$$
$$\cos \theta = 1 - 2\sin^2 \frac{\theta}{2}$$
$$2\sin^2 \frac{\theta}{2} = 1 - \cos \theta$$
$$\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$$

$$\boxed{\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}} \qquad (27)$$

To find a formula for $\cos \frac{\theta}{2}$, let's begin with Formula (25b) in the form

$$\cos 2m = 2\cos^2 m - 1$$

Let $m = \frac{\theta}{2}$ and then solve for $\cos \frac{\theta}{2}$.

$$\cos 2\left(\frac{\theta}{2}\right) = 2\cos^2 \frac{\theta}{2} - 1$$
$$\cos \theta = 2\cos^2 \frac{\theta}{2} - 1$$

$$\frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2}$$

$$\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}} \qquad (28)$$

Note: In both Formulas (27) and (28), the sign used depends on the quadrant in which $\frac{\theta}{2}$ lies.

Now find $\tan\frac{\theta}{2}$.

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} \qquad \text{(Identity 7)}$$

Multiply numerator and denominator by $2\sin\frac{\theta}{2}$.

$$\tan\frac{\theta}{2} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

Note that

$$\cos 2\left(\frac{\theta}{2}\right) = 1 - 2\sin^2\frac{\theta}{2}$$

That is,

$$2\sin^2\frac{\theta}{2} = 1 - \cos\theta$$

Also note that

$$\sin 2\left(\frac{\theta}{2}\right) = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

That is,

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

Thus:

$$\boxed{\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta}} \qquad (29)$$

EXAMPLE 7

Find $\cos\frac{\theta}{2}$ when $\sin\theta = -\frac{3}{5}$ (θ in the third quadrant).

If $\sin\theta = -\frac{3}{5}$ in the third quadrant as in Fig. 13.5, then $\cos\theta = -\frac{4}{5}$.

$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$$

$$= -\sqrt{\frac{1+\left(-\frac{4}{5}\right)}{2}}$$

$$= -\sqrt{\frac{1}{10}} \quad \text{or} \quad -\frac{\sqrt{10}}{10}$$

The sign is negative because if $180° < \theta < 270°$, then $90° < \frac{\theta}{2} < 135°$ where $\cos\frac{\theta}{2}$ is negative.

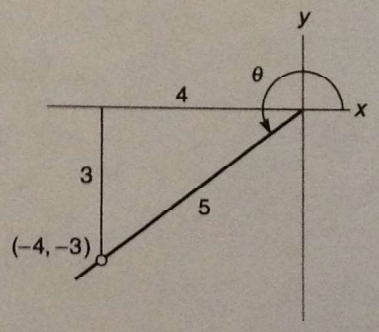


Figure 13.5

EXAMPLE 8

Prove $\frac{1-\cos 2x}{\sin 2x} = \tan x$.

$$\frac{1-\cos 2x}{\sin 2x} = \frac{1-(1-2\sin^2 x)}{2\sin x\cos x} \qquad \text{(Identities 25c and 24)}$$

$$= \frac{2\sin^2 x}{2\sin x\cos x}$$

$$= \frac{\sin x}{\cos x}$$

$$= \tan x \qquad \text{(Identity 7)}$$

Therefore, $\frac{1-\cos 2x}{\sin 2x} = \tan x$.

The following trigonometric formulas are given without proof for reference.

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \tag{30}$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) \tag{31}$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \tag{32}$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) \tag{33}$$

$$\sin(\theta+\phi) + \sin(\theta-\phi) = 2\sin\theta\cos\phi \tag{34}$$

$$\sin(\theta+\phi) - \sin(\theta-\phi) = 2\cos\theta\sin\phi \tag{35}$$

$$\cos(\theta+\phi) + \cos(\theta-\phi) = 2\cos\theta\cos\phi \tag{36}$$

$$\cos(\theta+\phi) - \cos(\theta-\phi) = -2\sin\theta\sin\phi \tag{37}$$

Exercises 13.3

Simplify each expression.

1. $2\sin\frac{x}{4}\cos\frac{x}{4}$

2. $20\sin^2 x\cos^2 x$

3. $1 - 2\sin^2 3x$

4. $\sqrt{\frac{1-\cos 6\theta}{2}}$

5. $\sqrt{\frac{1+\cos\frac{\theta}{4}}{2}}$

6. $\cos 2x + 2\sin^2 x$

7. $\cos^2\frac{x}{6} - \sin^2\frac{x}{6}$

8. $10\cos^2 44° - 10\sin^2 44°$

9. $20\sin 4\theta\cos 4\theta$

10. $2\sin 3x\cos 3x$

11. $-\sqrt{\frac{1+\cos 250°}{2}}$

12. $\sqrt{\frac{1-\cos 16\theta}{2}}$

13. $4 - 8\sin^2\theta$

14. $1 - 2\sin^2 7t$

15. $100\sin 30t\cos 30t$

16. $15\sin\frac{x}{6}\cos\frac{x}{6}$

17. Find $\sin 2\theta$ when $\cos\theta = \frac{3}{5}$ (θ in the first quadrant).

18. Find $\cos 2\theta$ when $\sin\theta = \frac{3}{5}$ (θ in the second quadrant).

19. Find $\tan 2\theta$ when $\cos\theta = -\frac{12}{13}$ (θ in the third quadrant).

20. Find $\sin\frac{\theta}{2}$ when $\sin\theta = -\frac{5}{13}$ (θ in the fourth quadrant).

21. Find $\cos\frac{\theta}{2}$ when $\sin\theta = -\frac{2}{3}$ (θ in the third quadrant).

22. Find $\tan\frac{\theta}{2}$ when $\tan\theta = -\frac{3}{4}$ (θ in the fourth quadrant).

Prove each identity.

23. $(\sin x + \cos x)^2 = 1 + \sin 2x$

24. $\frac{2\tan x}{\sin 2x} = 1 + \tan^2 x$

25. $\cos^4 x - \sin^4 x = \cos 2x$

26. $\sin 2x = \frac{2\tan x}{1 + \tan^2 x}$

27. $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$

28. $2\tan x \csc 2x = \sec^2 x$

29. $\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$

30. $\sec 2x = \frac{\sec^2 x}{2 - \sec^2 x}$

31. $\tan x + \cot 2x = \csc 2x$

32. $\tan(x + 45°) + \tan(x - 45°) = 2\tan 2x$

33. $\sin^2\frac{x}{2} = \frac{\sec x - 1}{2\sec x}$

34. $2\cos^2\frac{\theta}{2} = \frac{1 + \sec\theta}{\sec\theta}$

35. $\sec^2\frac{x}{2} = \frac{2}{1 + \cos x}$

36. $\csc^2\frac{x}{2} = \frac{2\sec x}{\sec x - 1}$

37. $\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}$

38. $\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \frac{1 + \sin\theta}{\cos\theta}$

39. $2\cos\frac{x}{2} = (1 + \cos x)\sec\frac{x}{2}$

40. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2$

41. $\tan\frac{x}{2} + \cot\frac{x}{2} = 2\csc x$

42. $\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = 1 + \frac{1}{2}\sin 2x$

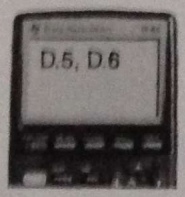
43. $\left(\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right)^2 = 1 - \sin\theta$

44. $8\sin^2\frac{x}{2}\cos^2\frac{x}{2} = 1 - \cos 2x$

45. $\sin 3x = 3\sin x - 4\sin^3 x$

46. $\cos 3x = 4\cos^3 x - 3\cos x$

13.4 TRIGONOMETRIC EQUATIONS



A trigonometric equation that is not an identity may have a number of solutions. The solutions may be given in terms of degrees or radians. There are no general procedures to follow when solving a trigonometric equation. You may try algebraic methods or use identities to write the equation in terms of a single trigonometric function.

EXAMPLE 1

Solve $2\sin\theta + 1 = 0$ for $0° \le \theta < 360°$.

$$2\sin\theta + 1 = 0$$

$$\sin\theta = -\frac{1}{2}$$

$\theta = 210°, 330°$ (Sine is negative in third and fourth quadrants.)

Note: If we did not have the restriction on θ, $0° \le \theta < 360°$, there would be an infinite number of solutions; that is, angles that are coterminal with 210° and 330°.

EXAMPLE 2

Solve $\sin 2\theta - \sin \theta = 0$ for $0 \le \theta < 2\pi$.

Replace $\sin 2\theta$ by $2 \sin \theta \cos \theta$ [Formula (24)] and factor.

$$2 \sin \theta \cos \theta - \sin \theta = 0$$
$$\sin \theta (2 \cos \theta - 1) = 0$$

$$\sin \theta = 0 \quad \text{or} \quad \cos \theta = \frac{1}{2}$$

$$\theta = 0, \pi \quad \text{or} \quad \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

EXAMPLE 3

Solve $2 \cos^2 x + 5 \sin x = 4$ for $0° \le x < 360°$.

Replace $\cos^2 x$ by $1 - \sin^2 x$ [from Formula (9)].

$$2(1 - \sin^2 x) + 5 \sin x = 4$$
$$0 = 2 \sin^2 x - 5 \sin x + 2$$
$$0 = (2 \sin x - 1)(\sin x - 2) \qquad \text{(Factor.)}$$

$$\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = 2$$

$x = 30°, 150°$ No solution because $-1 \le \sin x \le 1$

EXAMPLE 4

Solve $2 \cos^2 x + 6 \cos x + 3 = 0$ for $0° \le x < 360°$.

Since the left-hand side does not factor, we must use the quadratic formula with $a = 2$, $b = 6$, and $c = 3$.

$$\cos x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$\cos x = \frac{-6 \pm \sqrt{6^2 - 4(2)(3)}}{2(2)}$$
$$= \frac{-6 \pm \sqrt{12}}{4}$$
$$= \frac{-6 \pm 2\sqrt{3}}{4}$$
$$= \frac{-3 \pm \sqrt{3}}{2}$$

Changing each solution to a decimal value, we have

$$\cos x = \frac{-3 + \sqrt{3}}{2} = -0.6340 \quad \text{or} \quad \cos x = \frac{-3 - \sqrt{3}}{2} = -2.366$$

$x = 129.3°, 230.7°$ No solution because $-1 \le \cos x \le 1$

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EXAMPLE 5

Solve $\tan 2\theta + \cot 2\theta + 2 = 0$ for $0 \le \theta < 2\pi$.

Replace $\cot 2\theta$ by $\dfrac{1}{\tan 2\theta}$ [from Formula (4)].

$$\tan 2\theta + \frac{1}{\tan 2\theta} + 2 = 0$$

$$\tan^2 2\theta + 1 + 2\tan 2\theta = 0 \qquad \text{(Multiply each side by } \tan 2\theta\text{.)}$$

$$(\tan 2\theta + 1)^2 = 0$$

$$\tan 2\theta + 1 = 0$$

$$\tan 2\theta = -1$$

Hence,

$$2\theta = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$$

$$\theta = \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}$$

Note: The restriction $0 \le \theta < 2\pi$ is on θ not 2θ.

EXAMPLE 6

Solve $\cos x = \sin \dfrac{x}{2}$ for $0 \le x < 2\pi$.

Replace $\sin \dfrac{x}{2}$ by $\pm\sqrt{\dfrac{1 - \cos x}{2}}$ [from Formula (27)].

$$\cos x = \pm\sqrt{\frac{1 - \cos x}{2}}$$

$$\cos^2 x = \frac{1 - \cos x}{2} \qquad \text{(Square each side.)}$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0 \qquad \text{(Factor.)}$$

$$\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = -1$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3} \quad \text{or} \quad x = \pi$$

Since we squared each side, we *must* check for possible extraneous roots.

(a) $\cos \dfrac{\pi}{3} = \sin \dfrac{\frac{\pi}{3}}{2}$

$\cos \dfrac{\pi}{3} = \sin \dfrac{\pi}{6}$

$\dfrac{1}{2} = \dfrac{1}{2}$

$\dfrac{\pi}{3}$ is a solution.

(b) $\cos \dfrac{5\pi}{3} = \sin \dfrac{\frac{5\pi}{3}}{2}$

$\cos \dfrac{5\pi}{3} = \sin \dfrac{5\pi}{6}$

$\dfrac{1}{2} = \dfrac{1}{2}$

$\dfrac{5\pi}{3}$ is a solution.

(c) $\cos \pi = \sin \dfrac{\pi}{2}$

$-1 \neq 1$

π is not a solution.

Therefore, the solutions are $\dfrac{\pi}{3}$ and $\dfrac{5\pi}{3}$.

Exercises 13.4

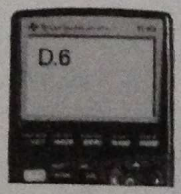
Solve each trigonometric equation for $0° \le x < 360°$.

1. $\cos x - 1 = 0$
2. $2 \sin x + \sqrt{3} = 0$
3. $\tan x - 1 = 0$
4. $2 \sin^2 x - 1 = 0$
5. $4 \cos^2 x - 3 = 0$
6. $\tan^2 x - 3 = 0$
7. $\sin 2x = 1$
8. $2 \cos 3x + 1 = 0$
9. $3 \tan^2 3x - 1 = 0$
10. $4 \sin^2 2x = 3$
11. $\sin 2x + \cos x = 0$
12. $\cos 2x - \sin x = 0$
13. $\sin^2 x + \sin x = 0$
14. $\cos x \tan x + \cos x = 0$
15. $2 \sin^2 x = 1 - 2 \sin x$
16. $\cos^2 x - 3 \cos x + 1 = 0$

Solve each trigonometric equation for $0 \le x < 2\pi$.

17. $2 \cos^2 x + \sin x = 1$
18. $\tan x + 2 \cos x = \sec x$
19. $\cos^2 x - \cos x \sec x = 0$
20. $\sin^2 x + \sin x \cos x = 0$
21. $2 \sin^2 x + 2 \cos 2x = 1$
22. $6 \sin^2 x + \cos 2x = 4$
23. $2 \sin^2 2x - \sin 2x - 1 = 0$
24. $\cos 2x - 2 \sin^2 2x + 1 = 0$
25. $4 \tan^2 x = 3 \sec^2 x$
26. $2 \sin x - \tan x = 0$
27. $4 \sin 2x \cos 2x = 1$
28. $\cos^2 3x - \sin^2 3x = 1$
29. $\cos x = \cos \frac{x}{2}$
30. $\sin x = \cos \frac{x}{2}$
31. $\cos \frac{x}{2} = 1 + \cos x$
32. $\sin \frac{x}{2} = 1 - \cos x$
33. $1 + \cos^2 \frac{x}{2} = 2 \cos x$
34. $1 + \sin^2 \frac{x}{2} = \cos x$

13.5 INVERSE TRIGONOMETRIC RELATIONS



In Section 4.1 we defined a relation as a set of ordered pairs of the form (x, y). In Section 9.2 we defined the inverse of a given equation to be the resulting equation when the variables x and y are interchanged. Recall that

$$\text{the inverse of} \quad y = b^x \quad \text{is} \quad x = b^y$$

Similar relationships are as follows:

The inverse of:	*Is:*
$y = x^2$	$x = y^2$
$y = 3x^2 + 4x + 7$	$x = 3y^2 + 4y + 7$
$y = \dfrac{x + 4}{3x - 1}$	$x = \dfrac{y + 4}{3y - 1}$
$y = \log_b x$	$x = \log_b y$

Likewise, each basic trigonometric equation has an inverse:

The inverse of:	*Is:*
$y = \sin x$	$x = \sin y$
$y = \cos x$	$x = \cos y$
$y = \tan x$	$x = \tan y$
$y = \cot x$	$x = \cot y$
$y = \sec x$	$x = \sec y$
$y = \csc x$	$x = \csc y$

Also, recall from the study of logarithms that we found it necessary to write the logarithmic equation solved for y as follows.

Exponential equation:	$y = b^x$
Inverse equation (logarithmic equation in exponential form):	$x = b^y$
Inverse equation (logarithmic equation in logarithmic form):	$y = \log_b x$

There are two common forms of the inverse trigonometric equations solved for y.

		Solved for y:	
The inverse of:	*Is:*	*Is:*	*Is:*
$y = \sin x$	$x = \sin y$	$y = \arcsin x$*	$y = \sin^{-1} x$†
$y = \cos x$	$x = \cos y$	$y = \arccos x$	$y = \cos^{-1} x$
$y = \tan x$	$x = \tan y$	$y = \arctan x$	$y = \tan^{-1} x$
$y = \cot x$	$x = \cot y$	$y = \operatorname{arccot} x$	$y = \cot^{-1} x$
$y = \sec x$	$x = \sec y$	$y = \operatorname{arcsec} x$	$y = \sec^{-1} x$
$y = \csc x$	$x = \csc y$	$y = \operatorname{arccsc} x$	$y = \csc^{-1} x$

EXAMPLE 1

What is the meaning of each equation?
(a) $y = \arctan x$
(b) $y = \arccos 3x$
(c) $y = 4 \operatorname{arccsc} 5x$

(a) y is the angle whose tangent is x.
(b) y is the angle whose cosine is $3x$.
(c) y is four times the angle whose cosecant is $5x$.

*This is read, "y equals the arcsine of x" and means that y is the angle whose sine is x.
†This notation will not be used here because of the confusion caused by the fact that -1 is not an exponent.

Remember, $x = \sin y$ and $y = \arcsin x$ express the same relationship. The first form expresses the relationship in terms of the function (sine) of the angle; the second form expresses the relationship in terms of the angle itself.

EXAMPLE 2

Given the equation $y = \arcsin x$, find y when $x = \frac{1}{2}$. (Give the answer in degrees.)

Substituting $x = \frac{1}{2}$, we have

$$y = \arcsin \frac{1}{2}$$

which means that y is the angle whose sine is $\frac{1}{2}$. We know that $\sin 30° = \frac{1}{2}$, so $y = 30°$. But we also know that

$$\sin 150° = \frac{1}{2} \quad \text{so} \quad y = 150°$$

$$\sin 390° = \frac{1}{2} \quad \text{so} \quad y = 390° \qquad (390° = 30° + 360°)$$

$$\sin 510° = \frac{1}{2} \quad \text{so} \quad y = 510° \qquad (510° = 150° + 360°)$$

$$\vdots \qquad\qquad \vdots$$

$$\sin(-210°) = \frac{1}{2} \quad \text{so} \quad y = -210°$$

$$\sin(-330°) = \frac{1}{2} \quad \text{so} \quad y = -330°$$

$$\vdots \qquad\qquad \vdots$$

Thus, there are infinitely many angles whose sine is $\frac{1}{2}$. We saw this when we graphed $y = \sin x$. That is,

$$y = 30° + n \cdot 360°$$
$$y = 150° + n \cdot 360°$$

for every integer n.

EXAMPLE 3

Given the equation $y = \arctan x$, find y when $x = -1$ for $0 \le y < 2\pi$.

Substituting $x = -1$, we have

$$y = \arctan(-1)$$

which means that y is the angle whose tangent is -1. We know that

$$\tan \frac{3\pi}{4} = -1 \quad \text{so} \quad y = \frac{3\pi}{4}$$

and

$$\tan \frac{7\pi}{4} = -1 \quad \text{so} \quad y = \frac{7\pi}{4}$$

for $0 \le y < 2\pi$.

EXAMPLE 4

Find $\arccos \frac{\sqrt{3}}{2}$ for the smallest positive angle in degrees.

Let $y = \arccos \frac{\sqrt{3}}{2}$, which means that y is the (smallest positive) angle whose cosine is $\frac{\sqrt{3}}{2}$. We know that

$$\cos 30° = \frac{\sqrt{3}}{2}$$

Thus, $\arccos \frac{\sqrt{3}}{2} = 30°$ (for the smallest positive angle).

EXAMPLE 5

Find $\operatorname{arcsec}(-2)$ for the smallest positive angle in radians.

Let $y = \operatorname{arcsec}(-2)$, which means that y is the (smallest positive) angle whose secant is -2. We know that

$$\sec 120° = -2$$

Therefore, $\operatorname{arcsec}(-2) = \frac{2\pi}{3}$ (for the smallest positive angle).

EXAMPLE 6

Find $\operatorname{arccot} \frac{1}{\sqrt{3}}$ for all angles in radians.

Let $y = \operatorname{arccot} \frac{1}{\sqrt{3}}$, which means that y is the angle whose cotangent is $\frac{1}{\sqrt{3}}$. We know that

$$\cot \frac{\pi}{3} = \frac{1}{\sqrt{3}} \quad \text{and} \quad \cot \frac{4\pi}{3} = \frac{1}{\sqrt{3}}$$

for $0 \le y < 2\pi$. Thus,

$$\operatorname{arccot} \frac{1}{\sqrt{3}} = \begin{cases} \frac{\pi}{3} + 2n\pi \\ \\ \frac{4\pi}{3} + 2n\pi \end{cases}$$

for every integer n.

EXAMPLE 7

Solve the equation $y = \cos 2x$ for x.

The equation $y = \cos 2x$ is equivalent to

$$\arccos y = 2x$$

So

$$x = \frac{1}{2} \arccos y$$

EXAMPLE 8

Solve the equation $2y = \arctan 3x$ for x.

The equation $2y = \arctan 3x$ is equivalent to

$$\tan 2y = 3x$$

So

$$x = \frac{1}{3} \tan 2y$$

EXAMPLE 9

Solve the equation $y = \frac{1}{3} \operatorname{arcsec} 2x$ for x.

First, multiply both sides by 3.

$$3y = \operatorname{arcsec} 2x$$

The equation $3y = \operatorname{arcsec} 2x$ is equivalent to

$$\sec 3y = 2x$$

Thus,

$$x = \frac{1}{2} \sec 3y$$

Exercises 13.5

Write the meaning of each equation.

1. $y = \arcsin x$ **2.** $y = \operatorname{arcsec} x$ **3.** $y = \operatorname{arccot} 4x$

4. $y = \arccos 2x$ **5.** $y = 3 \operatorname{arccsc} \frac{1}{2}x$ **6.** $y = \frac{1}{2} \arctan 3x$

Solve each equation for $0 \leq y < 2\pi$.

7. $y = \arcsin\left(\frac{\sqrt{3}}{2}\right)$ **8.** $y = \arccos\left(\frac{1}{2}\right)$ **9.** $y = \arctan 1$

10. $y = \operatorname{arccot}(-1)$ **11.** $y = \operatorname{arcsec}\left(-\frac{2}{\sqrt{3}}\right)$ **12.** $y = \operatorname{arccsc} \sqrt{2}$

13. $y = \arccos\left(-\frac{1}{2}\right)$ **14.** $y = \arcsin 0$

Solve each equation for $0° \le y < 360°$.

15. $y = \arccos\left(-\frac{1}{\sqrt{2}}\right)$ **16.** $y = \arctan(-\sqrt{3})$ **17.** $y = \arcsin\left(-\frac{1}{\sqrt{2}}\right)$

18. $y = \operatorname{arccot}\sqrt{3}$ **19.** $y = \operatorname{arcsec} 1$ **20.** $y = \arcsin(-1)$

21. $y = \arctan(1.963)$ **22.** $y = \arccos(-0.9063)$

Find the smallest positive angle in degrees for each expression.

23. $\arcsin\left(\frac{1}{2}\right)$ **24.** $\arccos 0$ **25.** $\operatorname{arccot}\left(-\frac{1}{\sqrt{3}}\right)$

26. $\arctan\sqrt{3}$ **27.** $\operatorname{arcsec}\sqrt{2}$ **28.** $\operatorname{arccsc}(-\sqrt{2})$

29. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ **30.** $\arcsin\left(\frac{1}{\sqrt{2}}\right)$

Find the smallest positive angle in radians for each expression.

31. $\arctan(-\sqrt{3})$ **32.** $\operatorname{arccsc} 2$ **33.** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

34. $\operatorname{arcsec}\left(\frac{2}{\sqrt{3}}\right)$ **35.** $\arcsin(-1)$ **36.** $\arctan 0$

37. $\operatorname{arcsec}(-1)$ **38.** $\arcsin\left(\frac{1}{\sqrt{2}}\right)$

Find all angles in degrees for each expression.

39. $\arcsin\left(\frac{1}{\sqrt{2}}\right)$ **40.** $\arctan\left(\frac{1}{\sqrt{3}}\right)$ **41.** $\operatorname{arccot}\sqrt{3}$

42. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ **43.** $\operatorname{arcsec}\left(\frac{2}{\sqrt{3}}\right)$ **44.** $\arctan(-\sqrt{3})$

45. $\arcsin(0.7431)$ **46.** $\arccos(-0.9397)$

Find all angles in radians for each expression.

47. $\arccos\left(\frac{1}{2}\right)$ **48.** $\arctan 1$ **49.** $\arccos 1$

50. $\arcsin 1$ **51.** $\operatorname{arcsec} 2$ **52.** $\arccos\left(-\frac{1}{\sqrt{2}}\right)$

53. $\arcsin(-0.8572)$ **54.** $\operatorname{arccot}(-0.8195)$

Solve each equation for x.

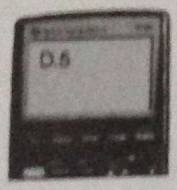
55. $y = \sin 3x$ **56.** $y = \tan 4x$ **57.** $y = 4\cos x$

58. $y = 3\sec x$ **59.** $y = 5\tan\frac{x}{2}$ **60.** $y = \frac{1}{2}\cos 3x$

61. $y = \frac{3}{2}\cot\frac{x}{4}$ **62.** $y = \frac{5}{2}\sin\frac{2x}{3}$ **63.** $y = 3\sin(x - 1)$

64. $y = 4\tan(2x + 1)$ **65.** $y = \frac{1}{2}\cos(3x + 1)$ **66.** $y = \frac{1}{3}\sec(1 - 4x)$

13.6 INVERSE TRIGONOMETRIC FUNCTIONS



To graph the inverse trigonometric relation

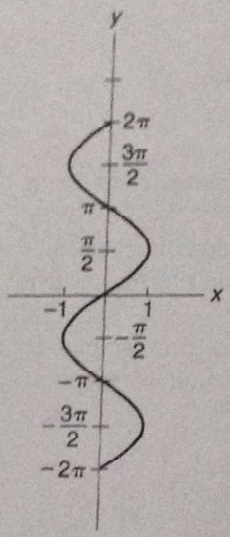
$$y = \arcsin x$$

first solve for x; that is,

$$x = \sin y$$

Then plot ordered pairs of solutions of this equation as in Fig. 13.6. In a similar manner, we can graph all six inverse trigonometric relations. They are shown in Fig. 13.7.

As you can see from the graphs, each x-value in each domain corresponds to (infinitely) many values of y. Thus, none of the inverse trigonometric relations is a function.



$y = \arcsin x$

Figure 13.6

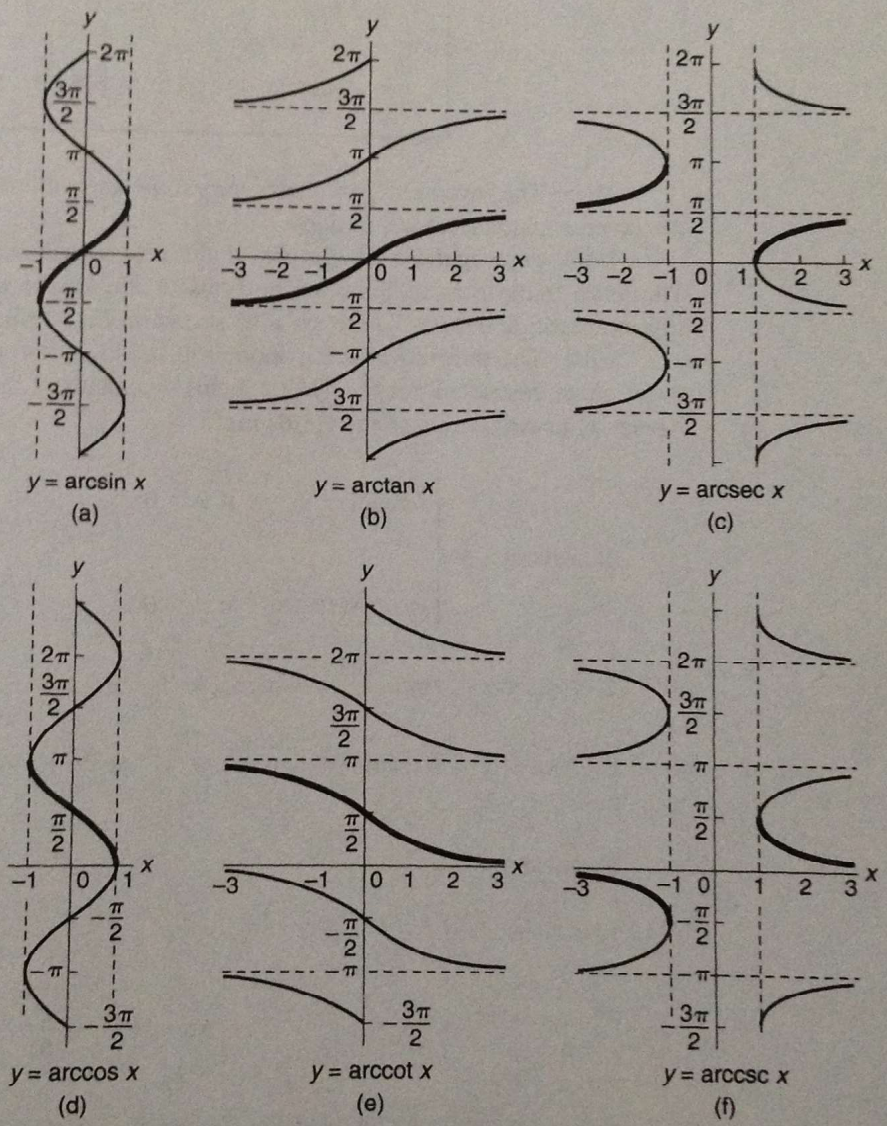


Figure 13.7

However, if we restrict the y-values of each inverse trigonometric relation, we can define an inverse that is also a function. While this could be done in any of several ways, it is customary to restrict the y-values as follows.

INVERSE TRIGONOMETRIC FUNCTIONS

$y = \text{Arcsin } x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \text{Arccos } x$	$0 \le y \le \pi$
$y = \text{Arctan } x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{Arccot } x$	$0 < y < \pi$
$y = \text{Arcsec } x$	$0 \le y \le \pi \quad y \ne \frac{\pi}{2}$
$y = \text{Arccsc } x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad y \ne 0$

Note: The inverse trigonometric *functions* are *capitalized* to distinguish them from the inverse trigonometric relations.

Look once again at the graphs of the inverse trigonometric relations in Fig. 13.7. The extra thick lines indicate the portions of the graphs that correspond to the inverse trigonometric *functions*. These are also shown in Fig. 13.8.

Note: The three inverse trigonometric functions on calculators are programmed to these same restricted ranges. When using a calculator to find the value of Arccot x, Arcsec x, or Arccsc x, use the following:

1. $$\text{Arccot } x = \begin{cases} \text{Arctan } \dfrac{1}{x} & \text{if } x > 0 \\ \pi + \text{Arctan } \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

2. $\text{Arcsec } x = \text{Arccos } \dfrac{1}{x}$ where $x \ge 1$ or $x \le -1$

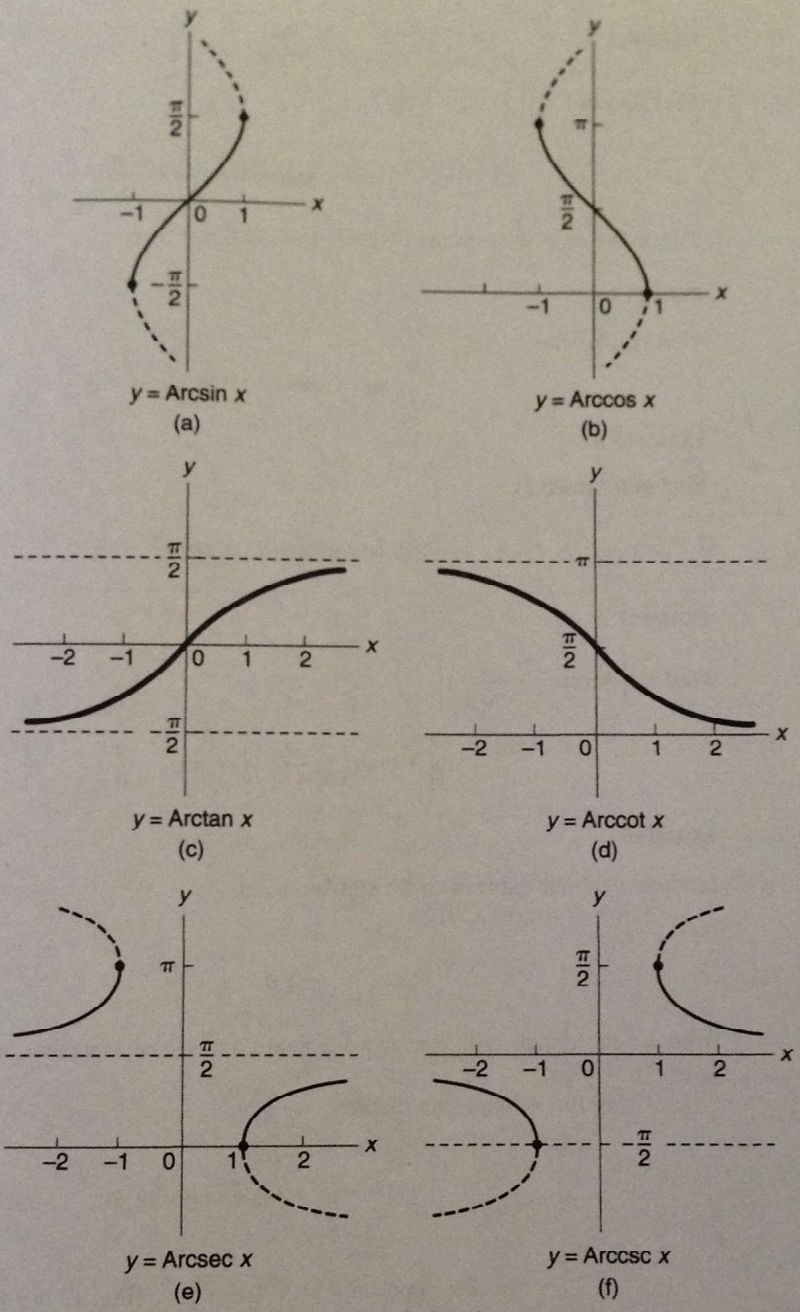
3. $\text{Arccsc } x = \text{Arcsin } \dfrac{1}{x}$ where $x \ge 1$ or $x \le -1$

EXAMPLE 1

Find $\text{Arcsin}\left(\frac{1}{2}\right)$.

$$\text{Arcsin}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

This is the only value in the defined range of $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.



Inverse trigonometric functions

Figure 13.8

EXAMPLE 2

Find Arctan(-1).

$$\text{Arctan}(-1) = -\frac{\pi}{4}$$

This is the only value in the defined range of $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

EXAMPLE 3

Find $\text{Arccos}\left(-\frac{1}{2}\right)$.

$$\text{Arccos}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

This is the only value in the defined range of $0 \le y \le \pi$.

EXAMPLE 4

Find $\tan[\text{Arccos}(-1)]$.

$$\tan[\text{Arccos}(-1)] = \tan \pi = 0$$

EXAMPLE 5

Find $\cos(\text{Arcsec } 2)$.

$$\cos(\text{Arcsec } 2) = \cos \frac{\pi}{3} = \frac{1}{2}$$

EXAMPLE 6

Find $\sin\left[\text{Arctan}\left(-\frac{1}{\sqrt{3}}\right)\right]$.

$$\sin\left[\text{Arctan}\left(-\frac{1}{\sqrt{3}}\right)\right] = \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

EXAMPLE 7

Find an algebraic expression for $\sin(\text{Arccos } x)$.

Let $\theta = \text{Arccos } x$. Then

$$\cos \theta = x = \frac{x}{1}$$

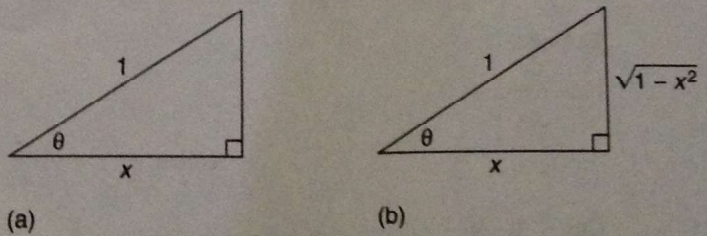
Draw a right triangle with θ as an acute angle, x as the adjacent side, and 1 as the hypotenuse as in Fig. 13.9(a).

Using the Pythagorean theorem, we have

$$c^2 = a^2 + b^2$$
$$1^2 = x^2 + (\text{side opposite } \theta)^2$$

and

$$\text{side opposite } \theta = \sqrt{1 - x^2} \qquad \text{[Fig. 13.9(b)]}$$



(a) (b)

Figure 13.9

Now we see that

$$\begin{aligned}\sin(\text{Arccos } x) &= \sin\theta \\ &= \frac{\text{side opposite } \theta}{\text{hypotenuse}} \\ &= \frac{\sqrt{1-x^2}}{1} \\ &= \sqrt{1-x^2}\end{aligned}$$

EXAMPLE 8

Find an algebraic expression for sec(Arctan x).

Let $\theta = \text{Arctan } x$. Then

$$\tan\theta = x = \frac{x}{1}$$

Draw a right triangle with θ as an acute angle, x as the opposite side, and 1 as the adjacent side as in Fig. 13.10. Using the Pythagorean theorem, we find that the hypotenuse is $\sqrt{x^2+1}$.

$$\begin{aligned}\sec(\text{Arctan } x) &= \sec\theta \\ &= \frac{\text{hypotenuse}}{\text{side adjacent to } \theta} \\ &= \frac{\sqrt{x^2+1}}{1} \\ &= \sqrt{x^2+1}\end{aligned}$$

EXAMPLE 9

Find an algebraic expression for cos(2 Arcsin x).

Let $\theta = \text{Arcsin } x$. Then

$$\sin\theta = \frac{x}{1}$$

Draw a right triangle with θ as an acute angle, x as the opposite side, and 1 as the hypotenuse as in Fig. 13.11. Using the Pythagorean theorem, we find that the side adjacent to θ is $\sqrt{1-x^2}$.

$$\begin{aligned}\cos(2\ \text{Arcsin } x) &= \cos 2\theta \\ &= 1 - 2\sin^2\theta \qquad \text{(Formula 25c)} \\ &= 1 - 2x^2\end{aligned}$$

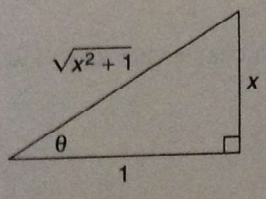


Figure 13.10

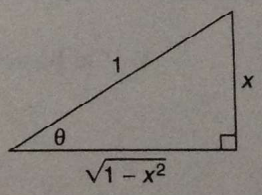


Figure 13.11

Exercises 13.6

Find the value of each expression in radians.

1. $\text{Arcsin}\left(\frac{\sqrt{3}}{2}\right)$
2. $\text{Arccos}\left(\frac{1}{2}\right)$
3. $\text{Arctan}\left(-\frac{1}{\sqrt{3}}\right)$
4. $\text{Arcsin}\left(-\frac{1}{2}\right)$
5. $\text{Arccos}\left(-\frac{\sqrt{3}}{2}\right)$
6. $\text{Arctan}\left(\frac{1}{\sqrt{3}}\right)$
7. $\text{Arcsec}(-2)$
8. $\text{Arccot}(-1)$
9. $\text{Arccsc}\ \sqrt{2}$
10. $\text{Arcsin}(-1)$
11. $\text{Arctan}\ \sqrt{3}$
12. $\text{Arccos}\ 0$
13. $\text{Arccos}\left(\frac{1}{\sqrt{2}}\right)$
14. $\text{Arcsin}\ 1$
15. $\text{Arcsin}\left(-\frac{\sqrt{3}}{2}\right)$
16. $\text{Arctan}(-\sqrt{3})$
17. $\text{Arccot}\ 0.35$
18. $\text{Arcsec}\ 1.357$
19. $\text{Arccot}(-1.5)$
20. $\text{Arccsc}\ 2.5$
21. $\text{Arcsec}(-3.2)$
22. $\text{Arcsin}\ 0.75$
23. $\text{Arccsc}(-1.15)$
24. $\text{Arccos}(-0.55)$

Find the value of each expression.

25. $\cos(\text{Arctan}\ \sqrt{3})$
26. $\tan\left[\text{Arcsin}\left(\frac{1}{\sqrt{2}}\right)\right]$
27. $\sin\left[\text{Arccos}\left(-\frac{1}{\sqrt{2}}\right)\right]$
28. $\sin[\text{Arctan}(-1)]$
29. $\tan[\text{Arccos}(-1)]$
30. $\sec\left[\text{Arccos}\left(-\frac{1}{2}\right)\right]$
31. $\sin\left[\text{Arcsin}\left(\frac{\sqrt{3}}{2}\right)\right]$
32. $\tan[\text{Arctan}(-\sqrt{3})]$
33. $\cos\left[\text{Arcsin}\left(\frac{3}{5}\right)\right]$
34. $\tan\left[\text{Arcsin}\left(\frac{12}{13}\right)\right]$
35. $\tan[\text{Arcsin}(-0.1560)]$
36. $\sin[\text{Arccot}(1.635)]$

Find an algebraic expression for each.

37. $\cos(\text{Arcsin}\ x)$
38. $\tan(\text{Arccos}\ x)$
39. $\sin(\text{Arcsec}\ x)$
40. $\cot(\text{Arcsec}\ x)$
41. $\sec(\text{Arccos}\ x)$
42. $\sin(\text{Arctan}\ x)$
43. $\tan(\text{Arctan}\ x)$
44. $\sin(\text{Arcsin}\ x)$
45. $\cos(\text{Arcsin}\ 2x)$
46. $\tan(\text{Arccos}\ 3x)$
47. $\sin(2\ \text{Arcsin}\ x)$
48. $\cos(2\ \text{Arctan}\ x)$

Graph the following.

49. $y = \text{Arcsin}\ 2x$
50. $y = 3\ \text{Arccos}\ x$
51. $y = 2\ \text{Arctan}\ 3x$

CHAPTER SUMMARY

1. The **basic identities** developed in this chapter are as follows:

$$\sin\theta = \frac{1}{\csc\theta} \qquad (1)$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad (2)$$

$$\tan\theta = \frac{1}{\cot\theta} \tag{3}$$

$$\cot\theta = \frac{1}{\tan\theta} \tag{4}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{5}$$

$$\csc\theta = \frac{1}{\sin\theta} \tag{6}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{7}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{8}$$

$$\sin^2\theta + \cos^2\theta = 1 \tag{9}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{10}$$

$$\cot^2\theta + 1 = \csc^2\theta \tag{11}$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{12}$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{13}$$

$$\sin(-\theta) = -\sin\theta \tag{14}$$

$$\cos(-\theta) = \cos\theta \tag{15}$$

$$\tan(-\theta) = -\tan\theta \tag{16}$$

$$\cot(-\theta) = -\cot\theta \tag{17}$$

$$\sec(-\theta) = \sec\theta \tag{18}$$

$$\csc(-\theta) = -\csc\theta \tag{19}$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi \tag{20}$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \tag{21}$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \tag{22}$$

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} \tag{23}$$

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{24}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{25a}$$

$$= 2\cos^2\theta - 1 \tag{25b}$$

$$= 1 - 2\sin^2\theta \tag{25c}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{26}$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}} \tag{27}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}} \tag{28}$$

$$\tan\frac{\theta}{2} = \frac{1 - \cos\theta}{\sin\theta} \tag{29}$$

2. The general methods for proving a trigonometric identity are as follows:
   (a) Substitute one expression for another using a known identity.
   (b) Factor one side of the identity.
   (c) Multiply numerator and denominator of a fractional expression by some trigonometric expression.
   (d) Express all functions on one side in terms of sines and cosines, and use one of the preceding methods.
3. The **inverse trigonometric functions** are defined as follows:

$$y = \text{Arcsin } x \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$y = \text{Arccos } x \qquad 0 \le y \le \pi$$

$$y = \text{Arctan } x \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$y = \text{Arccot } x \qquad 0 < y < \pi$$

$$y = \text{Arcsec } x \qquad 0 \le y \le \pi \quad y \ne \frac{\pi}{2}$$

$$y = \text{Arccsc } x \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2} \quad y \ne 0$$

See Fig. 13.8 for the graphs of the inverse trigonometric functions.

CHAPTER 13 REVIEW

Prove each identity.

1. $\sec x \cot x = \csc x$
2. $\sec^2 \theta + \tan^2 \theta + 1 = \dfrac{2}{\cos^2 \theta}$
3. $\dfrac{\cos \theta}{\cos \theta + \sin \theta} = \dfrac{\cot \theta}{1 + \cot \theta}$
4. $\cos\left(\theta - \dfrac{3\pi}{2}\right) = -\sin \theta$
5. $\left(\sin \dfrac{1}{2}x + \cos \dfrac{1}{2}x\right)^2 = 1 + \sin x$
6. $2 \cos^2 \dfrac{\theta}{2} = \dfrac{1 + \sec \theta}{\sec \theta}$
7. $\dfrac{2 \cot \theta}{1 + \cot^2 \theta} = \sin 2\theta$
8. $\csc x - \cot x = \tan \dfrac{1}{2}x$
9. $\tan 2x = \dfrac{2 \cos x}{\csc x - 2 \sin x}$
10. $\tan^2 \dfrac{x}{2} + 1 = 2 \tan \dfrac{x}{2} \csc x$

Simplify each expression.

11. $\sin \theta \cos \theta$
12. $\cos^2 3\theta - \sin^2 3\theta$
13. $\dfrac{1 + \cos 4\theta}{2}$
14. $1 - 2 \sin^2 \dfrac{\theta}{3}$
15. $\cos 2x \cos 3x - \sin 2x \sin 3x$
16. $\sin 2x \cos x - \cos 2x \sin x$
17. Find $\sin 2\theta$ when $\cos \theta = -\dfrac{5}{13}$ (θ in the second quadrant).
18. Find $\cos \dfrac{\theta}{2}$ when $\tan \theta = \dfrac{4}{3}$ (θ in the third quadrant).

Solve each trigonometric equation for $0 \le x < 2\pi$.

19. $2\cos^2 x = \cos x$

20. $4\sin^2 x - 1 = 0$

21. $2\cos^2 3x + \sin 3x - 1 = 0$

22. $\tan^2 x = \sin^2 x$

23. $\sin\frac{x}{2} + \cos\frac{x}{2} = 0$ (*Hint:* Square each side.)

24. $\sin 2x = \cos^2 x - \sin^2 x$

Solve each equation for $0 \le y < 2\pi$.

25. $y = \arcsin\left(\frac{1}{2}\right)$

26. $y = \arctan\left(-\frac{1}{\sqrt{3}}\right)$

Solve each equation for $0° \le y < 360°$.

27. $y = \arccos\left(\frac{1}{\sqrt{2}}\right)$

28. $y = \operatorname{arcsec}(-2)$

29. Find the smallest positive angle in degrees for $\arcsin\left(-\frac{1}{2}\right)$.

30. Find the smallest positive angle in radians for $\arctan(-\sqrt{3})$.

31. Find all angles in degrees for $\operatorname{arccot}(-\sqrt{3})$.

32. Find all angles in radians for $\arcsin(-1)$.

33. Solve for x: $y = \frac{1}{2}\sin\frac{3x}{4}$.

Find the value of each expression in radians.

34. $\operatorname{Arcsin}\left(\frac{1}{\sqrt{2}}\right)$

35. $\operatorname{Arctan}\left(-\frac{1}{\sqrt{3}}\right)$

36. $\operatorname{Arcsec}(-1)$

37. $\operatorname{Arccos}\left(-\frac{1}{2}\right)$

Find the value of each expression.

38. $\sin\left[\operatorname{Arccos}\left(-\frac{1}{2}\right)\right]$

39. $\tan(\operatorname{Arctan}\sqrt{3})$

40. Find an algebraic expression for $\sin(\operatorname{Arccot} x)$.

41. Graph $y = 1.5\operatorname{Arccos} 2x$

20.1 SEQUENCES AND SERIES

Suppose an object starts from rest and travels in a straight-line path with a constant acceleration of $4\ \text{m/s}^2$. It is shown in physics that the distance s (in meters) of the object from its starting point at time t (in seconds) is given by the function $s = f(t) = 2t^2$. Let us restrict t to positive integral values; that is, we take the domain of f to be the positive integers. We find that if $t = 1$, then $s = 2$; if $t = 2$, then $s = 8$; and so on. Thus we have the following correspondence:

Domain	1	2	3	4	$\cdots$	n	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$	
Range	2	8	18	32	$\cdots$	$2n^2$	$\cdots$.

In mathematics we use the term **sequence** to describe a function, such as the preceding one, whose domain consists of the positive integers. Thus the function

$$f(n) = 2n^2,$$

where n is any positive integer, is a sequence. With sequences, instead of using the function-value notation $f(n)$, we commonly use the letter a along with subscript notation, so we write $f(n)$ as a_n. That is, when the input is n, the output is a_n. For

example, $f(1)$ becomes a_1 and $f(2)$ becomes a_2. If $a_n = 2n^2$, we have $a_1 = 2(1)^2 = 2$, $a_2 = 2(2)^2 = 8$, $a_3 = 2(3)^2 = 18$, and so on.

When we list the function values of a sequence in the order

$$a_1, a_2, a_3, \ldots, a_n, \ldots,$$

this arrangement also defines the sequence, and the values are called **terms** of the sequence. The *first term* is a_1, the *second term* is a_2, and so on. The ***n*th term**, or **general term**, is a_n. When specific values for $a_1, a_2, \ldots$ are listed, a_n is usually indicated by a rule for the sequence. For example, if the general term is $a_n = 2n^2$, then the sequence is

$$2, 8, 18, 32, \ldots, 2n^2, \ldots. \tag{1}$$

A sequence with general term a_n is often denoted by the symbol $\{a_n\}$. For example, the sequence in (1) can be denoted $\{2n^2\}$.

We also use the term *sequence* when the domain is a *finite* set of consecutive positive integers. For example, if $a_n = n^2$ and $n = 3, 4, 5$, and 6, then the sequence is

$$9, 16, 25, 36. \tag{2}$$

The sequence in (1) is called an **infinite sequence**, and the sequence in (2) is called a **finite sequence** because it has a first term *and* a last term. Unless otherwise specified, we shall assume that the sequences we deal with have domains consisting of the positive integers.

Example 1

Find the first four terms of the sequence with the given general term.

a. $a_n = \dfrac{2n}{3n+1}$.

Solution In the expression $\dfrac{2n}{3n+1}$, we successively replace n by the integers 1, 2 3, and 4.

$$n = 1, \quad a_1 = \frac{2(1)}{3(1)+1} = \frac{2}{4} = \frac{1}{2}.$$

$$n = 2, \quad a_2 = \frac{2(2)}{3(2)+1} = \frac{4}{7}.$$

$$n = 3, \quad a_3 = \frac{2(3)}{3(3)+1} = \frac{6}{10} = \frac{3}{5}.$$

$$n = 4, \quad a_4 = \frac{2(4)}{3(4)+1} = \frac{8}{13}.$$

Thus the first four terms are $\frac{1}{2}, \frac{4}{7}, \frac{3}{5}$, and $\frac{8}{13}$.

b. $a_n = (-1)^{n+1}2^n$.

Solution

$$n = 1, \quad a_1 = (-1)^{1+1}2^1 = (1)(2) = 2.$$
$$n = 2, \quad a_2 = (-1)^{2+1}2^2 = (-1)(4) = -4.$$
$$n = 3, \quad a_3 = (-1)^{3+1}2^3 = (1)(8) = 8.$$
$$n = 4, \quad a_4 = (-1)^{4+1}2^4 = (-1)(16) = -16.$$

The first four terms are 2, -4, 8, and -16. Note that these terms alternate in sign because of the factor $(-1)^{n+1}$.

Example 2

Find a general term for the given sequence.

a. 2, 4, 6, 8, ...

Solution Here $a_1 = 2$, $a_2 = 4$, $a_3 = 6$, and $a_4 = 8$. Looking for a pattern, we observe that $a_1 = 2(1)$, $a_2 = 2(2)$, $a_3 = 2(3)$, and $a_4 = 2(4)$. That is, the value of each term is twice the number of that term. Thus a general term is given by $a_n = 2n$.

b. $x, x^2, x^3, x^4, \ldots$.

Solution By inspecting the terms, we see that a general term is $a_n = x^n$.

If we have a sequence, then an indicated sum of its terms is called a **series**. For example, given the (finite) sequence

$$2, 3, 4,$$

the corresponding series is

$$2 + 3 + 4.$$

Similarly, the series that corresponds to

$$x, x^2, x^3, \ldots, x^n$$

is

$$x + x^2 + x^3 + \cdots + x^n.$$

To represent such sums we can use **sigma** (or **summation**) **notation**, so named because the Greek letter Σ (sigma) is used. For example, the symbol

$$\sum_{k=1}^{3} (2k + 5)$$

denotes the sum of those numbers obtained from the expression $2k + 5$ by first replacing k by 1, then by 2, and finally by 3. That is, k takes on consecutive integer values from 1 to 3. Thus

$$\sum_{k=1}^{3} (2k + 5) = [2(1) + 5] + [2(2) + 5] + [2(3) + 5]$$

$$= 7 + 9 + 11 = 27.$$

The expression $\sum_{k=1}^{3} (2k + 5)$ may be read *the sum of* $2k + 5$ *from* $k = 1$ *to* $k = 3$. The letter k is called the **index of summation**; the numbers 1 and 3 are called the **limits of summation** (1 is the *lower limit* and 3 is the *upper limit*). The letter used for the index is a "dummy" symbol in the sense that it does not affect the sum of the terms. Any other letter may be used. For example,

$$\sum_{j=1}^{3} (2j + 5) = 7 + 9 + 11 = \sum_{k=1}^{3} (2k + 5).$$

With sigma notation, the series $2 + 3 + 4$ may be written

$$\sum_{n=2}^{4} n.$$

We say that the **expanded form** of $\sum_{n=2}^{4} n$ is $2 + 3 + 4$. The series $x + x^2 + \cdots + x^n$ may be written

$$\sum_{k=1}^{n} x^k.$$

Example 3

Evaluate each of the following.

a. $\sum_{k=4}^{7} \frac{k^2 + 3}{2}$.

Solution Here the sum begins with $k = 4$.

$$\sum_{k=4}^{7} \frac{k^2 + 3}{2} = \frac{4^2 + 3}{2} + \frac{5^2 + 3}{2} + \frac{6^2 + 3}{2} + \frac{7^2 + 3}{2}$$

$$= \frac{19}{2} + \frac{28}{2} + \frac{39}{2} + \frac{52}{2} = \frac{138}{2} = 69.$$

b. $\sum_{n=0}^{2}(-1)^n(2^n-1)$.

Solution Here the sum begins with $n = 0$.

$$\sum_{n=0}^{2}(-1)^n(2^n-1) = (-1)^0(2^0-1) + (-1)^1(2^1-1) + (-1)^2(2^2-1)$$
$$= (1)(0) + (-1)(1) + (1)(3)$$
$$= 0 - 1 + 3 = 2.$$

c. $\sum_{n=1}^{4} 2$.

Solution The 2 can be thought of as $2 + 0n$. Thus

$$\sum_{n=1}^{4} 2 = 2 + 2 + 2 + 2 = 8.$$

Problem Set 20.1

In Problems **1–14**, *find the first four terms of the sequence that has the given general term.*

1. $a_n = 3n$.
2. $a_n = \frac{1}{2}n$.
3. $a_n = n^2 + n$.
4. $a_n = 2n^2 + 4$.
5. $a_n = \dfrac{n^2-1}{n^2+2}$.
6. $a_n = \dfrac{n}{2^n}$.
7. $a_n = x^{2n}$.
8. $a_n = (2n)^2$.
9. $a_n = (-1)^{n+1}n^2$.
10. $a_n = (-1)^n(2n)^2$.
11. $a_n = \sin\dfrac{n\pi}{2}$.
12. $a_n = \cos\left(\dfrac{n\pi}{2}\right)$.
13. $a_n = \dfrac{e^n}{2}$.
14. $a_n = \dfrac{\sin nx}{n^2}$.

In Problems **15–22**, *find a general term for the given sequence.*

15. 1, 2, 3, 4, ...
16. 0, −1, −2, −3, ...
17. 3, 6, 9, 12, ...
18. 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, ...
19. 1, 3, 9, 27, ...
20. $x^2, x^4, x^6, x^8, \ldots$.
21. $\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$.
22. $-\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$.

In Problems **23–32**, *evaluate the given sum.*

23. $\sum_{k=1}^{5}(k+4)$.
24. $\sum_{k=4}^{7}(5-2k)$.

25. $\sum_{n=0}^{3}(3n^2-7)$.

26. $\sum_{n=2}^{4}\frac{n+1}{n-1}$.

27. $\sum_{n=1}^{3}3$.

28. $\sum_{n=1}^{5}1$.

29. $\sum_{k=3}^{4}\frac{(-1)^k(k+1)}{2^k}$.

30. $\sum_{n=1}^{4}(n^2+n)$.

31. $\sum_{n=1}^{3}2\left(\frac{1}{2}\right)^n$.

32. $\sum_{n=0}^{3}(-1)^n\frac{1}{n+1}$.

In Problems **33** *and* **34**, *write the series in expanded form.*

33. $\sum_{k=1}^{4}\sin kx$.

34. $\sum_{k=1}^{4}\frac{\cos kx}{k}$.

20.2 ARITHMETIC PROGRESSIONS

If you look for the pattern of the terms in the sequence

$$1, 3, 5, 7,$$

you may notice that by adding 2 to any term, you get the next term. This means that the difference between any term and the term preceding it is 2:

$$3-1=2, \qquad 5-3=2, \qquad 7-5=2.$$

We call the sequence 1, 3, 5, 7 an *arithmetic progression* (or *arithmetic sequence*) with *common difference* 2. Here is a more general definition.

> A sequence in which each term after the first can be obtained by adding the constant d to the preceding term is called an **arithmetic progression** with **common difference** d.

Example 1

Suppose the first term of an arithmetic progression is 6 and the common difference d is 5. Then

$$a_1 = 6,$$
$$a_2 = 6+5 = 11,$$
$$a_3 = 11+5 = 16,$$
$$a_4 = 16+5 = 21, \qquad \text{and so on.}$$

Thus the arithmetic progression is

$$6, 11, 16, 21, \ldots.$$

Example 2

a. In the arithmetic progression

$$\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \ldots$$

we can find d by choosing *any* two consecutive terms and subtracting the first one from the second one. For example, subtracting the first term from the second term gives

$$d = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

b. The sequence

$$2, -1, -4, -7, \ldots$$

is an arithmetic progression, because if we add -3 to *any* term, we get the next term:

$$2 + (-3) = -1, \qquad -1 + (-3) = -4, \qquad \text{and so on.}$$

We can find a formula for the nth term of an arithmetic progression with first term a_1 and common difference d. In this progression we have

$$a_2 = a_1 + d$$
$$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$$
$$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d.$$

Remaining terms can be obtained in a similar manner. Thus the arithmetic progression is

$$a_1, \quad a_1 + d, \quad a_1 + 2d, \quad a_1 + 3d, \ldots.$$

Each term is the sum of a_1 and a number that is a multiple of d. That number is d times one *less* than the number of the term. For example, the tenth term is

$$a_{10} = a_1 + (10 - 1)d = a_1 + 9d.$$

More generally, we have the following formula for the nth term of an arithmetic progression.

> The nth term, a_n, of an arithmetic progression with first term a_1 and common difference d is given by
>
> $$\boldsymbol{a_n = a_1 + (n - 1)d.} \qquad (1)$$

In Example 2, the nth term of the arithmetic progression in Part a is $a_n = \frac{1}{3} + (n - 1)\frac{1}{3} = \frac{1}{3} + \frac{1}{3}(n - 1)$; in Part b it is $a_n = 2 + (n - 1)(-3) = 2 - 3(n - 1)$.

Example 3

Find the eighteenth term of the sequence

$$7, 13, 19, \ldots.$$

Solution This sequence is an arithmetic progression with $a_1 = 7$ and $d = 6$. To find a_{18}, we use Eq. 1 with $n = 18$.

$$a_n = a_1 + (n - 1)d$$
$$a_{18} = 7 + (18 - 1)6 = 7 + (17)6 = 109.$$

Example 4

The third term of an arithmetic progression is 2 and the tenth term is -26. Find a_1 and d.

Solution We have $a_3 = 2$ and $a_{10} = -26$. Using Eq. 1 with $n = 3$ and $a_3 = 2$ gives

$$2 = a_1 + (3 - 1)d$$
$$2 = a_1 + 2d. \tag{2}$$

Using Eq. 1 with $n = 10$ and $a_{10} = -26$ gives

$$-26 = a_1 + (10 - 1)d$$
$$-26 = a_1 + 9d. \tag{3}$$

Equations 2 and 3 form a system of equations.

$$\begin{cases} 2 = a_1 + 2d & (4) \\ -26 = a_1 + 9d. & (5) \end{cases}$$

Multiplying Eq. 4 by -1 and adding it to Eq. 5, we have

$$-28 = 7d$$
$$-4 = d.$$

Substituting -4 for d in Eq. 4 gives

$$2 = a_1 + 2(-4)$$
$$2 = a_1 - 8$$
$$10 = a_1.$$

Thus $a_1 = 10$ and $d = -4$.

Example 5

A particle has an initial speed of 5 m/s and travels in a straight line with an acceleration of 2 m/s^2. The values of the speed of the particle, in meters per second, at positive integral values

of time t (in seconds) form the arithmetic progression

$$7, 9, 11, 13, \ldots, 5 + 2t, \ldots.$$

Find the speed of the particle when $t = 100$ s.

Solution Here the first term $a_1 = 7$ corresponds to the time $t = 1$, the second term $a_2 = 9$ corresponds to $t = 2$, and so on, and $d = 2$. We need to find a_{100}, which corresponds to $t = 100$. In Eq. 1 we set $a_1 = 7$, $d = 2$, and $n = 100$.

$$a_n = a_1 + (n - 1)d$$

$$a_{100} = 7 + (100 - 1)(2) = 205.$$

Thus the speed is 205 m/s.

We can find a formula for the sum S_n of the first n terms of an arithmetic progression. Suppose that $a_1, a_2, \ldots$ is an arithmetic progression. Then $a_2 = a_1 + d$, $a_3 = a_1 + 2d$, and so on. Thus

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

or

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n - 1)d]. \tag{6}$$

In an arithmetic progression, we can find a term by subtracting d from the term that follows it. Thus the terms in the sum S_n can be written in reverse order to give

$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + [a_n - (n - 1)d]. \tag{7}$$

Adding corresponding sides of Eq. 6 and Eq. 7 gives

$$\begin{array}{rl} S_n = & a_1 \quad + (a_1 + d) + \cdots + [a_1 + (n - 1)d] \\ S_n = & a_n \quad + (a_n - d) + \cdots + [a_n - (n - 1)d] \\ \hline 2S_n = & (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n). \end{array}$$

On the right side of the last equation, the number $a_1 + a_n$ occurs n times. Thus

$$2S_n = n(a_1 + a_n).$$

By dividing both sides by 2, we have the following result.

> The sum S_n of the first n terms of an arithmetic progression is given by
>
> $$\boldsymbol{S_n = \frac{n}{2}(a_1 + a_n),} \tag{8}$$
>
> where a_1 is the first term and a_n is the nth term.

Example 6

Find the sum of the first eight terms of the series

$$1 + 4 + 7 + \cdots.$$

Solution The terms form an arithmetic progression with $a_1 = 1$ and $d = 3$. The eighth term is given by

$$a_n = a_1 + (n - 1)d$$
$$a_8 = 1 + (8 - 1)3 = 22.$$

We can now find the sum by using Eq. 8.

$$S_8 = \frac{8}{2}(a_1 + a_8) = 4(1 + 22) = 92.$$

Example 7

Find the sum of the odd integers between 20 and 60.

Solution The odd integers between 20 and 60 form an arithmetic progression with $d = 2$. Here $a_1 = 21$ and $a_n = 59$. To find n, we have

$$a_n = a_1 + (n - 1)d$$
$$59 = 21 + (n - 1)2$$
$$38 = 2n - 2$$
$$40 = 2n$$
$$20 = n.$$

That is, 59 is the twentieth term of the arithmetic progression. Using Eq. 8 gives

$$S_{20} = \frac{20}{2}(21 + 59) = 800.$$

We can find another formula for S_n besides Eq. 8. Replacing a_n in that equation by $a_1 + (n + 1)d$, we have

$$S_n = \frac{n}{2}(a_1 + a_n)$$
$$= \frac{n}{2}[a_1 + a_1 + (n - 1)d].$$

Thus

$$\boxed{S_n = \frac{n}{2}[2a_1 + (n-1)d].}$$

To redo Example 6 by using this formula, we have

$$S_8 = \frac{8}{2}[2(1) + (8-1)3] = 4[2 + 21] = 92.$$

Example 8

Derive a formula for the sum of the first k positive integers.

Solution The terms of the series

$$\sum_{n=1}^{k} n = 1 + 2 + 3 + \cdots + k$$

form an arithmetic progression with $a_1 = 1$, $d = 1$, and k terms. Thus, by Eq. 8, we have

$$\boxed{\sum_{n=1}^{k} n = \frac{k(k+1)}{2}.} \tag{9}$$

For example, to find the sum of the first 25 positive integers we set $k = 25$:

$$\sum_{n=1}^{25} n = \frac{25(25+1)}{2} = 325.$$

Problem Set 20.2

In Problems **1–10**, *for those sequences that are arithmetic progressions, find the indicated term.*

1. 2, 6, 10, ...; a_6.
2. 4, 9, 14, ...; a_5.
3. 13, 1, −11, ...; a_8.
4. 13, 0, −13, ...; a_7.
5. 4, 1, −1, ...; a_5.
6. 4, 1, $\frac{1}{4}$, ...; a_7.
7. $\frac{1}{3}$, $\frac{2}{3}$, 1, ...; a_{11}.
8. a, $-a$, $-3a$, ...; a_9.
9. 0.4, 0.8, 1.2, ...; a_{10}.
10. −3, −1, 1, ...; a_6.

In Problems **11–20**, *find the indicated sum.*

11. $2 + 4 + 6 + \cdots$; S_{10}.

12. $1 + 5 + 9 + \cdots$; S_9.

13. $15 + 10 + 5 + \cdots$; S_{16}.

14. $-\frac{1}{3} + \frac{1}{3} + 1 + \cdots$; S_{12}.

15. $2 + 0 - 2 - 4 - \cdots$; S_{12}.

16. $14 + 7 + 0 + \cdots$; S_9.

17. $-10 + 6 + 22 + \cdots$; S_{10}.

18. $\frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \cdots$; S_9.

19. $3 + 6 + 9 + \cdots + 27$.

20. $7 + 9 + 11 + \cdots + 33$.

21. The fourth term of an arithmetic progression with $d = 14$ is 86. Find (a) the first term and (b) the eighth term.

22. The first term of an arithmetic progression is 16. The tenth term is 10. Find d.

23. The sixteenth term of an arithmetic progression is 28. The first term is -4. Find d.

24. In an arithmetic progression, $a_1 = 6$, $a_n = 26$, and $d = 4$. Find n.

25. The fourth term of an arithmetic progression is 11 and the eleventh term is 32. Find a_1 and d.

26. The third term of an arithmetic progression is -2 and the tenth term is 12. Find the first term.

27. Find the sum of the first 100 positive integers.

28. Find the sum of the odd integers between 30 and 80.

In Problems **29–32**, *find the values of the indicated quantities for the arithmetic progression that has the given properties.*

29. $a_{13} = 42$, $d = 2$; a_{16}, S_{16}.

30. $a_9 = 20$, $d = 1$; a_{16}, S_{16}.

31. $a_1 = -30$, $d = 3$, $S_n = 69$; n, a_n.

32. $a_1 = 12$, $a_n = 42$, $d = 2$; n, S_n.

33. The length of the first swing of a pendulum is 10 cm, and because of resistance, each succeeding swing is $\frac{1}{4}$ cm less. What is the length of the thirteenth pendulum swing?

34. How many swings of the pendulum in Problem 33 are completed before the pendulum comes to rest?

35. If a person saves 1¢ the first day, 2¢ the next day, 3¢ the next day, and so on, how much money will the person have at the end of 30 days?

36. In a vacuum, an object falls approximately 4.9 m the first second, 14.7 m the next, 24.5 m the next, and so on. How far does it fall in 10 s?

37. A display in a supermarket consists of boxes piled up in the form of a pyramid. There are 20 boxes on the bottom layer and each successive layer has two fewer boxes. How many boxes are in the display?

38. It can be shown that the sum of the squares of the first k positive integers is given by

$$\sum_{n=1}^{k} n^2 = \frac{k(k+1)(2k+1)}{6}.$$

Find the sum of the squares of the first ten positive integers.

39. It can be shown that the sum of the cubes of the first k positive integers is given by

$$\sum_{n=1}^{k} n^3 = \frac{k^2(k+1)^2}{4}.$$

Find the sum of the cubes of the first ten positive integers.

20.3 GEOMETRIC PROGRESSIONS

In the sequence

$$3, 6, 12, 24, 48, \ldots$$

each term, after the first, can be obtained by multiplying the preceding term by 2. This means that the ratio (or quotient) of every two consecutive terms is 2:

$$\frac{6}{3} = 2, \qquad \frac{12}{6} = 2, \qquad \text{and so on.}$$

We call the sequence 3, 6, 12, 24, 48 a *geometric progression* (or *geometric sequence*) with *common ratio* 2. Here is a more general definition.

> A sequence in which each term, after the first, can be obtained by multiplying the preceding term by the constant r is called a **geometric progression** with **common ratio** r.

Example 1

Suppose the first term of a geometric progression is 3 and the common ratio r is $-\frac{1}{2}$. Then

$$a_1 = 3$$

$$a_2 = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$$

$$a_3 = \left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right) = \frac{3}{4}$$

$$a_4 = \frac{3}{4}\left(-\frac{1}{2}\right) = -\frac{3}{8} \qquad \text{and so on.}$$

Thus the geometric progression is

$$3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$$

Example 2

a. The sequence

$$4, 0.4, 0.04, 0.004, \ldots$$

is a geometric progression with $r = 0.1$. For example, the ratio of the second term to the first term is $0.4/4 = 0.1$.

b. The sequence

$$\frac{1}{2}, 1, 2, 4, \ldots$$

is a geometric progression with $r = 2$. Verify this.

We can find a formula for the nth term of a geometric progression. Suppose a geometric progression has first term a_1 and common ratio r. Then

$$a_2 = a_1 r$$

$$a_3 = a_2 r = (a_1 r) r = a_1 r^2$$

$$a_4 = a_3 r = (a_1 r^2) r = a_1 r^3, \qquad \text{and so on.}$$

Thus the geometric progression is

$$a_1, a_1 r, a_1 r^2, a_1 r^3, \ldots.$$

In each term the exponent for r is one less than the number of the term. For example, the tenth term is $a_{10} = a_1 r^{10-1} = a_1 r^9$. More generally, we have the following formula.

> The nth term, a_n, of a geometric progression with first term a_1 and common ratio r is given by
>
> $$a_n = a_1 r^{n-1}. \tag{1}$$

Example 3

Find the fifth term of the geometric progression

$$2, -\frac{1}{2}, \frac{1}{8}, \ldots.$$

Solution Here $a_1 = 2$ and $r = (-\frac{1}{2})/2 = -\frac{1}{4}$. Using Eq. 1 with $n = 5$ gives

$$a_n = a_1 r^{n-1}$$

$$a_5 = 2\left(-\frac{1}{4}\right)^{5-1} = 2\left(-\frac{1}{4}\right)^4 = \frac{1}{128}.$$

Example 4

The first term of a geometric progression is 3 and the sixth term is $\frac{3}{32}$. Find the common ratio r.

Solution Here $a_1 = 3$ and $a_6 = \frac{3}{32}$. Using Eq. 1 with $n = 6$, we have

$$a_1 r^{n-1} = a_n$$

$$3r^5 = \frac{3}{32}$$

$$r^5 = \frac{1}{32}$$

$$r = \frac{1}{2} \qquad \text{[taking the fifth root of both sides]}.$$

Example 5

A person invests a principal of \$100 and earns interest of 6% at the end of each year. If none of the interest is withdrawn, how much is the investment worth at the end of 5 years?

Solution At the end of the first year, the value of the investment is the sum of the original investment and the interest:

$$100 + 100(0.06) = 100(1 + 0.06) \qquad \text{[factoring]}$$
$$= 100(1.06).$$

This amount is the principal on which interest is earned for the second year. At the end of that year, the value of the investment is the sum of principal and interest, or

$$100(1.06) + [100(1.06)]0.06 = 100(1.06)[1 + 0.06] \qquad \text{[factoring]}$$
$$= 100(1.06)^2.$$

Similarly, the value of the investment at the end of the third year is $100(1.06)^3$, and so on. These values,

$$100(1.06),\ 100(1.06)^2,\ 100(1.06)^3, \ldots,$$

form a geometric progression with common ratio 1.06. In general, the value of the investment at the end of n years is

$$100(1.06)^n.$$

Thus, at the end of 5 years, the value is $100(1.06)^5$ dollars. (A calculator value is \$133.82.) Because the interest is reinvested, the person is earning "interest on interest," called *compound interest*.

We can find a formula for the sum S_n of the first n terms of a geometric progression. Since

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1}, \tag{2}$$

multiplying both sides by r gives

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^n. \tag{3}$$

Subtracting Eq. 3 from Eq. 2 and factoring, we have

$$S_n - rS_n = a_1 - a_1r^n$$
$$S_n(1 - r) = a_1(1 - r^n).$$

If $r \neq 1$, then dividing both sides by $1 - r$ gives the following result.

The formula

$$\boldsymbol{S_n = \frac{a_1(1 - r^n)}{1 - r}} \tag{4}$$

gives the sum S_n of the first n terms of a geometric progression with first term a_1 and common ratio r.

From Eq. 4 we can get another formula for S_n that involves a_n. We have

$$S_n = \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1 - a_1r^n}{1 - r}$$
$$= \frac{a_1 - (a_1r^{n-1})r}{1 - r}.$$

From Eq. 1, $a_1r^{n-1} = a_n$, so we have

$$\boldsymbol{S_n = \frac{a_1 - ra_n}{1 - r}}. \tag{5}$$

If $r = 1$, then Eqs. 4 and 5 cannot be used. However, in that case we have

$$S_n = a_1 + a_1 + \cdots + a_1 = na_1.$$

Example 6

Find the sum of the first 10 terms of the geometric progression 12, 6, 3, ...

Solution Here $a_1 = 12$, $r = \frac{6}{12} = \frac{1}{2}$, and $n = 10$. From Eq. 4,

$$S_{10} = \frac{a_1(1 - r^{10})}{1 - r} = \frac{12[1 - (\frac{1}{2})^{10}]}{1 - \frac{1}{2}}$$
$$= \frac{12[1 - \frac{1}{1024}]}{\frac{1}{2}} = \frac{3069}{128}.$$

Example 7

Find the sum of the series $1 + \frac{1}{3} + (\frac{1}{3})^2 + \cdots + (\frac{1}{3})^5$.

Solution The terms of the series form a geometric progression with $a_1 = 1$ and $r = \frac{1}{3}$. This series is called a *geometric series*. Here $n = 6$ (not 5). From Eq. 4 we have

$$S_6 = \frac{a_1(1 - r^6)}{1 - r} = \frac{1[1 - (\frac{1}{3})^6]}{1 - \frac{1}{3}}$$

$$= \frac{\frac{728}{729}}{\frac{2}{3}} = \frac{364}{243}.$$

Since the last term in the sum is known to be $(\frac{1}{3})^5$, we could have used Eq. 5 to obtain the same result.

Problem Set 20.3

In Problems **1–10**, *for those sequences that are geometric progressions, find the indicated term.*

1. $3, 6, 12, \ldots;$ a_5.
2. $4, 1, \frac{1}{4}, \ldots;$ a_7.
3. $6, -3, \frac{3}{2}, \ldots;$ a_6.
4. $-1, 3, -9, \ldots;$ a_6.
5. $2, 4, 6, \ldots;$ a_5.
6. $\frac{1}{3}, \frac{2}{3}, 1, \ldots;$ a_5.
7. $-4, 2, -1, \ldots;$ a_9.
8. $0.3, 0.03, 0.003, \ldots;$ a_7.
9. $\frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \ldots;$ a_5.
10. $12, -4, \frac{4}{3}, \ldots;$ a_6.

In Problems **11–18**, *find the indicated sum.*

11. $3 + 9 + 27 + \cdots;$ S_6.
12. $1 - \frac{1}{4} + \frac{1}{16} - \cdots;$ S_6.
13. $6 - 12 + 24 - \cdots;$ S_6.
14. $0.1 + 0.02 + 0.004 + \cdots;$ S_5,
15. $0.3 + 0.03 + 0.003 + \cdots;$ S_7.
16. $24 + 12 + 6 + \cdots;$ S_7.
17. $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{128}$.
18. $1 - \frac{1}{2} + \frac{1}{4} - \cdots + \frac{1}{64}$.
19. The first term of a geometric progression is 16. The fourth term is $\frac{1}{4}$. Find r.
20. The first term of a geometric progression is 18 and the fourth term is $\frac{2}{81}$. Find the second term.
21. If a person saves 1¢ the first day, 2¢ the next day, 4¢ the next day, 8¢ the next day, and so on, how much money will the person have at the end of 30 days?
22. Find $\sum_{n=1}^{6} \left(\frac{1}{2}\right)^n$.

In Problems **23–28**, *find the indicated quantities for the geometric progression that has the given properties.*

23. $r = \frac{1}{2}, S_6 = 126;$ a_1, a_6.
24. $a_1 = 1, a_6 = 32;$ r, S_6.
25. $a_1 = \frac{3}{4}, a_n = -96, S_n = -\frac{255}{4};$ r.
26. $a_1 = \frac{1}{2}, a_{10} = 256;$ a_{12}, S_{12}.
27. $r = \frac{1}{3}, S_5 = 121;$ a_1, a_5.
28. $a_6 = 1, a_8 = 9;$ S_5.

29. By means of a pump, air is being removed from a container in such a way that each second, one-tenth of the remaining air in the container is removed. After 5 s, what percentage of air is left? Give your answer to the nearest percentage.

30. A tank full of alcohol is emptied of one-fourth its contents. The tank is then filled with water. This is repeated three more times. What part of the tank is now alcohol?

31. Find the value of an investment of \$250 after n years if interest is compounded annually at the rate of 6.5%.

32. Bacteria growing in a culture double their number every day. If there were originally 10,000 bacteria, how many will there be after 8 days?

33. A distant star now has a surface temperature of 10,000°C, and observations indicate that the temperature decreases by 10% every 1000 years. What will be the temperature 4000 years from now?

34. For the star of Problem 33, in how many years will the surface temperature be 7290°C?

20.4 LIMITS OF SEQUENCES

We now consider one of the very fundamental ideas of mathematics—the concept of a limit. The ideas and techniques we shall develop here are useful in both the main topics of calculus, namely, finding derivatives and finding integrals.

We begin by examining rather closely a sequence that conveniently typifies a situation which arises quite naturally in mechanics. Consider a pendulum that is displaced from its equilibrium (rest) position and then released. The pendulum will oscillate back and forth. Let us assume that the length of the first swing was $\frac{1}{2}$ m and that the length of any subsequent swing was exactly one-half the length of the swing that preceded it. If we let a_1 be the length of the first swing, a_2 the length of the second swing, and so on, we have

$$a_1 = \frac{1}{2} = \frac{1}{2^1}$$

$$a_2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{2^2}$$

$$a_3 = \frac{1}{2}\left(\frac{1}{2^2}\right) = \frac{1}{2^3}$$

$$a_4 = \frac{1}{2}\left(\frac{1}{2^3}\right) = \frac{1}{2^4}.$$

The pattern continues. For the nth swing of the pendulum,

$$a_n = \frac{1}{2^n}.$$

Thus the sequence whose terms correspond to the lengths of the swings is

$$\left\{\frac{1}{2^n}\right\} = \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \cdots, \frac{1}{2^n}, \cdots$$

$$= \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots, \frac{1}{2^n}, \cdots.$$

Some terms of this sequence are shown on the real number line in Fig. 20.1. Observe that as n gets larger, the terms get closer to zero. Moreover, although $1/2^n$ will never

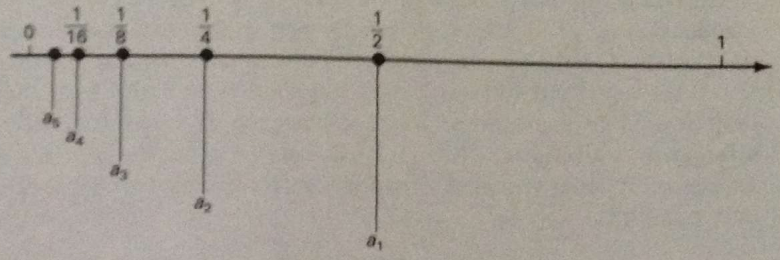


FIGURE 20.1

have a value of zero, for large enough values of n the corresponding terms will be as close to zero as we wish. For example, suppose we wish to find when the length of a swing will be less than $\frac{1}{10{,}000}$ m. Then we must have the inequality

$$\frac{1}{2^n} < \frac{1}{10{,}000},$$

which is true for $n > 13$ (verify this). That is, beginning with the fourteenth term (the fourteenth swing of the pendulum), each term in the sequence will be less than $\frac{1}{10{,}000}$ m.

The ideas of the preceding discussion are verbally expressed by saying that as n increases indefinitely, the sequence $\{1/2^n\}$ has zero for a **limit**, or $\{1/2^n\}$ *converges* to zero. Symbolically we write

$$\lim_{n\to\infty} \frac{1}{2^n} = 0. \tag{1}$$

Here the notation $n \to \infty$ means that n is increasing indefinitely through positive integral values. Equation 1 can be read *the limit of the sequence* $\{1/2^n\}$ as n *increases without bound is equal to zero.* This means that every interval containing zero, no matter how small, contains all the terms of the sequence $\{1/2^n\}$ from some term on.

More generally, to say that

$$\lim_{n\to\infty} a_n = a$$

means that the limit of the sequence $\{a_n\}$ as $n \to \infty$ is the number a. That is, every interval containing a contains all the terms of the sequence $\{a_n\}$ from some term on. For n sufficiently large, a_n is arbitrarily close to a.

Not every sequence has a limit. For example, the terms of the sequence $\{n^2\} = 1, 4, 9, \ldots$ increase without bound as $n \to \infty$, so the sequence has no limit. We denote this situation by writing

$$\lim_{n \to \infty} n^2 = \infty,$$

which can be read *as n increases without bound, the terms of the squence* $\{n^2\}$ *increase without bound*. The use of the equals sign in this situation does not mean that the limit exists. On the contrary, the symbolism here (∞) is a way of saying specifically that there is no limit and it indicates **why** there is no limit. A sequence that has a finite limit is said to **converge** or to be **convergent**. If a sequence does not converge, it **diverges** or is **divergent**. Thus $\{1/2^n\}$ is a convergent sequence, but $\{n^2\}$ is a divergent sequence. When a sequence converges, its limit must be unique. That is, there can be only one limit.

Example 1

The sequence

$$3, 2, -5, -5, \ldots, -5, \ldots$$

converges to -5, since *every* interval containing -5 must contain all the terms of the sequence from the third term on. For most convergent sequences, however, you may have to ignore many, perhaps a million, of the terms in the sequence before all the remaining terms lie in a given interval if the interval is quite small.

Example 2

a. The terms of the arithmetic sequence $\{-5n\} = -5, -10, -15, \ldots$ decrease without bound, so the sequence diverges; that is, it has no finite limit. Symbolically, we denote this situation by writing

$$\lim_{n \to \infty} (-5n) = -\infty.$$

In fact, because of the common difference d, where $|d| > 0$, *every arithmetic progression is divergent.*

b. The terms of the sequence $\{(-1)^n\} = -1, 1, -1, 1, \ldots$ oscillate between 1 and -1. Thus the sequence has no limit as $n \to \infty$ and hence is divergent.

Example 3

The terms of the sequence

$$\left\{\frac{n+1}{n}\right\} = 2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \cdots$$

appear to be getting closer to 1 as n increases. In fact, for $n = 1000$,

$$a_{1000} = \frac{1001}{1000} = 1 + \frac{1}{1000}.$$

In Example 6 it will be shown that 1 is indeed the limit.

Without going into their proofs, we shall state some theorems on limits and illustrate their use in determining the behavior of various sequences. We begin with the following.

THEOREM 1.

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. If $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist and $\{r^n\}$ diverges.

THEOREM 2.

$\lim_{n\to\infty} c = c$, where c is a constant.

THEOREM 3.

$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$ if $\{a_n\}$ is convergent.

THEOREM 4.

If each term of a divergent sequence is multiplied by the same nonzero constant, the resulting sequence is also divergent.

Example 4

Determine whether the given sequence converges or diverges. In the case of convergence, find the limit.

a. $\{(-\frac{1}{2})^n\} = -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$.

Solution We apply Theorem 1 with $r = -\frac{1}{2}$. Since $|-\frac{1}{2}| < 1$, we have

$$\lim_{n\to\infty} (-\tfrac{1}{2})^n = 0.$$

The sequence converges to 0.

b. $\{3\} = 3, 3, 3, \ldots$.

Solution From Theorem 2 with $3 = c$, we have

$$\lim_{n\to\infty} 3 = 3.$$

The sequence converges to 3.

c. $\left\{\frac{3}{4^n}\right\}$.

Solution Note that $3/4^n = 3(\frac{1}{4})^n$. Since $|\frac{1}{4}| < 1$, by Theorem 1 the sequence $\{(\frac{1}{4})^n\}$ converges to 0. Thus, by Theorem 3,

$$\lim_{n\to\infty} \frac{3}{4^n} = 3 \lim_{n\to\infty} \left(\frac{1}{4}\right)^n = 3(0) = 0.$$

The sequence converges to 0.

d. $\{2(\frac{4}{3})^n\}$.

Solution Because $|\frac{4}{3}| > 1$, by Theorem 1 the sequence $\{(\frac{4}{3})^n\}$ diverges. By Theorem 4, $\{2(\frac{4}{3})^n\}$ also diverges. Note how the terms in the sequence increase without bound as $n \to \infty$:

$$\left\{2\left(\frac{4}{3}\right)^n\right\} = \frac{8}{3}, \frac{32}{9}, \frac{128}{27}, \frac{512}{81}, \ldots.$$

Here are some additional theorems.

THEOREM 5.

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, then

a. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$,

b. $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$,

c. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $b_n \neq 0$ and $\lim_{n\to\infty} b_n \neq 0$.

THEOREM 6.

a. For $a_n > 0$, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = \infty$.

b. For $a_n > 0$, $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Example 5

a. Because $\lim_{n\to\infty} n^3 = \infty$, then by Theorem 6(b), $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

b. $\lim_{n\to\infty}\left(2 + \frac{1}{n^3}\right) = \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n^3}$ [Theorem 5(a)]

$= 2 + 0 = 2$ [Theorem 2 and Example 5a].

Example 6

Find $\lim_{n\to\infty} \frac{n+1}{n}$.

Solution As $n \to \infty$, the numerator and denominator get arbitrarily large. To describe this we say that the quotient $(n + 1)/n$ has the form ∞/∞. However, we can find the limit by first using some algebra to rewrite $(n + 1)/n$.

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(\frac{n}{n} + \frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} \quad \text{[Theorem 5(a)]}$$

$$= 1 + 0 = 1 \quad \text{[Theorems 2 and 6(b)]}.$$

Our result means that as $n \to \infty$, the number $(n + 1)/n$ gets arbitrarily close to 1.

Example 7

Find $\lim_{n\to\infty} \frac{2n^2 + 3}{3n^2 + 4n}$.

Solution As $n \to \infty$, the quotient takes on the form ∞/∞. No algebraic simplification is obvious. A situation like this can be remedied by dividing the numerator and denominator by

the greatest power of n that occurs in the denominator (in our case, n^2) and applying Theorem 5. We have

$$\lim_{n\to\infty}\frac{2n^2+3}{3n^2+4n}=\lim_{n\to\infty}\frac{\frac{2n^2+3}{n^2}}{\frac{3n^2+4n}{n^2}}$$

$$=\lim_{n\to\infty}\frac{2+\frac{3}{n^2}}{3+\frac{4}{n}}$$

$$=\frac{\lim_{n\to\infty}\left(2+\frac{3}{n^2}\right)}{\lim_{n\to\infty}\left(3+\frac{4}{n}\right)} \qquad \text{[Theorem 5(c)]}$$

$$=\frac{\lim_{n\to\infty}2+3\lim_{n\to\infty}\frac{1}{n^2}}{\lim_{n\to\infty}3+4\lim_{n\to\infty}\frac{1}{n}} \qquad \text{[Theorems 5(a) and 3]}$$

$$=\frac{2+3(0)}{3+4(0)}=\frac{2}{3} \qquad \text{[Theorems 2 and 6(b)].}$$

Example 8

Find $\lim_{n\to\infty}\dfrac{2n^2}{n^3+1}$

Solution The quotient has the form ∞/∞ as $n\to\infty$. We first divide the numerator and denominator by n^3.

$$\lim_{n\to\infty}\frac{2n^2}{n^3+1}=\lim_{n\to\infty}\frac{\frac{2n^2}{n^3}}{\frac{n^3+1}{n^3}}=\lim_{n\to\infty}\frac{\frac{2}{n}}{1+\frac{1}{n^3}}$$

$$=\frac{2\lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n^3}}$$

$$=\frac{2\cdot 0}{1+0}=\frac{0}{1}=0.$$

Our approach to the concept of a limit has been intuitive and is in no way rigorous. However, because the limit concept is fundamental in mathematics and lies at the very foundation of higher mathematics, you should have some "feeling" for this notion.

Problem Set 20.4

Find the following limits. In the case of divergence, so state.

1. $\lim_{n\to\infty} (2n)$.
2. $\lim_{n\to\infty} (3n + 4)$.
3. $\lim_{n\to\infty} \left(\frac{6}{n}\right)$.
4. $\lim_{n\to\infty} (7 + \frac{1}{3})$.
5. $\lim_{n\to\infty} (3 - \frac{1}{4})$.
6. $\lim_{n\to\infty} \left(\frac{-6}{n}\right)$.
7. $\lim_{n\to\infty} (\frac{3}{4})^n$.
8. $\lim_{n\to\infty} (\frac{17}{16})^n$.
9. $\lim_{n\to\infty} [3(\frac{1}{2})^n]$.
10. $\lim_{n\to\infty} \left(\frac{n+1}{2n}\right)$.
11. $\lim_{n\to\infty} \left(\frac{3n-1}{2n}\right)$.
12. $\lim_{n\to\infty} \left(1 + \frac{n-1}{n}\right)$.
13. $\lim_{n\to\infty} \left(\frac{n+3}{4-n}\right)$.
14. $\lim_{n\to\infty} \left(\frac{n^2-n+1}{2n^2}\right)$.
15. $\lim_{n\to\infty} \left[\frac{1}{5}\left(\frac{7}{4}\right)^n\right]$.
16. $\lim_{n\to\infty} \left[8\left(\frac{1}{3^n}\right)\right]$.
17. $\lim_{n\to\infty} \frac{(100)^n}{(101)^n}$.
18. $\lim_{n\to\infty} \left(\frac{3n^2+2n+5}{4n^2}\right)$.
19. $\lim_{n\to\infty} \left(\frac{2n^4-6n^2+5}{n^5}\right)$.
20. $\lim_{n\to\infty} \frac{8n^5-6n^4+3n^3+1}{n^6-4n^3+1}$.

20.5 THE INFINITE GEOMETRIC SERIES

If we are given an infinite sequence, then the indicated sum of its terms

$$a_1 + a_2 + a_3 + \cdots$$

is called an **infinite series** because there are infinitely many terms.

You might at first be alarmed by the notion of an "infinite sum" as just indicated. The word *sum* probably has meaning to you only as far as a finite number of quantities is concerned, and the thought of infinitely many additions may seem awesome. Be assured that we can, in a reasonable way, attach a meaning to such a sum. In fact, it will soon be done, in part, in terms of finite sums. But first, a word about notation.

An infinite series can be represented by the sigma notation

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots,$$

where the index n is replaced successively by the integers 1, 2, 3, ... For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots.$$

If we are given the series $\sum_{n=1}^{\infty} a_n$, then the sum of its first n terms is called the ***n*th partial sum** of the series and is denoted by S_n:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 = S_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 = S_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + \cdots + a_n = S_{n-1} + a_n, \end{aligned}$$

where S_1 is the first partial sum, S_2 the second partial sum, and so on. Observe that $S_2 = S_1 + a_2$, $S_3 = S_2 + a_3$, ..., $S_n = S_{n-1} + a_n$, and $S_n = \sum_{k=1}^{n} a_k$. The sequence

$$\{S_n\} = S_1, S_2, S_3, \ldots, S_n, \ldots,$$

is called the **sequence of partial sums** of the series.

Example 1

Determine the first four terms of the sequence of partial sums for the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Solution

$$S_1 = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$S_2 = S_1 + \frac{1}{2(2+1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

$$S_3 = S_2 + \frac{1}{3(4)} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$$

$$S_4 = S_3 + \frac{1}{4(5)} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5}.$$

Thus the first four terms of $\{S_n\}$ are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, and $\frac{4}{5}$.

If the sequence $\{S_n\}$ of partial sums of the series $\sum_{n=1}^{\infty} a_n$ converges to a finite number L, then L is called the **sum** of the series. That is,

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = L.$$

In this case $\sum_{n=1}^{\infty} a_n$ is said to **converge** to L, or to be **convergent**. This means that by adding up a sufficient number of terms of the series, we can approach L as arbitrarily closely as we wish. If $\lim_{n\to\infty} S_n$ does not exist, the series is said to **diverge** or be **divergent**.

We shall now determine the sum of the **infinite geometric series**

$$\sum_{n=1}^{\infty} a_1 r^{n-1} = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1} + \cdots, \qquad \text{where } a_1 \neq 0^*.$$

The nth partial sum, S_n, is

$$S_n = a_1 + a_1 r + a_1 r^2 + \cdots + a_1 r^{n-1},$$

which was previously found to be

$$S_n = \frac{a_1(1 - r^n)}{1 - r}.$$

The limit of S_n as $n \to \infty$ is now considered for the possible values of r.

If $|r| < 1$, then by Theorem 1, $\lim_{n\to\infty} r^n = 0$ and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1}{1 - r}.$$

Thus if $|r| < 1$, the geometric series converges to the sum $a_1/(1 - r)$. But if $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist, so neither does $\lim_{n\to\infty} S_n$. If $r = 1$, then

$$S_n = a_1 + a_1 + \cdots + a_1 = na_1,$$

which has no limit as $n \to \infty$. If $r = -1$, then

$$S_n = a_1 - a_1 + a_1 - a_1 + \cdots + (\pm a_1).$$

* If $a_1 = 0$, then the series is $0 + 0 + \cdots + 0 + \cdots$. We shall not consider this uninteresting case.

This is zero if n is even and is a_1 if n is odd. Hence the series diverges, because a limit of a sequence must be unique. In summary, we have:

> The infinite geometric series $\sum_{n=1}^{\infty} a_1 r^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series is convergent, its sum S is given by
>
> $$S = \frac{a_1}{1 - r}. \tag{1}$$

Example 2

Test the following series for convergence or divergence. In the case of convergence, find the sum.

a. $\sum_{n=1}^{\infty} 8(\frac{1}{2})^n = 4 + 2 + 1 + \frac{1}{2} + \cdots.$

Solution The series is geometric with $a_1 = 4$ and $r = \frac{1}{2}$. Since $|r| < 1$, the series converges and by Eq. 1 the sum S is given by

$$S = \frac{a_1}{1 - r} = \frac{4}{1 - \frac{1}{2}} = \frac{4}{\frac{1}{2}} = 8.$$

b. $6 - 1 + \frac{1}{6} - \frac{1}{36} + \cdots = \sum_{n=1}^{\infty} 6(-\frac{1}{6})^{n-1}.$

Solution The series is geometric with $a_1 = 6$ and $r = -\frac{1}{6}$. Because $|-\frac{1}{6}| < 1$, the series converges and the sum is

$$\frac{a_1}{1 - r} = \frac{6}{1 - (-\frac{1}{6})} = \frac{6}{\frac{7}{6}} = \frac{36}{7}.$$

c. $\sum_{n=1}^{\infty} 4(\frac{5}{3})^n.$

Solution The series is geometric with $r = \frac{5}{3}$. Since $|r| > 1$, the series diverges.

d. $\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots.$

Solution The series is geometric with $a_1 = \frac{1}{4}$ and $r = \frac{1}{4}$. Since $|r| < 1$, the series converges and the sum is

$$\frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

Example 3

The pendulum discussed in Sec. 20.4 had an initial swing of $\frac{1}{2}$ m, and the length of any subsequent swing was one-half the length of the previous swing. How far did the tip of the pendulum move before coming to rest?

Solution We showed that the sequence whose terms give the lengths of the swings is

$$\left\{\frac{1}{2^n}\right\} = \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \cdots$$

The corresponding series, which is geometric, is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n} + \cdots.$$

The sum of this series gives the total distance the pendulum tip moves. Here we have $a_1 = \frac{1}{2}$ and $r = \frac{1}{2}$. Thus the sum S is given by

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \text{ m}.$$

Example 4

Determine the rational number that corresponds to the repeating decimal $0.\overline{123}$.

Solution The line above the digits 123 is called a **vinculum** and indicates those digits that repeat. Hence

$$0.\overline{123} = 0.123123123\ldots$$

$$= 0.123 + 0.000123 + 0.000000123 + \cdots.$$

Thus the repeating decimal has been expressed as an infinite geometric series with $a_1 = 0.123$ and $r = 0.001$. Therefore, its sum is

$$\frac{0.123}{1 - 0.001} = \frac{0.123}{0.999} = \frac{123}{999} = \frac{41}{333}.$$

That is, $0.\overline{123} = 41/333$.

Problem Set 20.5

In Problems **1–24**, *find the sum of the series if the sum exists.*

1. $\sum_{n=1}^{\infty} (\frac{1}{2})^n$.
2. $\sum_{n=1}^{\infty} (\frac{3}{4})^n$.
3. $\sum_{n=1}^{\infty} (-\frac{2}{3})^n$.
4. $\sum_{n=1}^{\infty} (1.2)^n$.
5. $\sum_{n=1}^{\infty} \left(\frac{7}{6}\right)^n$.
6. $\sum_{n=1}^{\infty} 4\left(\frac{1}{1.1}\right)^n$.
7. $\sum_{k=3}^{\infty} (\frac{3}{5})^{k+2}$.
8. $\sum_{i=5}^{\infty} (-\frac{4}{9})^i$.
9. $\sum_{n=1}^{\infty} 3(-0.2)^n$.
10. $\sum_{n=1}^{\infty} 3(1 - \frac{1}{3})^n$.
11. $\sum_{n=1}^{\infty} 100\left(\frac{1}{4^n}\right)$.
12. $\sum_{n=1}^{\infty} \frac{1}{4}(3^n)$.

13. $\sum_{n=1}^{2} (\frac{1}{3})^n$.

14. $\sum_{n=1}^{\infty} \frac{9}{4}(\frac{4}{9})^{n+3}$.

15. $3 + \frac{3}{2} + \frac{3}{4} + \cdots$.

16. $4 + 1 + \frac{1}{4} + \cdots$.

17. $12 + 4 + \frac{4}{3} + \cdots$.

18. $\frac{1}{1.2} + \frac{1}{(1.2)^2} + \frac{1}{(1.2)^3} + \cdots$.

19. $-4 + 2 - 1 + \cdots$.

20. $100 - 10 + 1 - 0.1 + \cdots$.

21. $\frac{5}{3} + \frac{1}{6} + \frac{1}{60} + \cdots$.

22. $\frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots$.

23. $\frac{1}{0.1} + \frac{1}{(0.1)^2} + \frac{1}{(0.1)^3} + \cdots$.

24. $0.02 + 0.002 + 0.0002 + \cdots$.

In Problems **21–26**, *determine the rational number that is represented by the given repeating decimal.*

25. $0.\overline{24}$.

26. $0.\overline{42}$.

27. $3.\overline{212}$.

28. $2.0\overline{46}$.

29. $0.021\overline{32}$.

30. $0.15\overline{6}$.

31. When dropped from a height of 4 m, a ball after the first bounce reaches a height of 2 m, after the second bounce a height of 1 m, and so on. What is the total distance traveled by the ball before coming to rest?

32. The tip of a pendulum moves through a distance of 6 cm, after which the distance is constantly decreased on each swing by 10%. Before coming to rest, through what total distance has the tip moved?

33. The first oscillation of a mass suspended on a vertical spring is 20 cm long. If it is observed that the length of each succeeding oscillation decreases by 20%, how far does the mass travel before coming to rest?

34. When a motor is turned off, a flywheel attached to the motor is observed to "coast" to a stop. In the first second it made 190 revolutions, and in each succeeding second it made nine-tenths as many revolutions as the preceding second. How many revolutions did the flywheel make before coming to rest?

35. When a small object is projected up an inclined plane, it is observed to move 10 m in the first second and in any succeeding second it moves four-fifths as far as it did in the preceding second. How far does it travel before coming to rest?

36. The midpoints of the sides of a 10-cm square are joined to form an inscribed square, and this process is continued without end. Find the sum of the areas of all the squares, including the original one.

20.6 THE BINOMIAL THEOREM

In this section we shall develop a formula that gives the series of terms obtained by finding the product or *expansion* of $(a + b)^n$. By using previous rules and direct multiplication, we can find some positive integral powers of the binomial $a + b$.

$$(a + b)^2 = a^2 + 2ab + b^2.$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Looking at our results, we can point out similarities in the expansion of $(a + b)^n$ for $n = 2, 3, 4$.

1. The expansion of $(a + b)^n$ has $n + 1$ terms. For example, in $(a + b)^4$ we have $n = 4$ and the expansion has five terms.
2. The first term is a^n and the last term is b^n.
3. As we move from one term to the next, the exponent for a successively *decreases* by 1 (from n to 1). From the second term on, the exponent for b successively *increases* by 1 (from 1 to n).
4. In each term involving a and b, the sum of the exponents for a and b is n. For example, the second term in the expansion of $(a + b)^4$ contains a^3b^1, and $3 + 1 = 4 = n$.
5. In any term involving b, the exponent for b is one less than the number of the term. For example, the *third* term for $(a + b)^4$ contains b^2, and $2 = 3 - 1$.

We can also find a pattern for the coefficients of the terms in the above expansions of $(a + b)^n$. If the coefficient of any term is multiplied by the exponent for a in that term and then divided by the number of that term, the result is the coefficient of the next term. For example, the first term in the expansion of $(a + b)^4$ is $a^4 = 1 \cdot a^4$. Multiplying the coefficient 1 by the exponent of a (4) and dividing by the number of the term (1), we obtain

$$\frac{1 \cdot 4}{1} = \frac{4}{1},$$

which is the coefficient 4 of the second term. Continuing in this manner, we have

$$(a + b)^4 = a^4 + \frac{4}{1}a^3b + \frac{4}{1} \cdot \frac{3}{2}a^2b^2 + \frac{4}{1} \cdot \frac{3}{2} \cdot \frac{2}{3}ab^3 + b^4 \qquad (1)$$

$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

In Eq. 1, note that the numerator of the term in which the exponent for b is 2 contains the *two* factors $4 \cdot 3$, and the denominator contains the *two* factors $1 \cdot 2$. Similarly, the numerator of the term in which the exponent for b is 3 contains the *three* factors $4 \cdot 3 \cdot 2$, and the denominator contains the *three* factors $1 \cdot 2 \cdot 3$. Both of these denominators consist of the product of consecutive positive integers and are usually represented by **factorial notation**. If n is a positive integer, then the symbol $n!$, read ***n factorial***, represents the product of the first n positive integers:

$$n! = n(n - 1)(n - 2)(n - 3) \cdots (2)(1).$$

For example,

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 4 \cdot 3! = 4(6) = 24.$$

We define 0! to be 1. Using factorial notation, we can express the term involving b^r in the expansion of $(a + b)^n$ as

$$\frac{\overbrace{n(n-1)(n-2)\cdots(n-r+1)}^{r \text{ factors}}a^{n-r}b^r}{r!}.$$

All the patterns that we have observed carry over to the expansion of $(a + b)^n$, where n is *any* positive integer. In summary we have the **binomial theorem**:

BINOMIAL THEOREM

If n is a positive integer, then

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}b^r + \cdots + b^n.$$

Example 1

Use the binomial theorem to expand $(x + 2)^5$.

Solution We replace a by x, b by 2, and n by 5. The expansion will have six terms.

$$(x+2)^5 = x^5 + \frac{5}{1!}x^4(2) + \frac{5\cdot 4}{2!}x^3(2)^2 + \frac{5\cdot 4\cdot 3}{3!}x^2(2)^3 + \frac{5\cdot 4\cdot 3\cdot 2}{4!}x(2)^4 + (2)^5.$$

This simplifies to

$$(x+2)^5 = x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32.$$

Example 2

Write the first four terms in the expansion of $(a + b)^{20}$.

Solution Using the binomial theorem with $n = 20$ gives

$$(a+b)^{20} = a^{20} + \frac{20}{1!}a^{19}b + \frac{20\cdot 19}{2!}a^{18}b^2 + \frac{20\cdot 19\cdot 18}{3!}a^{17}b^3 + \cdots$$

$$= a^{20} + 20a^{19}b + 190a^{18}b^2 + 1140a^{17}b^3 + \cdots.$$

Example 3

Expand $(3x^2 - 1)^4$.

Solution Noting that $(3x^2 - 1)^4 = [3x^2 + (-1)]^4$, we use the binomial theorem with $a = 3x^2$, $b = -1$, and $n = 4$.

$$\begin{aligned}(3x^2 - 1)^4 &= (3x^2)^4 + \frac{4}{1!}(3x^2)^3(-1) + \frac{4 \cdot 3}{2!}(3x^2)^2(-1)^2 + \frac{4 \cdot 3 \cdot 2}{3!}(3x^2)(-1)^3 + (-1)^4 \\ &= 3^4x^8 + 4(3)^3x^6(-1) + 6(3)^2x^4(1) + 4(3)x^2(-1) + 1 \\ &= 81x^8 - 108x^6 + 54x^4 - 12x^2 + 1.\end{aligned}$$

Example 4

Find the sixth term in the expansion of $\left(x + \frac{2}{y}\right)^{15}$.

Solution Here x plays the role of a, $2/y$ plays the role of b, and $n = 15$. In the sixth term, the exponent for $2/y$ is 5, so the exponent for x is 10. The numerator of the coefficient consists of five decreasing factors beginning with 15, and the denominator is 5! This gives

$$\frac{(15)(14)(13)(12)(11)}{5!}x^{10}\left(\frac{2}{y}\right)^5,$$

which simplifies to

$$\frac{96{,}096x^{10}}{y^5}.$$

Setting $a = 1$ and $b = x$ in the binomial theorem, we obtain the **binomial series**:

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}x^r + \cdots,$$

which for $|x| < 1$ can be shown to be a valid equation for **any real number** n. When n is not a positive integer, the series is unending, but nevertheless we can get a reasonable approximation to $(1 + x)^n$ in most cases by considering a few terms only.

Example 5

Write the first four terms of $(1 + x)^{-2}$.

Solution This is the binomial series with $n = -2$.

$$\begin{aligned}(1 + x)^{-2} &= 1 + (-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \cdots \\ &= 1 - 2x + 3x^2 - 4x^3 + \cdots.\end{aligned}$$

Example 6

Approximate $\sqrt{104}$ to three decimal places by using the binomial series.

Solution We can write

$$\sqrt{104} = \sqrt{100 + 4} = \sqrt{100(1 + \tfrac{4}{100})}$$
$$= 10(1 + \tfrac{1}{25})^{1/2}.$$

Because $|\frac{1}{25}| < 1$, we can use the binomial series with $x = \frac{1}{25}$ and $n = \frac{1}{2}$:

$$\sqrt{104} = 10\left[1 + \frac{1}{2}\left(\frac{1}{25}\right) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}\left(\frac{1}{25}\right)^2 + \cdots\right]$$
$$= 10\left(1 + \frac{1}{50} - \frac{1}{5000} + \cdots\right)$$
$$= 10(1 + 0.02 - 0.0002 + \cdots)$$
$$\approx 10(1.0198),$$
$$\sqrt{104} \approx 10.198.$$

Considering more terms in this series would not have contributed to the accuracy of the desired approximation. Compare our answer to that obtained with a calculator.

Problem Set 20.6

In Problems **1–12**, *use the binomial theorem to expand each expression and simplify.*

1. $(x + 4)^3$.
2. $(x - 3)^3$.
3. $(y - 2)^4$.
4. $(y + 3)^5$.
5. $(3x + 1)^5$.
6. $(x + h)^6$.
7. $(2z - y)^4$.
8. $(z^2 - 2)^3$.
9. $\left(a - \frac{1}{b}\right)^5$.
10. $(x - 2y)^6$.
11. $\left(1 + \frac{x}{y^2}\right)^6$.
12. $\left(\frac{x}{y} + \frac{y}{x}\right)^4$.

In Problems **13–20**, *find the first three terms in the binomial expansion of each expression and simplify.*

13. $(x + 1)^{100}$.
14. $(x - 1)^{45}$.
15. $(2x - 3)^7$.
16. $(x + 2)^8$.
17. $(y^2 - 5x)^{10}$.
18. $(y^3 - 6)^{12}$.
19. $\left(3z^2 + \frac{x}{3}\right)^5$.
20. $(a^2 + b^2)^{20}$.

In Problems **21–28**, *find only the indicated term(s) and simplify.*

21. The fourth term of $(x + y)^{15}$.
22. The fifth term of $(\frac{1}{2}x - y)^{10}$.
23. The sixth term of $\left(2x - \frac{3}{y}\right)^8$.
24. The fourth term of $\left(\frac{1}{x^2} - y\right)^7$.

25. The middle term of $(x - 2y)^6$.

26. The middle terms of $\left(\frac{1}{x} + \frac{1}{y}\right)^5$.

27. The term involving x^6 in $(x^3 - y)^5$.

28. The term involving x^8 in $(2y - 3x^2)^7$.

In Problems **29–32**, *find the first four terms in the expansion of the given expression and simplify.*

29. $(1 + x)^{-3}$.

30. $(1 + x)^{2/3}$.

31. $(1 - y)^{1/2}$.

32. $(1 - x)^{1/4}$.

In Problems **33–40**, *approximate the given number to three decimal places by using the binomial series.*

33. $\sqrt{50}$.

34. $\sqrt{26}$.

35. $\sqrt{61}$.

36. $\sqrt{99}$.

37. $\sqrt[3]{29}$.

38. $\sqrt[3]{66}$.

39. $\sqrt[4]{80}$.

40. $(1.02)^{-4}$.

20.7 REVIEW

Important Terms and Symbols

Section 20.1 Sequence, term of sequence, general term of sequence, a_n, sigma notation, $\sum$, index of summation, upper limit of summation, lower limit of summation.

Section 20.2 Arithmetic progression, common difference (d).

Section 20.3 Geometric progression, common ratio (r).

Section 20.4 Limit of sequence, $\lim_{n \to \infty} a_n = a$, convergent sequence, divergent sequence.

Section 20.5 Infinite series, $\sum_{n=1}^{\infty} a_n$, nth partial sum, sequence of partial sums, sum of infinite series, convergent series, divergent series, infinite geometric series.

Section 20.6 Factorial notation, n factorial ($n!$), binomial theorem, binomial series.

Formula Summary

Arithmetic Progression

$$a_n = a_1 + (n - 1)d$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$= \frac{n}{2}[2a_1 + (n - 1)d]$$

Geometric Progression

$$a_n = a_1 r^{n-1}$$

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{if } r \neq 1$$

$$= na_1 \quad \text{if } r = 1$$

Infinite Geometric Series

$$S = \frac{a_1}{1 - r} \quad \text{if } |r| < 1$$

Binomial Theorem

$$(a + b)^n = a^n + \frac{n}{1!} a^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} a^{n-r}b^r + \cdots + b^n$$

Review Questions

1. An infinite sequence is a function whose domain is the set of _____.
2. The sixth term of the sequence $\left\{(-1)^{n+1}\frac{2n}{n+1}\right\}$ is _____.
3. The sequence $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \ldots$ has a general term given by _____.
4. The sequence $-6, 2, 10, \ldots$ is a(n) _____(a)_____ progression whose _____(b)_____ is 8.
5. For an arithmetic progression, the nth term is given by the formula _____(a)_____ and the sum of the first n terms is given by the formula _____(b)_____.
6. For a geometric progression, the nth term is given by the formula _____(a)_____ and the sum of the first n terms is given by the formula _____(b)_____.
7. The sixth term of the series $\sum_{i=1}^{10}(-1)^{i+1}x^{2i}$ is _____.
8. As n increases without bound, the sequence $\{1/n\}$ converges to _____.
9. The limit of any arithmetic progression with difference $d \neq 0$ _____(does)(does not)_____ exist.
10. $\sum_{n=1}^{\infty} r^n$ converges if _____.
11. The last term in the expansion of $(a + 2b)^{19}$ is _____.
12. True or false: For a sequence to have the number 2 as a limit, all the terms from some point on must be equal to 2. _____
13. The sequence $4, -4, 4, -4, \ldots$ _____(does)(does not)_____ converge.
14. For what value(s) of a does the sequence $a, -a, a, -a, \ldots$ converge? _____
15. The sum of the geometric series $2 + 1 + \frac{1}{2} + \cdots$ is _____.
16. The value of $4!$ is _____.

17. The number of terms in the expansion of $(4x - 1)^8$ is _____.

18. The geometric progression $1, 1 + r, (1 + r)^2, (1 + r)^3$ has a common ratio of _____.

Answers to Review Questions

1. Positive integers. 2. $-\frac{12}{7}$. 3. $n/2^n$. 4. (a) Arithmetic, (b) Common difference.
5. (a) $a_n = a_1 + (n - 1)d$, (b) $S_n = n(a_1 + a_n)/2$.
6. (a) $a_n = a_1 r^{n-1}$, (b) $S_n = a_1(1 - r^n)/(1 - r)$. 7. $-x^{12}$. 8. 0. 9. Does not.
10. $|r| < 1$. 11. $2^{19}b^{19}$. 12. False (to see this, consider the counterexample $\{2 - 1/n\}$).
13. Does not. 14. $a = 0$. 15. 4. 16. 24. 17. 9. 18. $1 + r$.

Review Problems

In Problems **1** *and* **2**, *find the tenth term of the given arithmetic progression.*

1. $3, 8, 13, \ldots$.

2. $7, 4, 1, \ldots$.

In Problems **3** *and* **4**, *find the sixth term of the given geometric progression.*

3. $8, -2, \frac{1}{2}, \ldots$.

4. $3, 12, 48, \ldots$.

In Problems **5** *and* **6**, *evaluate.*

5. $\sum_{n=1}^{5} (n^2 - 2n)$.

6. $\sum_{n=0}^{4} (-1)^n(2^n - 1)$.

7. Find $1 + \frac{1}{2} + (\frac{1}{2})^2 + \cdots + (\frac{1}{2})^6$.
8. Find $1 + 3 + 5 + \cdots + 33$.
9. How many terms of the sequence $-16, -12, -8, \ldots$ must be added to give a sum of 44?
10. Find the sum of all odd integers between 30 and 60.
11. The twelfth term of an arithmetic progression is -12 and the twenty-third term is 20. Find the sixteenth term.
12. The first term of a geometric progression is 36 and the third term is $\frac{1}{4}$. Find the fifth term.
13. Find the sum $1 + \frac{1}{4} + \frac{1}{16} + \cdots$.
14. Find the sum $1 - \frac{1}{4} + \frac{1}{16} - \cdots$.
15. A ball is released from an initial height of 8 m. After each contact with the floor, the ball rebounds to a height that is three-fourths of the height from which it last fell. What height does the ball reach on the fifth bounce?
16. For the ball in Problem 15, what is the total distance that the ball travels before coming to rest?

In Problems **17–20**, *find the limit, if it exists.*

17. $\lim_{n\to\infty} \frac{2}{n^2}$.

18. $\lim_{n\to\infty} \left(\frac{3}{2}\right)^n$.

19. $\lim_{n\to\infty} \frac{6n^2 + 5n}{1 - 2n^2}$.

20. $\lim_{n\to\infty} \frac{4n^2 + 2n + 5}{n^3 + 4n}$.

In Problems **21–24**, *use the binomial theorem to expand each expression.*

21. $(x - 4)^4$.

22. $(2x + 1)^5$.

23. $(x^3 + 2y)^3$.

24. $\left(1 - \frac{x}{3}\right)^6$.

25. Find the middle term of $(2 - \frac{1}{2}x^2)^{12}$.

26. By using the binomial series, approximate $\sqrt[3]{124}$ to three decimal places.

27. Find $\sum_{k=1}^{\infty} (\frac{4}{5})^k$.

28. In the series $1 + \frac{1}{2} + \frac{1}{4} + \cdots$, what is the numerical difference between the nth partial sum and the limit of the nth partial sum?

29. The sum of an infinite geometric series is 6 and the first term is 2. What is the second term?

30. The value of the fifth term in the expansion of $(a + 2)^5$ is 48 for a particular value of a. Find this value.

31. A 12-hour clock strikes once at 1 o'clock, twice at 2 o'clock, and so on. How many strikes will it make in 24 *consecutive* hours?

21.1 THE CONIC SECTIONS

In Chapter 8 the properties of straight lines were determined from their algebraic equations. In the following sections we extend the concepts of analytic geometry and concern ourselves with four curves that may be formed by the intersection of a plane with a *right circular cone*.

A **right circular cone** can be thought of as the surface generated by a rotating line that passes through a certain point, called the **vertex** of the cone, and a certain circle (see Fig. 21.1). Any line on the surface of the cone that intersects the vertex is called an **element**, or **generator**, of the cone. The **axis** of the cone is the line through the vertex that makes equal angles with all elements of the cone. The cone extends indefinitely on both sides of its vertex and can be spoken of as a *double right circular cone*. The part of the cone on one side of the vertex is called a **nappe**; the cone has an *upper nappe* and a *lower nappe*.

The **conic sections**, or **conics**, are the *circle*, the *parabola*, the *ellipse*, and the *hyperbola*. They are shown in Fig. 21.2, where the type of curve obtained depends on how the intersecting plane cuts the cone. An **ellipse** occurs when an intersecting plane is not parallel to any generator, as in Fig. 21.2(c). If, in addition, the intersecting plane is perpendicular to the axis of the cone, a **circle** occurs, as in (a). In a sense, a circle can be considered a special case of an ellipse. A **parabola** occurs when the intersecting plane is parallel to an element of the cone, as in (b). The **hyperbola** in (d) occurs when the intersecting plane cuts both nappes.

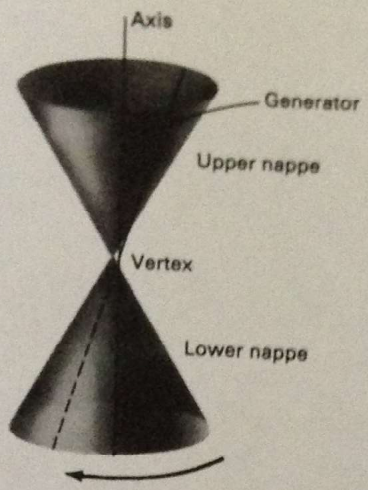


FIGURE 21.1

In comparison to these **regular conics**, certain special cuts in which the intersecting plane passes through the vertex give rise to the **degenerate conics**. If, in the case of Fig. 21.2(a), the intersecting plane also passes through the vertex, the resulting intersection is merely a point. The degenerate case of the hyperbola, namely two intersecting straight lines, occurs when the intersecting plane also contains the axis of the cone, as shown in Fig. 21.3(a). Finally, the parabola degenerates to a straight line if the intersecting plane also passes through the vertex, as shown in Fig. 21.3(b).

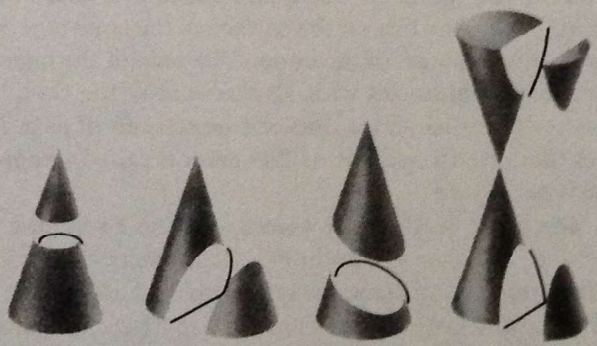
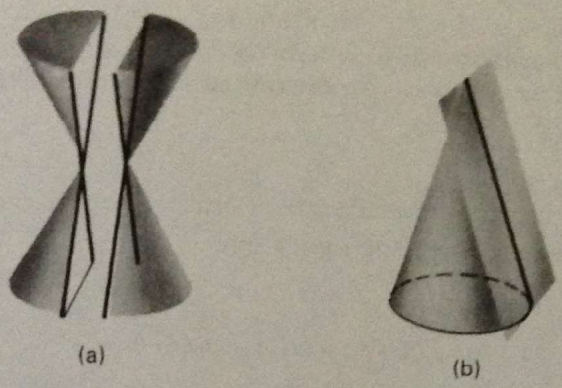


FIGURE 21.2



(a) (b)

FIGURE 21.3

21.2 THE DISTANCE FORMULA

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points in the xy-plane, a formula can be derived for the distance between them—that is, for the length d of the line segment joining the points (see Fig. 21.4). Through P_1 and P_2 we construct horizontal and vertical

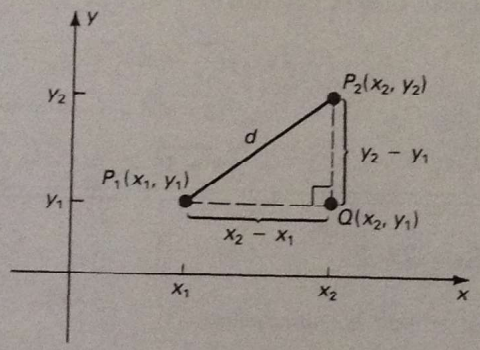


FIGURE 21.4

segments, respectively, which intersect at $Q(x_2, y_1)$. Thus a right triangle P_1QP_2 is formed; its hypotenuse has length d and its sides have lengths $x_2 - x_1$ and $y_2 - y_1$. By the Pythagorean theorem,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

Since d cannot be negative, we obtain the *distance formula*:

DISTANCE FORMULA

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Because $(x_2 - x_1)^2 = (x_1 - x_2)^2$ and $(y_2 - y_1)^2 = (y_1 - y_2)^2$, the order of the subscripts in the distance formula is immaterial. Thus, given any two points, the distance between these points can be determined by choosing either point as (x_1, y_1) in the formula.

Example 1

Find the distance between the given points.

a. $(5, -2)$ and $(-3, 4)$.

Solution Letting $(5, -2) = (x_1, y_1)$ and $(-3, 4) = (x_2, y_2)$, by the distance formula we have

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(-3 - 5)^2 + [4 - (-2)]^2} \\ &= \sqrt{(-8)^2 + (6)^2} = \sqrt{64 + 36} \\ &= \sqrt{100} = 10. \end{aligned}$$

If we choose $(-3, 4)$ as (x_1, y_1), we would get the same result.

b. $(3, -6)$ and the origin.

Solution Let $(0, 0) = (x_1, y_1)$ and $(3, -6) = (x_2, y_2)$. Then

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(3 - 0)^2 + (-6 - 0)^2} \\ &= \sqrt{9 + 36} = \sqrt{45} = 3\sqrt{5}. \end{aligned}$$

Problem Set 21.2

Find the distance between the given points.

1. $(2, 3), (5, 7)$.
2. $(4, 5), (12, 11)$.
3. $(0, 5), (2, -2)$.
4. $(1, 3), (1, 4)$.
5. $(-1, -2), (-3, -4)$.
6. $(-4, 4)$, origin.
7. $(2, -3), (-5, -3)$.
8. $(-\frac{3}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{2})$.
9. $(1, -\frac{1}{2})$, origin.
10. $(4, 0), (-1, \sqrt{11})$.
11. $(-5, 0), (0, -12)$.
12. $(0, -2), (-5, -2)$.

21.3 THE CIRCLE

A **circle** is the set of all points in a plane that are at a given distance from a fixed point in the plane.

The fixed point is called the **center**, and the given distance is the **radius** of the circle. In general, suppose that a circle has a radius of r and has its center at (h, k), as in Fig. 21.5. Let (x, y) be any point on the circle. Then its distance from (h, k)

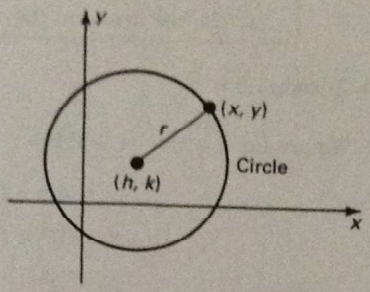


FIGURE 21.5

must be r. Applying the distance formula to (h, k) and (x, y) gives

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

Squaring both sides, we have

$$(x-h)^2 + (y-k)^2 = r^2.$$

Any point on the circle must satisfy this equation. Also, it can be shown that all points satisfying the equation lie on the circle. Thus we have the following formula.

$$(x-h)^2 + (y-k)^2 = r^2 \tag{1}$$

is the **standard form** of an equation of the circle with center (h, k) and radius r.

Example 1

Find the standard form of an equation of the circle with the given center and radius.

a. Center (0, 0) and radius $\sqrt{3}$.

Solution We use Eq. 1 with $h = 0$, $k = 0$, and $r = \sqrt{3}$.

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-0)^2 + (y-0)^2 = (\sqrt{3})^2$$

$$x^2 + y^2 = 3.$$

In general,

$$x^2 + y^2 = r^2$$

is the standard form of an equation of the circle with center at the origin and radius r. (2)

b. Center $(-3, 2)$ and radius 4.

Solution We use Eq. 1 with $h = -3$, $k = 2$, and $r = 4$.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$[x - (-3)]^2 + (y - 2)^2 = 4^2$$
$$(x + 3)^2 + (y - 2)^2 = 16.$$

Example 2

Describe the graph of the given equation.

a. $(x - 1)^2 + (y + 4)^2 = 9$.

Solution This has the form of Eq. 1. By writing this equation as

$$(x - 1)^2 + [y - (-4)]^2 = (3)^2,$$

we see that $h = 1$, $k = -4$, and $r = 3$. Thus the graph is a circle with center $(1, -4)$ and radius 3.

b. $x^2 + y^2 = 5$.

Solution This has the form of Eq. 2 with $r^2 = 5$. Thus $r = \sqrt{5}$. The graph is a circle with center at the origin and radius $\sqrt{5}$.

If we expand the terms in the standard form of a circle, we have

$$(x - h)^2 + (y - k)^2 = r^2$$
$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$
$$x^2 + y^2 + (-2h)x + (-2k)y + (h^2 + k^2 - r^2) = 0 \quad \text{[rearranging]}.$$

Because h, k, and r are constants, this equation has the form

$$x^2 + y^2 + Dx + Ey + F = 0, \tag{3}$$

where D, E, and F are constants. Equation 3 is called the **general form** of an equation of a circle. If an equation cannot be put in this form, its graph is not a circle.

Example 3

Find the general form of an equation of the circle with center (1, 4) and radius 2.

Solution The standard form is

$$(x-1)^2+(y-4)^2=2^2.$$

Expanding gives

$$x^2-2x+1+y^2-8y+16=4$$
$$x^2+y^2-2x-8y+13=0,$$

which is the general form.

We can determine the center and radius of a circle given in general form by using the method of completing the square to express the circle in standard form. For example, the equation

$$x^2+y^2-6x+10y+5=0$$

has the general form of Eq. 3. Regrouping gives

$$(x^2-6x)+(y^2+10y)=-5.$$

Completing the squares in x and y, we have

$$(x^2-6x+9)+(y^2+10y+25)=-5+9+25,$$

or the standard form

$$(x-3)^2+(y+5)^2=29.$$

Thus the graph of the given equation is a circle with center $(3,-5)$ and radius $\sqrt{29}$.

Example 4

Describe the graphs of the following equations.

a. $x^2+y^2-y=0$.

Solution We complete the square.

$$x^2+y^2-y=0$$
$$x^2+(y^2-y)=0$$
$$x^2+(y^2-y+\tfrac{1}{4})=\tfrac{1}{4} \qquad \text{[completing the square]}$$
$$(x-0)^2+(y-\tfrac{1}{2})^2=(\tfrac{1}{2})^2 \qquad \text{[standard form].}$$

The graph is a circle with center $(0,\frac{1}{2})$ and radius $\frac{1}{2}$.

b. $2x^2 + 2y^2 + 8x - 3y + 5 = 0$.

Solution To complete the squares, we first get the coefficients of the x^2- and y^2-terms to be 1. Thus we divide both sides by 2.

$$x^2 + y^2 + 4x - \tfrac{3}{2}y + \tfrac{5}{2} = 0$$

$$(x^2 + 4x) + (y^2 - \tfrac{3}{2}y) = -\tfrac{5}{2} \qquad \text{[regrouping]}$$

$$(x^2 + 4x + 4) + (y^2 - \tfrac{3}{2}y + \tfrac{9}{16}) = -\tfrac{5}{2} + 4 + \tfrac{9}{16} \qquad \text{[completing the squares]}$$

$$(x + 2)^2 + (y - \tfrac{3}{4})^2 = \tfrac{33}{16}.$$

Thus the graph is a circle with center $(-2, \frac{3}{4})$ and radius $\sqrt{\frac{33}{16}}$ or $\sqrt{33}/4$.

Every circle has an equation of the general form $x^2 + y^2 + Dx + Ey + F = 0$. But an equation of this form does not always have a circle as its graph. The graph could be a point, or there may not be any graph at all. For example, the equation

$$x^2 + y^2 - 2x - 6y + 10 = 0 \tag{4}$$

may be written

$$(x^2 - 2x + 1) + (y^2 - 6y + 9) = -10 + 1 + 9$$

$$(x - 1)^2 + (y - 3)^2 = 0.$$

This implies a center at (1, 3), but a radius of 0. Thus the graph is the single point (1, 3). Also, if the 10 in Eq. 4 were replaced by 11, then we would have the equation

$$(x - 1)^2 + (y - 3)^2 = -1.$$

This implies that r^2 is -1, a negative number. But, for a circle, r^2 cannot be negative. Thus the equation does not define a circle. In fact, because the left side of the equation is a sum of squares, it can never have a negative value. Therefore the equation has no graph.

Problem Set 21.3

In Problems **1–6**, *find the standard and general forms of an equation of the circle with the given center C and radius r.*

1. $C = (2, 3)$, $r = 6$.
2. $C = (4, -5)$, $r = 2$.
3. $C = (-1, 6)$, $r = 4$.
4. $C = (-2, -3)$, $r = 1$.
5. $C = (0, 0)$, $r = \frac{1}{2}$.
6. $C = (3, 0)$, $r = \sqrt{3}$.

In Problems **7–12**, *give the center C and radius r of the circle with the given equation. Also, sketch the circle.*

7. $x^2 + y^2 = 9$
8. $(x - 1)^2 + (y - 2)^2 = 1$.
9. $(x - 3)^2 + (y + 4)^2 = 2$.
10. $(x + 6)^2 + (y + 1)^2 = 3^2$.
11. $(x + 2)^2 + y^2 = 1$.
12. $x^2 + (y - 3)^2 = 16$.

In Problems **13–24**, *describe the graph of the given equation.*

13. $x^2 + y^2 - 2x - 4y - 4 = 0.$

14. $x^2 + y^2 + 4x - 6y + 9 = 0.$

15. $x^2 + y^2 + 6y + 5 = 0.$

16. $x^2 + y^2 - 12x + 27 = 0.$

17. $x^2 + y^2 + 2x - 2y + 3 = 0.$

18. $x^2 + y^2 + 6x - 2y - 15 = 0.$

19. $x^2 + y^2 - 14x + 4y + 38 = 0.$

20. $x^2 + y^2 + 4y - 8x + 21 = 0.$

21. $x^2 + y^2 - 3x + y + \frac{5}{2} = 0.$

22. $9x^2 + 9y^2 - 6x + 18y + 9 = 0.$

23. $2x^2 + 2y^2 - 4x + 7y + 2 = 0.$

24. $16x^2 + 16y^2 + 24x - 48y - 3 = 0.$

Recall that in Chapter 12 we considered nonlinear systems of equations. In Problems **25–28**, *you are given a system of two equations. Use a rough sketch to predict the number of different real solutions of each system.*

25. $\begin{cases} x^2 + y^2 = 4 \\ x^2 - 2x + y^2 - 2y = 2. \end{cases}$

26. $\begin{cases} x^2 - 4x + y^2 = -3 \\ x^2 - 10x + y^2 = -21. \end{cases}$

27. $\begin{cases} x^2 + y^2 = 1 \\ x^2 - 6x + y^2 = -8. \end{cases}$

28. $\begin{cases} x^2 + y^2 = 9 \\ x - y = 1. \end{cases}$

29. Which of the following equations are *not* equations of circles?

(a) $x^2 + y^2 = 4.$

(b) $x^2 + 2y^2 = 4.$

(c) $x^2 - (y - 2)^2 = 4.$

(d) $x^2 + 2xy + y^2 = 4.$

(e) $2x^2 + 2y^2 = 4.$

(f) $2x + 2y = 4.$

(g) $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{1}{4}.$

30. A particle moves in a coordinate plane so that it is always twice as far from (4, 0) as from (1, 0). Determine whether its path is a circle.

31. A racing automobile travels around a circular track at a speed of 52.80 m/s and makes one revolution every 39.25 s. Taking the origin of a coordinate system at the center of the track and approximating π by 3.14, determine an equation of the path of the automobile.

32. When a particle with mass m and charge q enters a magnetic field of induction B with a velocity v at right angles to B, it can be shown that the particle will travel in a circle of radius r, where

$$r = \frac{mv}{qB}.$$

This is one of the basic operating principles of a mass spectrograph and an important consideration in the design of cyclotrons. If a singly charged lithium ion enters a magnetic field of induction $B = 0.4$ webers/m^2 with a speed of 1.17×10^5 m/s, determine an equation of its circular path. For this ion, $q = 1.60 \times 10^{-19}$ C and $m = 1.16 \times 10^{-26}$ kg. (Assume that r is in meters.)

21.4 THE PARABOLA

A **parabola** is the set of all points in a plane that are equidistant from a given straight line and a given point not on the line.

The given point is called the **focus** and the given line the **directrix**. Let $2p$, where $p > 0$, be the distance from the focus to the directrix. In Fig. 21.6, for convenience we choose the focus at $F(0, p)$ and the directrix having equation $y = -p$.

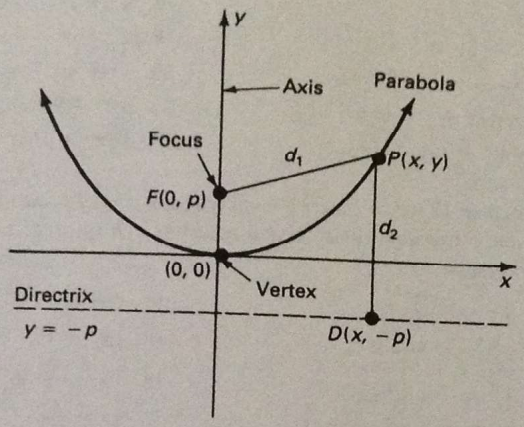


FIGURE 21.6

The line through the focus and perpendicular to the directrix is called the **axis** of the parabola. The axis is a *line of symmetry* in the sense that if a mirror were placed along the axis, then half of the parabola is the mirror image of the other half. The origin lies on the parabola, since it is the midpoint of the segment of the axis that connects the focus and the directrix.

To find an equation of the parabola, we first select an arbitrary point $P(x, y)$ on the parabola. It should be clear that the point of intersection of the directrix and the perpendicular to the directrix from P is $D(x, -p)$, as shown. From the definition of a parabola and the distance formula, we have

$$d_1 = d_2,$$

$$\sqrt{(x-0)^2 + (y-p)^2} = \sqrt{(x-x)^2 + (y+p)^2}.$$

Squaring both sides yields

$$x^2 + (y-p)^2 = (y+p)^2,$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2.$$

Simplifying gives

$$\boxed{x^2 = 4py.}$$

The coordinates of every point on the parabola satisfy this equation. It can also be shown that any point with coordinates that satisfy $x^2 = 4py$ is a point on the parabola. The equation $x^2 = 4py$ is called the **standard form** of an equation of the parabola with focus at $(0, p)$ and directrix $y = -p$.

The point at which a parabola intersects its axis is called the **vertex** of the parabola. It is the midpoint of the segment of the axis that joins the focus and the directrix. The vertex of the parabola described above is at the origin.

More generally, a parabola with vertex at the origin and a coordinate axis for its axis has four possible orientations, the equations of which can be derived in the manner above. The possibilities are indicated in Fig. 21.7. There is no need for you to

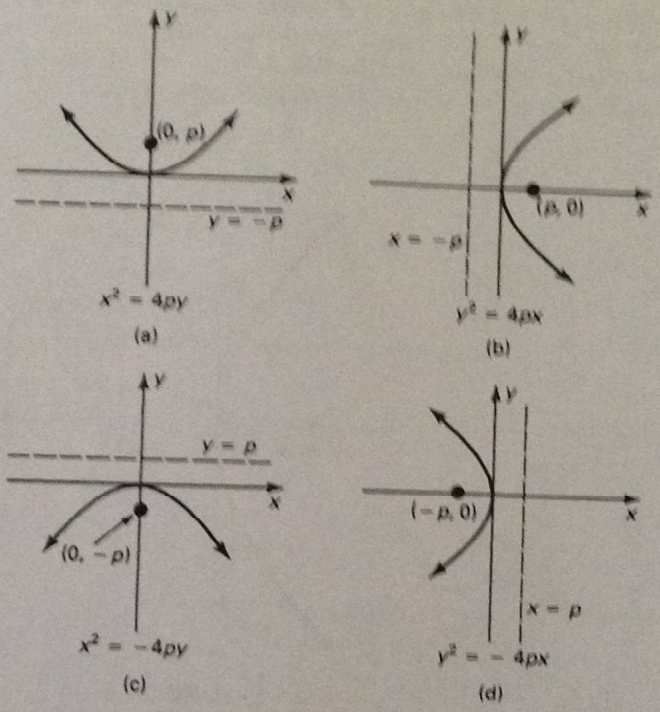


FIGURE 21.7

memorize which type of parabola corresponds to each equation, for the equation itself reveals the orientation of the parabola it represents. Consider, for example, Fig. 21.7(b) and the equation $y^2 = 4px$. Because p is considered positive and y^2 is never negative, it follows that x [or $y^2/(4p)$] is never negative. Thus the parabola opens to the right, as the graph indicates. You are urged to interpret mentally the remaining graphs and equations of Fig. 21.7 in a similar manner. In each case, *p is the distance from the vertex to the focus.* Note that Figs. 21.7(b) and (d) are *not* graphs of functions of x.

Example 1

Find the vertex, focus, and equation of the directrix of the parabola given by $y^2 = -16x$. Sketch the graph.

Solution This equation has the standard form $y^2 = -4px$, so the vertex of the parabola is at the origin. Because $x = y^2/-16$, x is never positive, so the parabola must open to the left, as shown in Fig. 21.8. Also,

$$4p = 16$$

$$p = 4.$$

Thus the distance from the vertex to the focus is 4, so the focus is at $(-4, 0)$. The equation of the directrix is $x = 4$. To aid you in sketching the graph, we remark that *the "width" of a parabola at its focus is 4p.** This means that the parabola $y^2 = -16x$ is 16 units wide along a line perpendicular to its axis and passing through $(-4, 0)$. From this information we plot the points $(-4, 8)$ and $(-4, -8)$. The graph is given in Fig. 21.8.

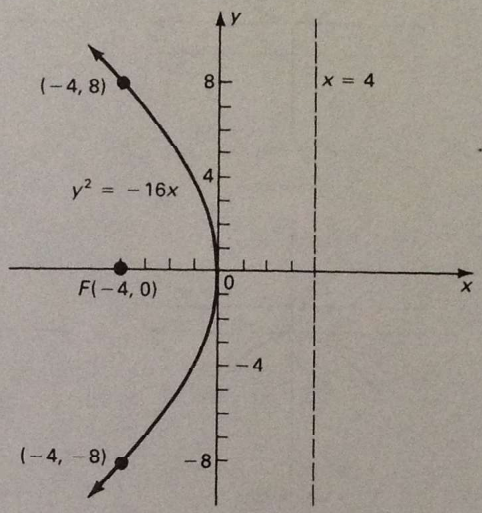


FIGURE 21.8

Example 2

Determine an equation of the parabola with vertex at $V(0, 0)$ and focus at $F(0, -\frac{3}{2})$.

Solution Refer to Fig. 21.9. From the locations of the focus and vertex, the parabola must open downward and, as a result, y is never positive. Hence, an equation of the parabola has the form $x^2 = -4py$. Because p, the distance from the focus to the vertex, is $\frac{3}{2}$, the required equation is

$$x^2 = -4(\tfrac{3}{2})y$$

$$x^2 = -6y.$$

* The line segment passing through the focus, perpendicular to the axis, and with its ends on the parabola is called the *latus rectum.* Thus the width of the latus rectum is $4p$.

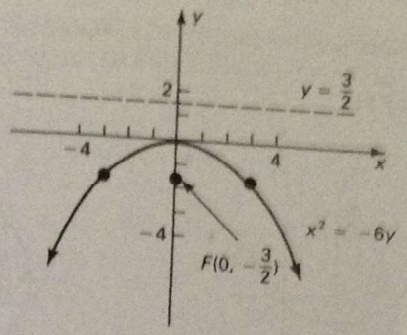


FIGURE 21.9

Consider the parabola with its axis parallel to the x-axis and its vertex at the point $V(h, k)$ as shown in Fig. 21.10, where $2p$, $p > 0$, is the distance from the focus to the directrix. The focus must be at $F(h + p, k)$ and the directrix must be the line $x = h - p$. From the dimensions shown in the diagram, we can deduce that the point of intersection of the directrix and a perpendicular to the directrix from an arbitary point (x, y) on the parabola is $D(h - p, y)$. From the definition of a parabola,

$$d_2 = d_1$$

$$\sqrt{[x - (h + p)]^2 + (y - k)^2} = \sqrt{[x - (h - p)]^2 + (y - y)^2}.$$

Squaring both sides and simplifying yields

$$y^2 - 2ky + k^2 = 4px - 4hp,$$

which can be written in the *standard form*

$$\boxed{(y - k)^2 = 4p(x - h).}$$

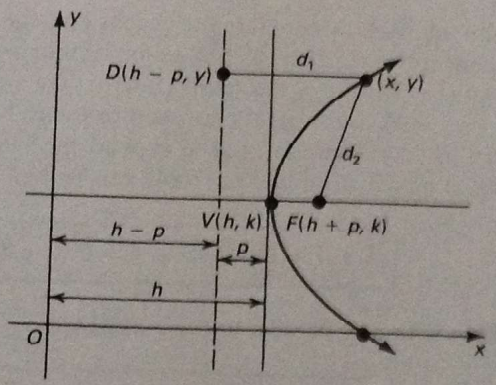


FIGURE 21.10

In a similar manner the standard forms of equations of other parabolas with vertex at (h, k) and axis parallel to a coordinate axis can be derived. The four possibilities are shown in Fig. 21.11.

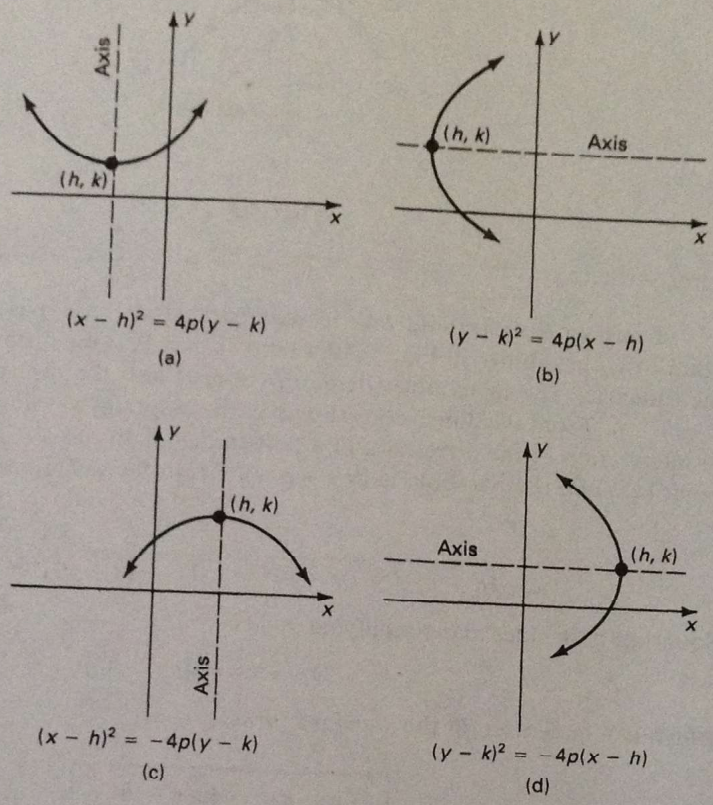


FIGURE 21.11

You should carefully compare the graphs and equations in Figs. 21.7 and 21.11 and observe the obvious patterns. For example, moving (or *translating*) the vertex of the parabola $y^2 = 4px$ from the point $(0, 0)$ to the point (h, k) gives the parabola $(y - k)^2 = 4p(x - h)$. Both parabolas open to the right.

It is left as an exercise for you to expand the standard forms and show that a general form of an equation of a parabola can be written

$$\boxed{Ax^2 + Dx + Ey + F = 0, \quad A, E \neq 0} \tag{1}$$

for a parabola with axis parallel to the y-axis, and

$$\boxed{Cy^2 + Dx + Ey + F = 0, \quad C, D \neq 0} \tag{2}$$

for a parabola with axis parallel to the x-axis. Conversely, for A, C, D, and E unequal to zero, Eqs. 1 and 2 satisfy the requirements of the locus for a parabola and can be transformed into standard form.

In the case that $E = 0$ in Eq. 1 or $D = 0$ in Eq. 2, three degenerate cases may arise. For example, if $E = 0$ in Eq. 1, then

$$Ax^2 + Dx + F = 0.$$

Let us consider different values of A, D, and F.

1. For $x^2 + 4x + 4 = 0$,

$$(x + 2)(x + 2) = 0$$
$$x = -2.$$

Thus the set of all points satisfying the given equation is the straight line $x = -2$.

2. For $x^2 - 2x - 8 = 0$,

$$(x - 4)(x + 2) = 0$$
$$x = 4 \quad \text{or} \quad x = -2.$$

Thus two distinct parallel straight lines occur. This is the only degenerate case that cannot be illustrated by a conic section.

3. For $x^2 + 4x + 4 = 0$,

$$x^2 + 4x + 4 = -1$$
$$(x + 2)^2 = -1,$$

so no parabola exists.

Example 3

Find the standard form of an equation of the parabola with vertex at $V(-1, 4)$ and focus at $F(-1, 1)$.

Solution By indicating the data in the plane (Fig. 21.12), we conclude that the parabola must open downward. The equation is of the form $(x - h)^2 = -4p(y - k)$ with $h = -1$, $k = 4$, and $p = 4 - 1 = 3$. Thus the equation is

$$(x + 1)^2 = -4(3)(y - 4)$$
$$(x + 1)^2 = -12(y - 4).$$

The graph and directrix are shown in Fig. 21.12. Note that the equation of the directrix is $y = 7$.

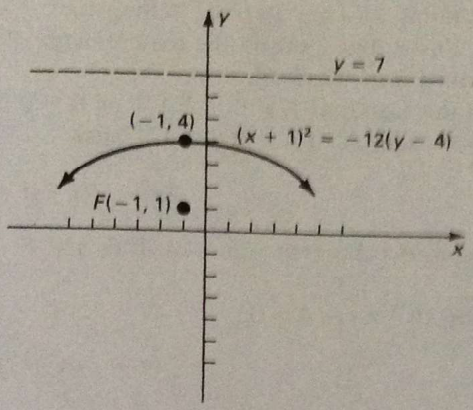


FIGURE 21.12

Example 4

Find the coordinates of the focus and vertex and the equation of the directrix of the parabola

$$y^2 - 8y - 8x + 24 = 0.$$

Solution Rearranging terms and completing the square in y, we have

$$y^2 - 8y + 16 = 8x - 24 + 16$$

$$= 8x - 8,$$

$$(y - 4)^2 = 8(x - 1).$$

Hence, the vertex is the point $(1, 4)$ and the parabola opens to the right. Also, since $4p = 8$, $p = 2$ and the focus lies two units to the right of the vertex at the point $(3, 4)$. A point on the directrix lies two units to the left of the vertex, so the equation of the directrix is $x = -1$.

Example 5

The height s of a ball thrown vertically upward from the ground is given by the equation of motion

$$s = 19.6t - 4.9t^2,$$

where s is in meters and t is elapsed time in seconds. Find the maximum height of the ball.

Solution Note that s is a quadratic function of t. Observe that we can write

$$s = 19.6t - 4.9t^2$$

$$s = -4.9(t^2 - 4t)$$

$$s - 19.6 = -4.9(t^2 - 4t + 4).$$

Thus

$$(t-2)^2 = -\frac{1}{4.9}(s-19.6).$$

This is the equation of a parabola, opening downward, with vertex at (2, 19.6), as shown in Fig. 12.13 in Sec. 12.4. Hence the maximum height of the ball is 19.6 m.

Example 5 supplements our discussion of quadratic functions in Sec. 12.4, where it was stated that the graph of a quadratic function is a parabola. It can be shown that if the graph of a quadratic function $y = f(x)$ crosses the x-axis at the points $(x_1, 0)$ and $(x_2, 0)$, then the variable y has its maximum (or minimum) value when

$$x = \frac{x_1 + x_2}{2}. \tag{3}$$

This fact will be used in the following example.

Example 6

Find the maximum value of the quadratic function $y = 20x - x^2$.

Solution To find the values of x where its graph crosses the x-axis, we set $y = 0$. Then

$$0 = 20x - x^2$$
$$0 = x(20 - x),$$

so

$$x = 0 \quad \text{or} \quad x = 20.$$

By Eq. 3, the maximum value is attained when

$$x = \frac{0 + 20}{2} = 10.$$

Hence the maximum value is

$$y = 20x - x^2$$
$$= 20(10) - (10)^2$$
$$= 100.$$

You should use this technique to verify the result in Example 5.

Problem Set 21.4

In Problems **1–12**, *find the coordinates of the vertex V and focus F, an equation of the directrix D, and sketch each parabola.*

1. $y^2 = 4x$.
2. $y^2 = -6x$.
3. $x^2 = -8y$.
4. $y^2 = -x$.
5. $x^2 = 2y$.
6. $(x + 2)^2 = 4(y - 7)$.
7. $(y + 2)^2 = \frac{1}{2}x$.
8. $4x^2 = 3(y - 1)$.
9. $(x - 4)^2 = 8(y + 3)$.
10. $(y - 7)^2 = 12(x - 4)$.
11. $(y + 2)^2 = -\frac{1}{4}(x - 1)$.
12. $y = x^2$.

In Problems **13–22**, *determine the standard form of an equation of the parabola that has the given properties.*

13. Focus (0, 3), directrix $y = -5$.
14. Vertex (2, 4), directrix $y = 6$.
15. Focus (1, 4), vertex (3, 4).
16. Focus (0, 5), directrix $x = -10$.
17. Vertex $(-3, 2)$, directrix $y = 4$.
18. Focus $(-2, -2)$, vertex $(-2, -6)$.
19. Focus $(0, \frac{3}{2})$, directrix $y = -\frac{3}{2}$.
20. Vertex (3, 1), directrix $x = 6$.
21. Focus $(-3, -2)$, vertex $(-4, -2)$.
22. Focus $(0, -4)$, directrix $y = 4$.

In Problems **23–27**, *transform the given equation into standard form and determine the coordinates of the focus and vertex.*

23. $y^2 - 6y + 4x + 1 = 0$.
24. $2y^2 + 4y - x - 4 = 0$.
25. $3x^2 - 12x - y + 12 = 0$.
26. $x^2 + 3y - 8x + 19 = 0$.
27. $y^2 + 4y - x + 5 = 0$.

In Problems **28–31**, *use a rough graph of the given system to predict how many different real solutions exist.*

28. $\begin{cases} y^2 = x - 1, \\ y = x^2 - 2x + 1. \end{cases}$
29. $\begin{cases} y = x^2 - 3x, \\ x + y = -4. \end{cases}$
30. $\begin{cases} x = (y - 1)^2, \\ x - y = 2. \end{cases}$
31. $\begin{cases} y^2 = 2x + 1, \\ x^2 + y^2 = 9. \end{cases}$

32. The power P developed in a resistor of resistance R ohms carrying a current of i amperes is given by $P = i^2R$. If a resistor has a resistance of $10\,\Omega$, sketch a graph of power as a function of current.

33. When an object is thrown straight upward with an initial velocity of 58.8 m/s, its height h, in meters, as a function of time t, in seconds, is given by $h = 58.8t - 4.9t^2$. At what time does the object reach its maximum height and what is that height?

34. The displacement s of an object from a reference point is given by the equation $s = 3t^2 - 24t + 103$, where s is in meters and t is in seconds. What is the minimum displacement of the particle from the reference point?

35. If a light source is placed between the focus and vertex of a parabolic mirror, the rays of light diverge after reflection, whereas if the light source is placed outside the focus, the rays converge after reflection. This is how automobile headlights can give a broad beam of light

(high beams) from one filament and a narrow beam of light (low beams) from a second filament. If the equation of a cross-sectional view of a parabolic mirror is $y^2 = 20x$, where x and y are in centimeters, what may the distance be between the vertex of the mirror and a filament if the light rays are to diverge after reflection from the mirror?

36. When an object of mass $\frac{1}{8}$ kg moves in a straight line on a horizontal surface 4 m above the ground, its energy E, in the absence of rotation, can be given by $E = 4.9 + \frac{1}{16}v^2$, where v is the speed of the object in meters per second, and E is measured in joules. The value of v is taken to be positive for one direction of motion and negative for the opposite direction. Sketch a graph of this energy function for the values of v from $v = -4$ m/s to $v = 4$ m/s.

21.5 THE ELLIPSE

> An **ellipse** is the set of all points in a plane such that for each point the sum of its distances from two fixed points is a constant.

The two fixed points in the definition are called the **foci** (the plural of focus) of the ellipse. Let $2c$, where $c > 0$, be the distance between the foci. For convenience we shall locate the foci on the x-axis at points $F_1(c, 0)$ and $F_2(-c, 0)$ (see Fig. 21.13).

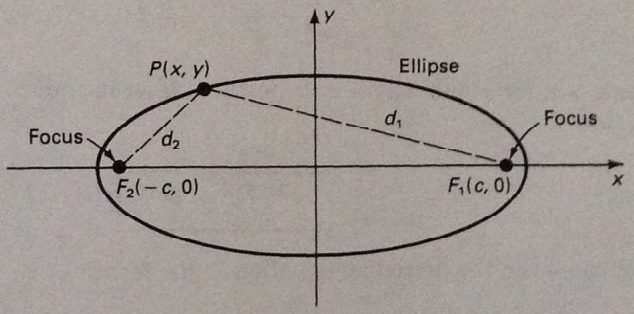


FIGURE 21.13

Furthermore, let the constant sum referred to in the definition be $2a$. To eliminate from our consideration those points on the line segment joining the foci, we must require that $2a > 2c$, or $a > c$. Selecting an arbitrary point $P(x, y)$ on the ellipse and applying the definition, we have

$$d_1 + d_2 = \text{constant}$$

$$\sqrt{(x-c)^2 + (y-0)^2} + \sqrt{(x+c)^2 + (y-0)^2} = 2a.$$

Rearranging terms, we have

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}.$$

Squaring both sides gives

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2,$$

which simplifies to

$$a^2 + cx = a\sqrt{(x+c)^2 + y^2}.$$

Squaring both sides, we have

$$a^4 + 2a^2cx + c^2x^2 = a^2(x^2 + 2cx + c^2 + y^2).$$

Rearranging terms and grouping give

$$a^4 - a^2c^2 + c^2x^2 - a^2x^2 - a^2y^2 = 0$$
$$a^2(a^2 - c^2) + x^2(c^2 - a^2) - a^2y^2 = 0.$$

Dividing both sides by $a^2(a^2 - c^2)$ gives

$$1 - \frac{x^2}{a^2} - \frac{y^2}{a^2 - c^2} = 0,$$

which can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

Because $a > c$, the number $a^2 - c^2$ is positive. If we introduce a new positive number b such that

$$\boxed{a^2 - c^2 = b^2,} \tag{1}$$

then we can write the preceding equation in the form

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.} \tag{2}$$

Thus the coordinates of every point on the given ellipse satisfy Eq. 2. Conversely, it can be shown that any point whose coordinates satisfy Eq. 2 is a point on the ellipse. Equation 2 is called the **standard form** of an equation of the ellipse with foci on the x-axis at $(c, 0)$ and $(-c, 0)$, where $a^2 = b^2 + c^2$. The equation $a^2 = b^2 + c^2$ implies that $a^2 > b^2$, so $a > b$.

The relationship between a, b, and c is easily seen from triangle OP_1F_1 in Fig. 21.14. From Eq. 2, if $y = 0$, then $x = \pm a$; thus the x-intercepts are $(\pm a, 0)$. If $x = 0$, then $y = \pm b$; the y-intercepts are $(0, \pm b)$. Segment V_1V_2, which has length $2a$ and passes through the foci, is called the **major axis** and its midpoint is called the **center** of

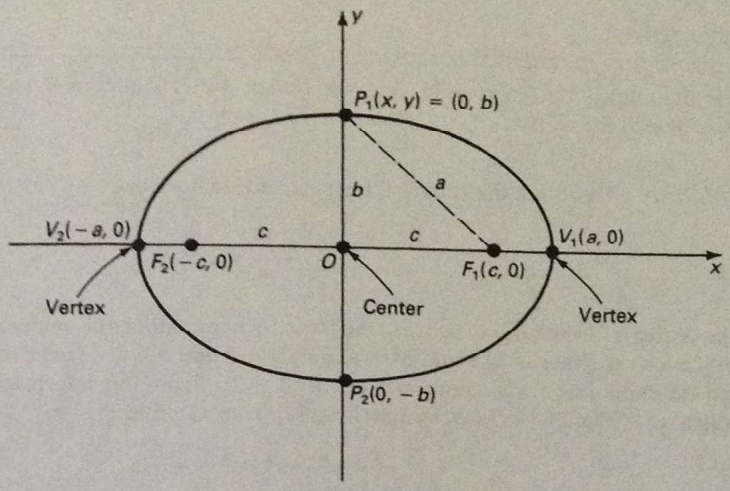


FIGURE 21.14

the ellipse. The ellipse shown has its center at the origin. Segment P_1P_2, which has length $2b$, is called the **minor axis**. We speak of a as the **semimajor axis**; *it is the distance from the center of the ellipse to the end of the major axis.* Similarly, we speak of b as the **semiminor axis**. Since $a > b$, the major axis is longer than the minor axis. The endpoints of the major axis, namely, V_1 and V_2, are called the **vertices** of the ellipse.

For an ellipse having center at the origin and foci on the y-axis [at $(0, c)$ and $(0, -c)$], the standard form is

$$\boxed{\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.} \tag{3}$$

Here, the major axis lies on the y-axis. The vertices are $(0, \pm a)$. Again, $b^2 = a^2 - c^2$.

In Eqs. 2 and 3, remember that since $a > b$, **the larger denominator is always a^2**. Thus, if the larger denominator is in the x^2-term, the major axis is horizontal. If it is in the y^2-term, the major axis is vertical.

In general, it can be shown that the standard form of an equation of an ellipse with center (h, k) and major axis horizontal is given by

$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,} \tag{4}$$

and for one where the major axis is vertical, the equation is

$$\boxed{\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.} \tag{5}$$

Example 1

For the ellipse $4x^2 + 9y^2 = 36$, find the center, vertices, foci, and endpoints of the minor axis. Sketch the ellipse.

Solution To obtain the standard form, we divide both sides by 36:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

The center is at the origin. Because the larger denominator (9) is under x^2, the major axis lies on the x-axis. Since $a^2 = 9$ and $b^2 = 4$, then $a = 3$ and $b = 2$. Hence, the vertices are $(\pm 3, 0)$ and the endpoints of the minor axis are at $(0, \pm 2)$. From this information we can sketch the graph, as in Fig. 21.15. Also, by rewriting Eq. 1 we can find c:

$$c^2 = a^2 - b^2 = 9 - 4 = 5$$

$$c = \sqrt{5}.$$

Thus the foci are at $(\pm\sqrt{5}, 0)$.

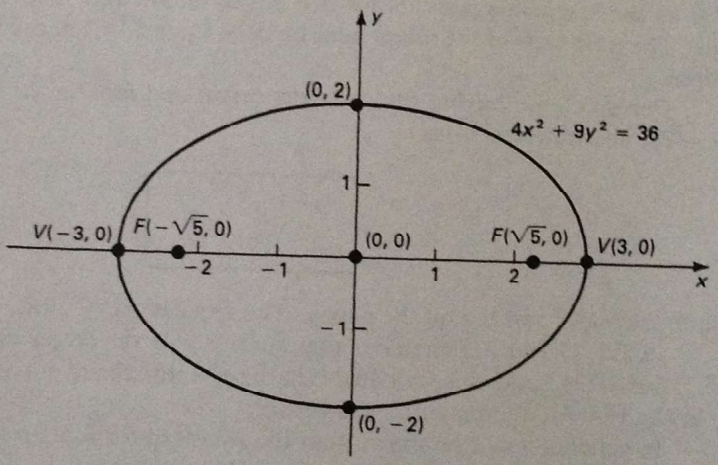


FIGURE 21.15

Example 2

Find an equation of the ellipse with major axis of length eight units, a focus at (2, 8), and center at (2, 5).

Solution An ellipse with a center at (2, 5) and a focus at (2, 8) must have a vertical major axis. The distance from the center to the given focus is three units. Thus $c = 3$. Also, since the

length of the major axis is $2a$ ($= 8$), then $a = 4$. Therefore, by Eq. 1, we have $b^2 = a^2 - c^2 = 16 - 9 = 7$. Thus, because $h = 2$ and $k = 5$, from Eq. 5 the standard form is

$$\frac{(x-2)^2}{7} + \frac{(y-5)^2}{16} = 1.$$

The graph is shown in Fig. 21.16.

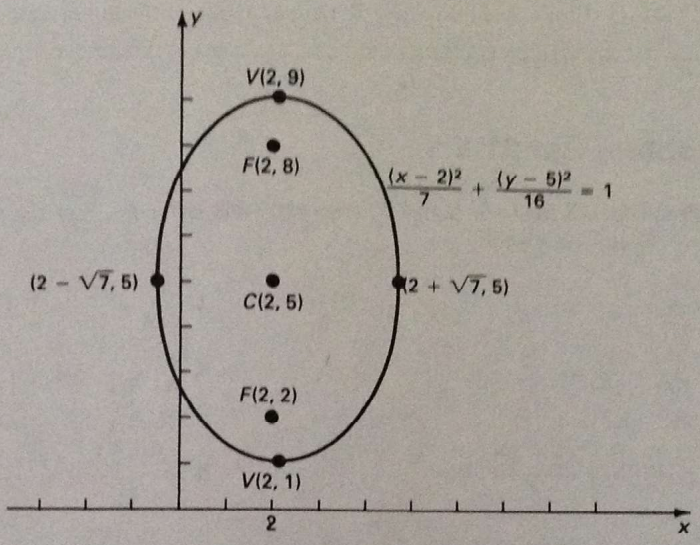


FIGURE 21.16

Example 3

Determine the center, foci, vertices, and endpoints of the minor axis of the ellipse

$$x^2 + 2y^2 + 4x - 4y + 2 = 0.$$

Solution First we put the equation into standard form. Rearranging and grouping terms, we can write

$$(x^2 + 4x) + 2(y^2 - 2y) = -2.$$

Completing the squares in x and y, we have

$$(x^2 + 4x + 4) + 2(y^2 - 2y + 1) = -2 + 4 + 2.$$

Note that when 1 was added to $y^2 - 2y$ to complete the square in y, it was equivalent to adding 2 to the left side of the equation. Hence 2 had to be added to the right side. This gives

$$(x + 2)^2 + 2(y - 1)^2 = 4.$$

Finally, dividing by 4 gives the standard form.

$$\frac{(x+2)^2}{4} + \frac{(y-1)^2}{2} = 1.$$

Clearly $a^2 = 4$, $b^2 = 2$, and $c^2 = a^2 - b^2 = 2$. Thus $a = 2$, $b = \sqrt{2}$, and $c = \sqrt{2}$. Because $h = -2$ and $k = 1$, the center is at $(-2, 1)$. Also, the major axis is horizontal. Because $c = \sqrt{2}$, the foci are at $(-2 \pm \sqrt{2}, 1)$. Also, because $a = 2$, the vertices are $(-2 \pm 2, 1)$, that is, at $(0, 1)$ and $(-4, 1)$. Lastly, because $b = \sqrt{2}$, the endpoints of the minor axis are at $(-2, 1 \pm \sqrt{2})$.

Problem Set 21.5

In Problems **1–8**, *find the center C, vertices V, and endpoints E of the minor axis of the given ellipse. Sketch each ellipse.*

1. $\frac{x^2}{25} + \frac{y^2}{16} = 1.$

2. $\frac{x^2}{144} + \frac{y^2}{169} = 1.$

3. $\frac{x^2}{100} + \frac{y^2}{144} = 1.$

4. $\frac{x^2}{49} + \frac{y^2}{4} = 1.$

5. $4x^2 + y^2 = 4.$

6. $16x^2 + 9y^2 = 144.$

7. $\frac{(x-2)^2}{9} + \frac{(y+3)^2}{4} = 1.$

8. $\frac{(x+2)^2}{9} + \frac{(y-3)^2}{25} = 1.$

In Problems **9–12**, *for each ellipse find the center C, vertices V, and endpoints E of the minor axis.*

9. $4x^2 + y^2 - 16x = 0.$

10. $2x^2 + y^2 + 8x + 4y + 6 = 0.$

11. $9x^2 + 25y^2 - 54x + 100y = 44.$

12. $x^2 + 2y^2 + 4x - 8y - 6 = 0.$

In Problems **13–21**, *determine the standard form of an equation of the ellipse with the given properties. Assume the center is at the origin unless otherwise stated.*

13. Vertex (6, 0), focus (5, 0).

14. Major axis of length 16, focus (6, 0).

15. Focus (−8, 1), minor axis of length 4, center (0, 1).

16. Focus (3, 3), center on y-axis, vertex (−5, 3).

17. Major axis of length 16 and horizontal, minor axis of length 8.

18. Vertex (3, 10), focus (3, −6), center on x-axis.

19. Vertex (0, 0), center (0, −8), minor axis of length 5.

20. Focus (−4, −5), major axis of length 14, center (−4, 1).

21. Minor axis horizontal and of length 4, focus (−2, −8), center (−2, −1).

22. The arch of a bridge has the shape of one-half an ellipse. The maximum height of the bridge is 50 m and the bridge has a span at water level of 120 m. If the origin of a coordinate system is midway between the ends of the bridge at water level, what is the equation of the ellipse? What is the equation if the origin of the coordinate system is at one end of the bridge at water level?

21.6 THE HYPERBOLA

A **hyperbola** is the set of all points in a plane such that for each point the difference between its distances from two fixed points, called **foci**, is a positive constant.

Suppose the distance between the two foci is $2c$, where $c > 0$. For convenience we shall locate the foci on the x-axis at points $F_1(c, 0)$ and $F_2(-c, 0)$ (see Fig. 21.17). Let the constant difference referred to in the definition be $2a$, where

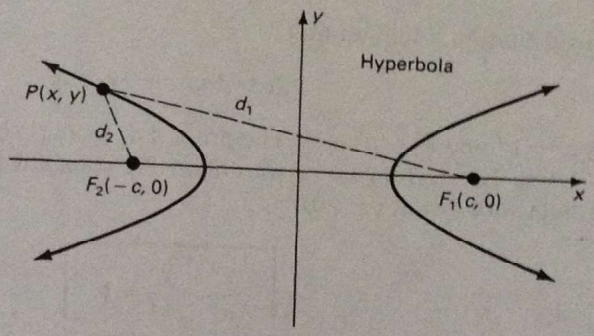


FIGURE 21.17

$a > 0$. Selecting a point $P(x, y)$ on the hyperbola, we have (by definition)

$$d_1 - d_2 = \pm 2a,$$

where the correct sign depends on which of d_1 or d_2 is the larger. Therefore, by the distance formula,

$$\sqrt{(x-c)^2 + (y-0)^2} - \sqrt{(x+c)^2 + (y-0)^2} = \pm 2a.$$

Using the same techniques as those used in the preceding section, we obtain

$$c^2x^2 - a^2x^2 - a^2y^2 = a^2c^2 - a^4,$$

or

$$x^2(c^2 - a^2) - a^2y^2 = a^2(c^2 - a^2).$$

Dividing both sides by $a^2(c^2 - a^2)$ gives

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1. \qquad (1)$$

As in the case of the derivation of an equation of an ellipse, it would seem natural to introduce a positive number b such that

$$\boxed{b^2 = c^2 - a^2.} \tag{2}$$

(Do not confuse this with $b^2 = a^2 - c^2$ for an ellipse.) However, we must be sure that $c^2 - a^2$ is always positive. It is shown in geometry that the difference between any two sides of a triangle is less than the third side. Hence, from Fig. 21.17, for the triangle F_1F_2P with $d_1 > d_2$ we have

$$2c > d_1 - d_2,$$

and by definition it follows that

$$2c > 2a, \quad \text{or} \quad c > a.$$

Thus $c^2 > a^2$ and $c^2 - a^2 > 0$. The proof if $d_2 > d_1$ is similar.

Using Eq. 2, from Eq. 1 we obtain the **standard form** of an equation of the hyperbola with foci at $(\pm c, 0)$:

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,} \tag{3}$$

where $c^2 = a^2 + b^2$.

When $y = 0$, then $x = \pm a$ and hence the hyperbola crosses the x-axis at the points $V_1(a, 0)$ and $V_2(-a, 0)$ (refer to Fig. 21.18). These points, those for which the hyperbola cuts the line through the two foci, are called the **vertices** of the hyperbola. The line segment V_1V_2 that joins the vertices is $2a$ units long and is called the **transverse axis**. The midpoint of the transverse is called the **center** of the hyperbola. In Fig. 21.18 the center is at the origin.

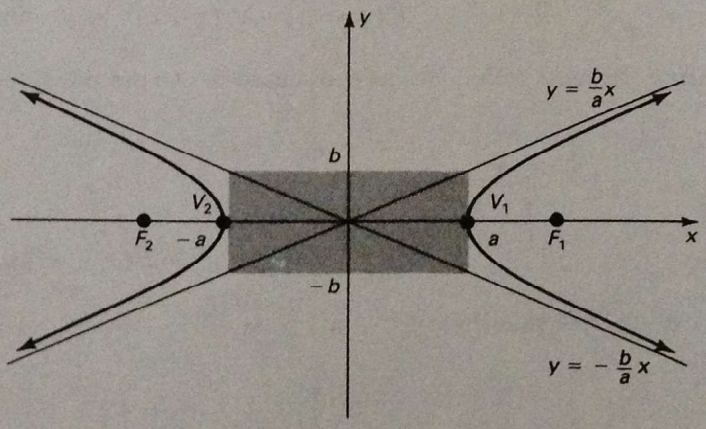


FIGURE 21.18

To determine more precisely the nature of a hyperbola and the geometrical significance of b, we solve Eq. 3 for y:

$$b^2x^2 - a^2y^2 = a^2b^2$$

$$y^2 = \frac{b^2x^2 - a^2b^2}{a^2} = \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right),$$

$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}.$$

As x increases or decreases without bound, then the values of a^2/x^2 get closer to zero and the expression under the radical sign approaches a value of 1. We can conclude that y will approach the values $(b/a)x$ and $-(b/a)x$. The straight lines $y = (b/a)x$ and $y = -(b/a)x$ are called **asymptotes** for the hyperbola. They are straight lines that provide a good approximation to the hyperbola when x is far from the origin. These results are indicated in Fig. 21.18.

From the diagram it should be clear that the asymptotes coincide, so to speak, with the diagonals of a rectangle of length $2a$ and width $2b$ and whose vertices are (a, b), $(-a, b)$, $(-a, -b)$, and $(a, -b)$. The line segment on the y-axis joining the points $(0, b)$ and $(0, -b)$ has length $2b$ and is called the **conjugate axis** of the hyperbola. Clearly, the asymptotes provide convenient guide lines for sketching a hyperbola. They are most easily sketched by drawing extended diagonals of the rectangle whose sides pass through the vertices and the endpoints of the conjugate axis. The rectangle and asymptotes are *not* part of the graph. It can be shown that *the equations of the asymptotes of the hyperbola can be found by replacing the 1 in the standard form by zero and solving for y.*

In a similar manner, it can be shown that an equation of a hyperbola with center at the origin, transverse axis along the y-axis and conjugate axis along the x-axis, is

$$\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.} \tag{4}$$

In this case the vertices are located at $V_1(0, a)$ and $V_2(0, -a)$, the foci are at $F_1(0, c)$ and $F_2(0, -c)$, and the lines $y = \pm ax/b$ are asymptotes.

Using the definition, we can show that the standard form of an equation of a hyperbola with center at (h, k) and transverse axis horizontal is given by

$$\boxed{\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,} \tag{5}$$

whereas if the transverse axis is vertical, the equation is

$$\boxed{\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.} \tag{6}$$

Note that a^2 is always associated with the positive term. Again, the equations of the asymptotes can be found by replacing the 1 in the standard form by zero and solving for y.

Finally, a general form of an equation of a hyperbola can be written

$$\boxed{Ax^2 + Cy^2 + Dx + Ey + F = 0,} \tag{7}$$

where A and C have opposite signs.

Example 1

Discuss and sketch the graph of the hyperbola $\dfrac{y^2}{4} - x^2 = 1$.

Solution From the standard forms of an equation of a hyperbola, a^2 is always associated with the positive term. In this case, $a^2 = 4$ and $b^2 = 1$; that is, $a = 2$, and $b = 1$. Also, $c^2 = a^2 + b^2 = 4 + 1 = 5$, so $c = \sqrt{5}$. Because the equation is of the form of Eq. 4, or, equivalently, the form of Eq. 6 with $h = k = 0$, we deduce that the hyperbola has vertices $V_1(0, 2)$ and $V_2(0, -2)$, foci $F_1(0, \sqrt{5})$ and $F_2(0, -\sqrt{5})$, a vertical transverse axis of length 4, and a horizontal conjugate axis of length 2. We can easily sketch the graph by first locating the vertices and endpoints of the conjugate axis. Then we construct the rectangle whose sides pass through these points. Next, we sketch the asymptotes by drawing extended diagonals of the rectangle. Finally, we draw the hyperbola which passes through the vertices and approaches the asymptotes. The results are shown in Fig. 21.19. Note that the equations of the asymptotes

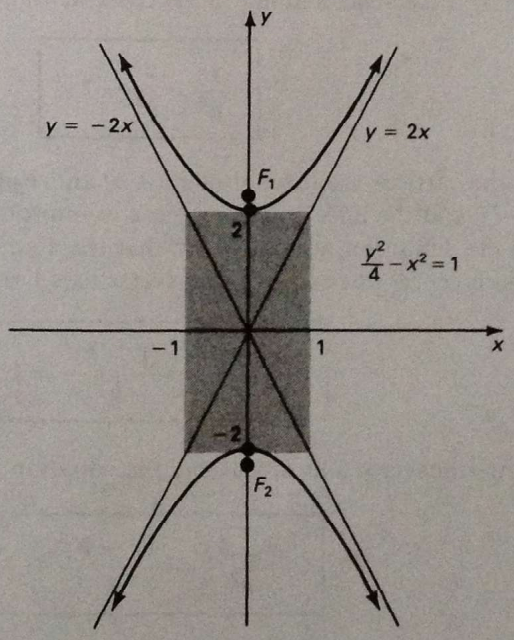


FIGURE 21.19

are $y = 2x$ and $y = -2x$. They are obtained by solving the equation $y^2/(4) - x^2 = 0$ for y.

$$\frac{y^2}{4} = x^2, \qquad y^2 = 4x^2, \qquad y = \pm 2x.$$

Example 2

Discuss and sketch the graph of the hyperbola

$$2x^2 - y^2 - 16x + 4y + 24 = 0.$$

Solution First we put the equation into standard form. The equation can be written

$$(2x^2 - 16x) - (y^2 - 4y) = -24.$$

Completing the squares and simplifying, we have

$$2(x^2 - 8x + 16) - (y^2 - 4y + 4) = -24 + 32 - 4$$

$$2(x - 4)^2 - (y - 2)^2 = 4$$

$$\frac{(x - 4)^2}{2} - \frac{(y - 2)^2}{4} = 1. \tag{8}$$

Equation 8 matches Eq. 5 where $h = 4$ and $k = 2$. Thus the center of the hyperbola is at (4, 2) and the transverse axis is horizontal. Since $a^2 = 2$ and $b^2 = 4$, then $a = \sqrt{2}$, $b = 2$, and $c^2 = 4 + 2$ or $c = \sqrt{6}$. In Fig. 21.20 we locate the center of the hyperbola, the vertices

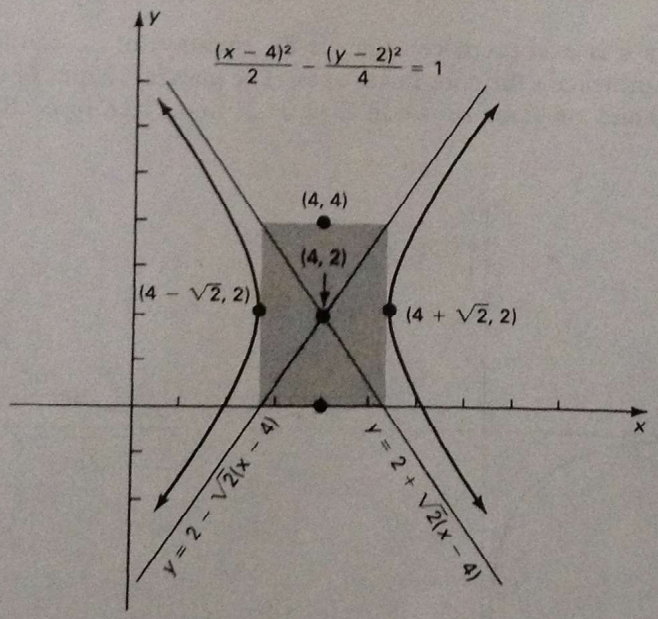


FIGURE 21.20

$(4 \pm \sqrt{2}, 2)$, and the endpoints of the conjugate axis $(4, 2 \pm 2)$. Then the corresponding rectangle is completed and the asymptotes are drawn. The graph is then sketched. As previously mentioned, to find the equations of the asymptotes we set the left side of Eq. 8 equal to zero and solve for y:

$$\frac{(y-2)^2}{4} = \frac{(x-4)^2}{2}$$

$$(y-2)^2 = 2(x-4)^2$$

$$y - 2 = \pm\sqrt{2}(x-4)$$

$$y = 2 \pm \sqrt{2}(x-4).$$

The standard form of an equation of a hyperbola with center at the origin and transverse axis along the x-axis is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

If $b = a$, the result $x^2 - y^2 = a^2$ is a hyperbola with mutally perpendicular asymptotes, $y = \pm x$. The corresponding rectangle is, in this case, a square and the curve is called an **equilateral hyperbola**.

We conclude our discussion here by stating that there is one special form of an equation of an equilateral hyperbola. The equation

$$xy = c, \tag{9}$$

where c is a nonzero constant, is an equation of an equilateral hyperbola whose asymptotes are the coordinate axes. The general shapes of the graphs of $xy = c$ for $c > 0$ and $c < 0$ are shown in Figs. 21.21 and 21.22, respectively. If c is positive, the

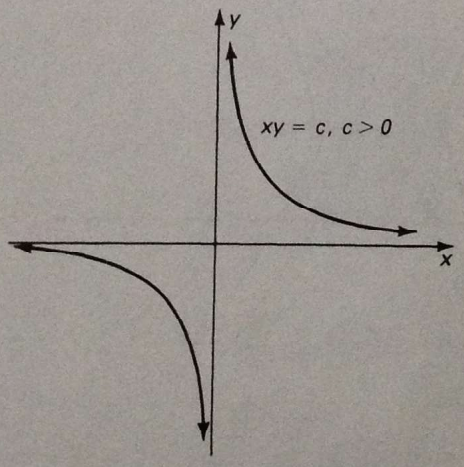


FIGURE 21.21

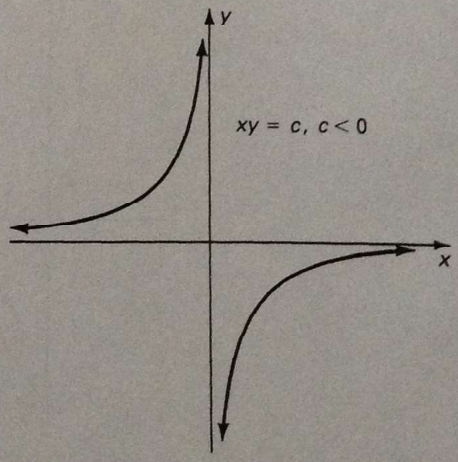


FIGURE 21.22

foci lie on the line $y = x$, while if c is negative, they lie on the line $y = -x$. The graph of a hyperbola having the form of Eq. 9 can be easily sketched by assuming values for one variable and determining the corresponding values of the other variable.

Problem Set 21.6

In Problems **1–10**, *find the center C, vertices V, focus F, endpoints E of the conjugate axis, and asymptotes A of each hyperbola. Sketch each hyperbola.*

1. $\frac{x^2}{16} - \frac{y^2}{9} = 1.$

2. $\frac{y^2}{25} - \frac{x^2}{144} = 1.$

3. $\frac{y^2}{36} - x^2 = 1.$

4. $4x^2 - 9y^2 = 36.$

5. $x^2 - y^2 = 4.$

6. $\frac{x^2}{16} - \frac{(y-2)^2}{16} = 1.$

7. $4(y+3)^2 - 25(x-2)^2 = 100.$

8. $(x+3)^2 - 2y^2 + 8 = 0.$

9. $9x^2 - 36x - 16y^2 - 32y - 124 = 0.$

10. $y^2 - 4x^2 - 10y + 16x - 7 = 0.$

In Problems **11–20**, *determined the standard form of an equation of the hyperbola satisfying the given conditions. Assume the center is at (0, 0) unless otherwise stated.*

11. Focus (0, 3), vertex (0, 2).

12. Transverse axis of length 10, focus (7, 0).

13. Vertex (4, 0), conjugate axis of length 2.

14. Conjugate axis of length 4, focus (0, 6).

15. Center (−4, 2), focus (−4, 6), vertex on x-axis.

16. Center (−7, 0), focus at origin, transverse axis of length 6.

17. Vertex on y-axis, center (−2, 4), conjugate axis of length 3.

18. Vertex at (4, 0), center (4, 2), conjugate axis of length 4.

19. Center on line $y = 4$, vertex (0, 2), focus (0, 1).

20. Vertex (2, 6), focus (4, 6), center on y-axis.

In Problems **21–26**, *sketch the graph of each hyperbola and state the equations of the asymptotes.*

21. $xy = 3.$

22. $xy = -3.$

23. $xy = -5.$

24. $5xy = 6.$

25. $4xy = 1.$

26. $xy = 120.$

27. The speed v of a wave is a function of the frequency f and the wavelength λ, where $v = f\lambda$. For visible light in a vacuum, $v = 3 \times 10^8$ m/s and values of λ range from about 4×10^{-7} to about 7×10^{-7} m. Sketch a graph of frequency versus wavelength for the visible portion of the spectrum.

28. The relationship between the index of refraction of a material, n, the wavelength of light in the material, λ, and the wavelength of light in air, λ_a, is $\lambda_a = n\lambda$. If the wavelength of blue-green light in air is 5000 angstroms, sketch a graph of λ versus n for values of n from 1.5 to 2.5, in increments of 0.1.

29. For an ideal gas at constant temperature, the product of the pressure p and volume v is a constant. A particular sample of such a gas has a pressure of 4 atmospheres and a volume of 5 liters. Sketch a graph of pressure versus volume.

21.7 SUMMARY OF CONIC SECTIONS

It is often important to be able to identify a curve from a brief examination of its equation. Assuming no degenerate cases, from the results of the preceding sections we conclude that the graph of the **general equation of a conic section**

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is:

1. A circle if $A = C$, and $B = 0$.
2. A parabola if $A = 0$ or $C = 0$ but not both, and $B = 0$.
3. An ellipse if $A \neq C$ but A and C have the same sign, and $B = 0$.
4. A hyperbola if either
   a. A and C have opposite signs, and $B = 0$, or
   b. $A = C = 0$, and $B \neq 0$.

Example 1

Classify each of the following equations as that of a circle, parabola, ellipse, or hyperbola.

a. $x^2 + y^2 + 3x - 6y - 7 = 0$.

Solution Because the coefficients of the x^2- and y^2-terms are equal ($A = C = 1$), the equation is that of a circle.

b. $2y^2 + x^2 - 4y - 4x - 10 = 0$.

Solution Because the coefficients of the x^2- and y^2-terms are unequal but have the same sign ($A = 1 \neq 2 = C$), the equation is that of an ellipse.

c. $2y^2 + 3y - 4x + 9 = 0$.

Solution As there is a y^2-term but no x^2-term ($A = 0$, $C = 2$), the equation is that of a parabola.

d. $5x^2 - 3y^2 - 5x + 2y + 16 = 0$.

Solution Because the x^2- and y^2-terms are opposite in sign, the equation is that of a hyperbola.

e. $xy = 7$.

Solution Because $A = C = 0$ and $B = 1 \neq 0$, the equation is that of an (equilateral) hyperbola.

Problem Set 21.7

By inspection, classify each of the following equations as a circle, parabola, ellipse, or hyperbola.

1. $3x^2 + 3y^2 + 2x + 5y - 6 = 0$.
2. $3x^2 + 2y^2 + 2x - 7 = 0$.
3. $2y^2 + 3x + 2y + 1 = 0$.
4. $3xy = 18.21$.
5. $x^2 + 3x + 14y - 17 = 0$.
6. $-2x^2 - 2y^2 + 3x - 4y + 17 = 0$.
7. $2x^2 - 3y^2 + 2x + 6 = 0$.
8. $3y^2 - 2x = 4$.
9. $4x^2 + 2y^2 + 3y = 8$.
10. $2x^2 + 3y = 7$.
11. $xy = -3$.
12. $4x^2 + 4y^2 - 6x + 3y + 7 = 0$.
13. $x^2 + 2y^2 + 3 = 2y - y^2$.
14. $4x^2 - 2y^2 - 3x + 4y + 182 = 0$.
15. $x^2 + 5y^2 + x + 17y = 1$.
16. $1.3x^2 - 7.2 = -8.6y^2$.
17. $x^2 + 3x - 4y + 6 = 4$.
18. $3x^2 + 3y^2 - 4 = 0$.
19. $4x^2 - 7y^2 = 16$.
20. $1.2x^2 = 3y + 7.4$.

21.8 PARAMETRIC EQUATIONS

Whenever we graphed an equation in the variables x and y, we obtained points (x, y) by assigning values to one variable and determining corresponding values of the other variable. It is often convenient, however, to express both x and y as functions of a third variable, called a **parameter**.

For example, the equations

$$x = t + 1 \tag{1}$$

$$y = t^2 - 3 \tag{2}$$

define a curve. Here t is the parameter and the pair of equations are called **parametric equations** of the curve.* If $t = 0$, then $x = 1$ and $y = -3$. Thus $(1, -3)$ is a point on the curve. By choosing other values for t, we obtain the table and graph in Fig. 21.23. We connected the points in order of increasing values of t, as shown by the arrows in the graph. Thus the curve is given a directional sense. This is meaningful because the parametric equations could be looked upon as giving the position of a moving particle at time t. Although Fig. 21.23 shows the t-values associated with some points, *the parameter usually does not appear in such a graph at all.*

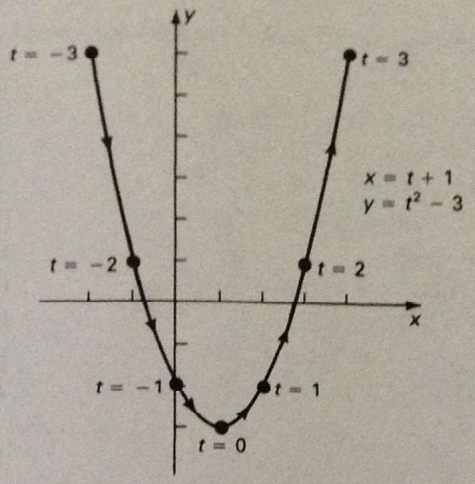
You may have noticed that the curve in Fig. 21.23 appears to be a parabola. To prove that this is indeed the case, we can find a single equation in x and y by *eliminating the parameter*. Because $t = x - 1$ from Eq. 1, by substituting into Eq. 2 we get the direct relationship between x and y

$$y = (x - 1)^2 - 3$$

$$y = x^2 - 2x - 2, \tag{3}$$

* See, for example, Eqs. 1 and 2 in Sec. 16.7, which concern Lissajous figures.

t	x	y
−3	−2	6
−2	−1	1
−1	0	−2
0	1	−3
1	2	−2
2	3	1
3	4	6



which is a familiar form of a parabola. Equation 3 is called a **rectangular form** of the curve. In many cases it is not so simple and sometimes next to impossible to eliminate the parameter. In fact, one of the advantages of parametric equations is that a curve with a complicated equation in x and y may have a very simple parameteric form.

When eliminating the parameter, you must be careful about the domain of the resulting rectangular form, as Example 1 will show.

Example 1

Sketch the graph of

$$x = t^2, \qquad y = t^2$$

and eliminate the parameter.

Solution The graph appears in Fig. 21.24(a). Because $x = t^2 = y$, a rectangular form is $y = x$, whose graph is the straight line in Fig. 21.24(b). Note that the graph in (a) is not the

t	x	y
−2	4	4
−1	1	1
0	0	0
1	1	1
2	4	4

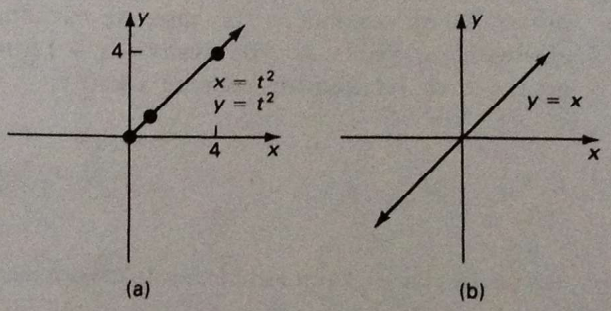


FIGURE 21.24

complete graph in (b), but only a portion of it. To see why, observe that in the parametric equations, for all values of t we have $x \geq 0$. In (b) we have $-\infty < x < \infty$. That is, we obtain the graph in (a) from the rectangular form $y = x$ by restricting the domain so that $x \geq 0$. The graph in (a) is the correct representation of the parametric equations. The proper rectangular form is $y = x$, *where* $x \geq 0$.

Example 2

Sketch the graph of

$$x = 3 \sin \theta, \qquad y = 4 \cos \theta,$$

and eliminate the parameter.

Solution Here the parameter is θ. Because the sine and cosine functions have period 2π, we need consider θ only in the interval from zero to 2π. We also note that $|x| \leq 3$ and $|y| \leq 4$, because the sine and cosine functions have values between -1 and 1. The graph appears in Fig. 21.25. To eliminate the parameter we write

$$\sin \theta = \frac{x}{3} \quad \text{and} \quad \cos \theta = \frac{y}{4}.$$

From the identity $\sin^2 \theta + \cos^2 \theta = 1$,

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{16} = 1.$$

This is a familiar form of an ellipse.

θ	x	y
0	0	4
$\frac{\pi}{4}$	2.1	2.8
$\frac{\pi}{2}$	3	0
$\frac{3\pi}{4}$	2.1	−2.8
π	0	−4
$\frac{5\pi}{4}$	−2.1	−2.8
$\frac{3\pi}{2}$	−3	0
$\frac{7\pi}{4}$	−2.1	2.8
2π	0	4

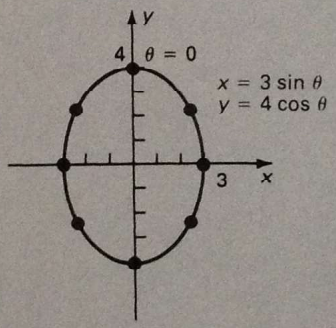


FIGURE 21.25

Example 3

Sketch the graph of

$$x = t^3 - 3t, \qquad y = t^2.$$

Solution See Fig. 21.26. We include the parameter values $\pm\sqrt{3}$ because they give x a value of zero, since $0 = t(t^2 - 3) = t(t + \sqrt{3})(t - \sqrt{3})$.

t	x	y
-2	-2	4
$-\sqrt{3}$	0	3
-1	2	1
0	0	0
1	-2	1
$\sqrt{3}$	0	3
2	2	4

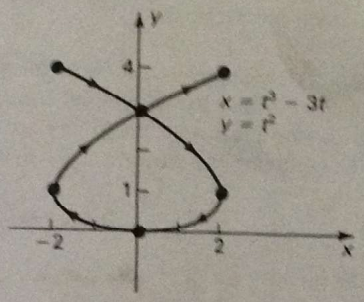


FIGURE 21.26

Example 4

Determine a set of parametric equations for

$$x^3 + y^3 = xy.$$

Solution Our technique is to let $y = tx$. Then

$$x^3 + t^3x^3 = x(tx),$$

or

$$x + t^3x = t$$

$$x = \frac{t}{1 + t^3}.$$

Hence,

$$y = tx$$

$$= \frac{t^2}{1 + t^3}.$$

The parametric equations are $x = \dfrac{t}{1 + t^3}$ and $y = \dfrac{t^2}{1 + t^3}$.

Example 5

In the absence of air resistance, when a projectile with initial velocity v_0 is fired at an angle α with the ground, the coordinates of its position at time t are given by the parametric equations

$$x = (v_0 \cos \alpha)t \tag{4}$$

$$y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2, \tag{5}$$

for which the graph is a parabola (Fig. 21.27). Here, g is the constant acceleration due to gravity, h is the maximum height of the projectile, and R is the (horizontal) range. If $v_0 = 20$ m/s, $\alpha = 30°$, and $g = 9.8$ m/s^2, (a) find when the projectile strikes the ground (that is, the time of flight), (b) find the range R, and (c) find the maximum height h.

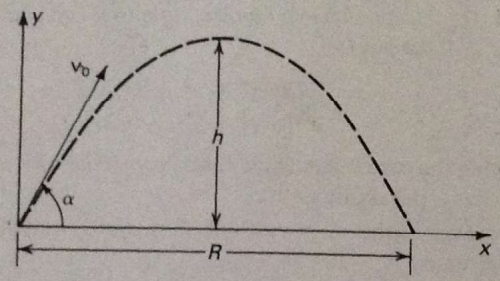


FIGURE 21.27

Solution Substituting the given data in Eqs. 4 and 5, we obtain

$$x = (20 \cos 30°)t \tag{6}$$

$$y = (20 \sin 30°)t - 4.9t^2. \tag{7}$$

a. When the projectile strikes the ground, then $y = 0$. Thus, from Eq. 7,

$$(20 \sin 30°)t - 4.9t^2 = 0$$

$$t(20 \sin 30° - 4.9t) = 0.$$

Hence either $t = 0$ or $t = (20 \sin 30°)/4.9 = 2.04$. We choose $t = 2.04$ s.

b. The range R is obtained by substituting $t = 2.04$ from Part a into Eq. 6:

$$R = (20 \cos 30°)(2.04) = 35.3 \text{ m}.$$

c. To find h we note that the path of the projectile is a parabola that meets the x-axis when $t = 0$ and $t = 2.04$. From Example 6 of Sec. 21.4 and the discussion immediately preceding it, we conclude that the maximum height is obtained when

$$t = \frac{0 + 2.04}{2} = 1.02 \text{ s}.$$

Thus, from Eq. 7, the maximum height h is given by

$$h = (20 \sin 30°)(1.02) - 4.9(1.02)^2 = 5.10 \text{ m}.$$

Problem Set 21.8

In Problems **1–10**, *sketch the graph of the curve represented by the given parametric equations. Eliminate the parameter in* **1–9**.

1. $x = 2t,\ y = t + 1.$

2. $x = t - 3,\ y = 3t + 1.$

3. $x = t,\ y = \dfrac{1}{t}.$

4. $x = t - 1,\ y = t^2 - 2t.$

5. $x = 3 - t,\ y = t - t^2.$

6. $x = \cos t,\ y = 2 \sin t.$

7. $x = 4 \cos t,\ y = 2 \cos t.$

8. $x = \cos 2t,\ y = \cos t.$

9. $x = 2 \cos t - 1,\ y = 2 \sin t - 2.$

10. $x = t^3 - 3t - 2,\ y = t^2 - t - 2.$

In Problems **11–16**, *find parametric representations for the curves given by the equations in the same manner as in Example* 4.

11. $x = y + xy.$

12. $x^2 + y^2 = 9x.$

13. $x^3 + y^3 = 2xy.$

14. $y^2 - 2x^2 = 8y.$

15. $x^2 + 2xy + y = x.$

16. $x^3 - 3xy + y^3 = 0.$

17. By eliminating the parameter, show that a projectile with motion as given by Eqs. 4 and 5 of Example 5 has a parabolic path.

18. A particle moves in the xy-plane such that the coordinates of its position as functions of time t are given by

$$x = r \cos \omega t,$$

$$y = r \sin \omega t,$$

where r and ω are constants. By eliminating the parameter, determine the type of path the particle follows.

19. Sketch the graph of the projectile in Example 5 with the path as given by Eqs. 6 and 7. Choose parameter values of $t = 0,\ 0.25,\ 0.50,\ 0.75$, and so on.

20. Prove that the parametric equations

$$x = a \cos \theta,$$

$$y = b \sin \theta$$

describe an ellipse for $a \neq b$.

21. A projectile is fired from ground level with an initial velocity of 300 m/s at an angle of 40° with the ground (see Example 5). Assume $g = 9.8\ \text{m/s}^2$.

(a) After 3 s, how far has the projectile traveled horizontally and how far vertically?

(b) Find when the projectile strikes the ground.

(c) Find the range (to the nearest meter).

(d) Find the maximum height (to the nearest meter) and the time when it occurs.

22. A body is fired from the ground at an angle of 45° with an initial velocity of 120 m/s. To the nearest meter, find the range of the body (see Example 5). Assume that $g = 9.8\ \text{m/s}^2$.

23. The coordinates of the position of a particle at time t are given parametrically by

$$x = 2t, \qquad y = t + 1,$$

where x and y are in meters and t is in seconds. Show that the distance d of the particle from the origin is given by $d = \sqrt{x^2 + y^2}$. Determine at what time(s) t the particle will be 4 m from the origin.

24. The coordinates of the position of a particle (in meters) as functions of time t (in seconds) are given parametrically by

$$x = 2 - t, \qquad y = 3t + 1.$$

For what value(s) of t will the particle be 3 m from the origin? (See Problem 23.)

21.9 REVIEW

Important Terms

Section 21.1	Right circular cone, conic sections, circle, parabola, ellipse, hyperbola, degenerate conics.
Section 21.2	Distance formula.
Section 21.3	Circle, center of circle, radius of circle, standard forms of equation of circle, general form of equation of circle.
Section 21.4	Parabola, focus of parabola, directrix, axis, vertex, standard forms of equation of parabola.
Section 21.5	Ellipse, foci of ellipse, center of ellipse, major axis, minor axis, semimajor axis, semiminor axis, vertices of ellipse, standard forms of equation of ellipse.
Section 21.6	Hyperbola, foci of hyperbola, vertices of hyperbola, center of hyperbola, transverse axis, conjugate axis, asymptotes for hyperbola, standard forms of equation of hyperbola, equilateral hyperbola.
Section 21.7	General equation of conic section.
Section 21.8	Parameter, parametric equations, eliminating the parameter.

Formula Summary

Distance Formula $\quad d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Conic Sections (Standard Forms)

Circle

Center (h, k), radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

Center (0, 0), radius r:

$$x^2 + y^2 = r^2$$

Parabola

Vertex (h, k), axis $x = h$, distance from vertex to focus $= p > 0$:

$$(x - h)^2 = 4p(y - k) \qquad \text{Opens up}$$

$$(x - h)^2 = -4p(y - k) \qquad \text{Opens down}$$

Vertex (h, k), axis $y = k$, distance from vertex to focus $= p > 0$:

$$(y - k)^2 = 4p(x - h) \qquad \text{Opens to right}$$

$$(y - k)^2 = -4p(x - h) \qquad \text{Opens to left}$$

Ellipse

Center (h, k):

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \qquad \text{Major axis horizontal}$$

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \qquad \text{Major axis vertical}$$

Hyperbola

Center (h, k):

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \qquad \text{Horizontal transverse axis}$$

$$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1 \qquad \text{Vertical transverse axis}$$

Review Questions

1. The set of all points in a plane that are equidistant from a fixed point in the plane is called a(n) _____.
2. The only conic section with asymptotes is the _____.
3. The equation $(x - 2)^2 + (y + 4)^2 = 3$ defines a circle of radius _____(a)_____ whose center is at _____(b)_____.
4. The graph of $y^2 = x$ is called a _____.
5. The vertex of the graph of $(y - 2)^2 = 9x$ is at _____.
6. The parabola $x^2 = -4y$ opens _____(downward)(to the left)_____.

7. The graph of $\frac{x^2}{4} - \frac{y^2}{9} = 1$ is called a(n) ___(a)___, and the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is called a(n) ___(b)___.

8. The major axis of $\frac{(x+3)^2}{4} + \frac{(y-2)^2}{9} = 1$ is ___(horizontal)(vertical)___.

9. The graph of $xy = -7$ is a(n) _____.

10. In sketching the graph determined by parametric equations, the parameter ___(does)(does not)___ have to appear in the graph.

11. The graph of a parabola is that of a function of x if the directrix is ___(parallel)(perpendicular)___ to the x-axis.

12. In a plane, the set of all points the sum of whose distances from two fixed points is a constant is called a(n) _____.

13. In a plane the equation $\frac{x^2}{5} + \frac{y^2}{5} = 1$ defines a(n) _____.

Answers to Review Questions

1. Circle. **2.** Hyperbola. **3.** (a) $\sqrt{3}$, (b) $(2, -4)$. **4.** Parabola. **5.** $(0, 2)$. **6.** Downward. **7.** (a) Hyperbola, (b) Ellipse. **8.** Vertical. **9.** Hyperbola (equilateral). **10.** Does not. **11.** Parallel. **12.** Ellipse. **13.** Circle.

Review Problems

In Problems **1–4**, *find the distance between the given points.*

1. $(1, 4), (-3, 2)$.
2. $(-1, -1), (-6, -1)$.
3. $(-8, 2), (0, -4)$.
4. $(-3, -4), (-1, 1)$.

In Problems **5–8**, *find the standard and general forms of an equation of a circle having the given center C and radius r.*

5. $C = (0, 0), r = 5$.
6. $C = (0, -2), r = \sqrt{2}$.
7. $C = (1, -1), r = \frac{1}{2}$.
8. $C = (\frac{1}{2}, \frac{1}{4}), r = 1$.

In Problems **9–36**, *discuss and sketch the graph of each equation.*

9. $x^2 + y^2 = 16$.
10. $(x-4)^2 + (y-3)^2 = 1$.
11. $(x+2)^2 + (y-1)^2 = 7$.
12. $x^2 + (y-1)^2 - 12 = 0$.
13. $9x^2 - 100y^2 = 900$.
14. $3x - 4y^2 = 0$.
15. $36x^2 + y^2 = 36$.
16. $36y^2 - x^2 = 4$.
17. $5x^2 + 2y = 0$.
18. $x^2 + \frac{1}{2}y^2 = 1$.
19. $6x^2 - 1 = -6y^2$.
20. $\frac{1}{49}x^2 - \frac{1}{49}y^2 = 1$.
21. $\frac{(x+1)^2}{2} - \frac{y^2}{8} = 2$.
22. $(y-6)^2 - (x+2)^2 = 1$.
23. $(y-2)^2 = 12x$.
24. $\frac{1}{2}y - (x-3)^2 = \frac{3}{2}y$.
25. $xy = -8$.
26. $x^2 = \frac{1}{3}y$.
27. $(x+5)^2 + \frac{1}{2}(y-2)^2 = 4$.
28. $x^2 = 36 - (y+2)^2$.

29. $x^2 + y^2 + 5x - 6y + 2 = 0.$

30. $x^2 + y^2 - x - 1 = 0.$

31. $9x^2 + 9y^2 - 18x - 6y + 10 = 0.$

32. $3x^2 + 3y^2 - 6x + 12y + 16 = 0.$

33. $y^2 - 2x - 2y = 4.$

34. $4x^2 + 9y^2 - 16x + 18y = 11.$

35. $2x^2 - 2y^2 + 4x + 10 = 0.$

36. $25x^2 - 2y^2 = -100.$

In Problems **37–39**, *find the standard form of an equation of the given conic.*

37. Parabola: vertex (2, 1) and focus (2, 4).

38. Hyperbola: vertices (1, 1) and (1, −5), and foci $(1, -2 \pm \sqrt{13})$.

39. Ellipse: vertices (−2, −2) and (−2, 8), and foci (−2, 0) and (−2, 6).

40. Sketch the curve and eliminate the parameter:

$$x = 2(2 - t),$$

$$y = t + t^2.$$

41. In the manner of Example 4 in Sec. 21.8, find a parametric representation for the curve given by the equation

$$2x^3 - y^3 = 3xy.$$

Every applied problem encountered thus far in our study of technical mathematics has had a solution in the real number system. There are, however, many problems in applied mathematics that have *no* solutions in that system. To solve such problems, we shall expand our number system and create a new one. Our new system, called the *system of complex numbers*, includes the real numbers and provides solutions to previously unsolvable problems.

Moreover, with some problems in science and engineering, a solution in the real number system exists, but the characteristics of that system make it tedious to obtain the solution. In such problems—for example, the analysis of ac circuits in electrical engineering—the complex number system allows a much simpler and compact solution. In the following section you will see that the need for complex numbers arises quite naturally from consideration of solving certain equations.

13.1 THE *j*-OPERATOR AND COMPLEX NUMBERS

In the previous chapter we saw that a quadratic equation may have no solution in the real number system. For example, consider the equation $x^2 = -1$. Since the square of any real number is nonnegative, there is no real number x whose square is -1. That is, the symbol $\sqrt{-1}$ does not make sense in the real number system. Thus $x^2 = -1$ has no real solution. Similarly, there are no real solutions of the equations $x^2 = -2$ and, more generally, $x^2 = -a$, where $-a$ is negative (note that if $-a$ is

negative, then a is positive). To overcome this situation, we shall extend the real number system so that the new system will indeed provide solutions to equations like $x^2 = -1$. In fact, by considering only $\sqrt{-1}$, we shall be able to handle a square root of *any* negative number.

To begin, we introduce (or invent, so to speak) a new type of number that, by its definition, is a solution of the equation $x^2 = -1$.

> The ***j*-operator**, denoted by j, is that number which when squared is equal to -1; that is,
>
> $$j^2 = -1.$$

From this definition, j is a square root of -1, so we write

$$j = \sqrt{-1}.$$

Since j is not a real number, we call it (according to custom) the **imaginary unit**. Do not let the term *imaginary* fool you. It does not mean that j is "impossible" or "does not exist" but means only that j does not belong to the set of real numbers. Indeed, the practical applications of j to topics such as electrical theory are extensive.*

By taking the product of j and a real number, for instance, $2j$, $-5j$, and $6.4j$, we obtain what is referred to as a *pure imaginary number*.

> **A pure imaginary number** is a number that can be expressed in the form bj, where b is a real number.†

In performing operations with j, we follow the usual algebraic rules for real numbers. To illustrate, we consider powers of j:

$$j^1 = j,$$

$$j^2 = -1 \qquad \text{by definition,}$$

$$j^3 = j^2 \cdot j = (-1) \cdot j = -j,$$

$$j^4 = j^2 \cdot j^2 = (-1)(-1) = 1,$$

$$j^5 = j^4 \cdot j = 1 \cdot j = j,$$

$$j^6 = j^4 \cdot j^2 = 1 \cdot (-1) = -1, \qquad \text{and so on.}$$

The preceding cyclic pattern continues, so any positive integral power of j must be one of four possible numbers: j, -1, $-j$, or 1. The pattern also suggests an easy way to simplify j^n, where n is a positive integer. First, we divide n by 4 and find the remainder r, which is either 1, 2, 3, or 0. Then we raise j to the rth power; that is, $j^n = j^r$.

* In mathematics, the symbol i is preferred instead of j. However, in electrical theory the use of j avoids confusion with the use of i for electrical current. We adopt that symbol here.

† In electrical theory, bj is usually written jb. For example, $2j$ is written $j2$, which should not be confused with j^2.

Example 1

Simplify the following powers of j.

a. j^{35}.

Solution Since $35 \div 4$ is 8 with a remainder of 3,

$$j^{35} = j^3 = -j.$$

In other words,

$$j^{35} = (j^4)^8 \cdot j^3 = (1)^8 \cdot j^3 = j^3 = -j.$$

b. j^{26}.

Solution Since $26 \div 4$ is 6 with a remainder of 2,

$$j^{26} = j^2 = -1.$$

c. j^{40}.

Solution Since 40 is a multiple of 4, dividing 40 by 4 results in a remainder of 0. Thus

$$j^{40} = j^0 = 1.$$

Our claim that $j^0 = 1$ is reasonable because

$$j^{40} = (j^4)^{10} = (1)^{10} = 1.$$

To generalize the notion of $\sqrt{-1}$, we express a square root of *any* negative number as the product of a real number and j in the following manner:

> If a is a positive number, then
>
> $$\sqrt{-a} = j\sqrt{a}.$$

Thus

$$\sqrt{-4} = j\sqrt{4} = j(2) = 2j.$$

Example 2

Write the following numbers in terms of j.

a. $\sqrt{-2} = j\sqrt{2}$.

b. $-\sqrt{-9} = -(j\sqrt{9}) = -[j(3)] = -3j$.

c. $\sqrt{-32} = j\sqrt{32} = j(4\sqrt{2}) = 4j\sqrt{2}$.

d. $\sqrt{-\frac{1}{16}} = j\sqrt{\frac{1}{16}} = \frac{1}{4}j = \frac{j}{4}$.

By combining the real number system with the concept of a pure imaginary number, we are led to the notion of a *complex number*, which is the sum of a real number and a pure imaginary number.

> A **complex number** is one of the form $a + bj$, where a and b are real numbers. The number a is called the **real part** of $a + bj$, and b is called the **imaginary part**.

Some examples of complex numbers are $2 + 3j$, $-4 + j$, and $-3 - 5j$, which is $-3 + (-5)j$. The real part of $-3 - 5j$ is -3 and the imaginary part is -5. If $a = 0$, then $a + bj$ is bj, a *pure imaginary number*. If $b = 0$, then $a + bj$ is a, a *real number*. Thus *every real number is also a complex number with imaginary part zero*. For example, the real number 3 is the complex number $3 + 0j$. Complex numbers that are not real are called **imaginary numbers**. Thus the complex number 7 is real, but the complex number $7 + 2j$ is imaginary.

The complex number $a - bj$ is called the **complex conjugate** (or simply the **conjugate**) of the complex number $a + bj$, and vice versa. For example, the conjugate of $2 + 3j$ is $2 - 3j$, the conjugate of $-3 - 4j$ is $-3 + 4j$, and the conjugate of 6 is 6 because the conjugate of $6 + 0j$ is $6 - 0j$, or 6.

Example 3

Complex Number	Real Part	Imaginary Part	Conjugate
a. $3 + 4j$	3	4	$3 - 4j$
b. $-3 - j$	-3	-1	$-3 + j$
c. -7	-7	0	-7
d. $-j$	0	-1	j
e. $\sqrt{-4}$	0	2	$-2j$
f. $\sqrt[3]{-1}$	-1	0	-1
g. 0	0	0	0

Observe in Part f that $\sqrt[3]{-1}$ is a real number; $(-1)^3 = -1$, so $\sqrt[3]{-1} = -1$. Note in Parts e, f, and g that a real number is equal to its conjugate. Finally, in Part g we note that *zero is the only complex number that is both real and pure imaginary.*

Now that we have defined a complex number, we must be precise regarding when two complex numbers are equal.

EQUALITY OF COMPLEX NUMBERS

$a + bj = c + dj$ if and only if $a = c$ and $b = d$.

This means that **two complex numbers are equal if and only if they have the same real parts and the same imaginary parts.** Thus $2 - 3j = -3j + 2$ and $0.2 + j = j + \frac{1}{5}$, but $4 + 2j \neq 2 + 4j$ and $3 - j \neq j - 3$.

Example 4

Solve each of the following equations if x and y are real numbers.

a. $2 - 5j = x + yj$.

Solution By the definition of equality of complex numbers, we can equate the real parts of both sides of the equation:

$$2 = x.$$

Equating imaginary parts gives

$$-5 = y.$$

The solution is $x = 2$ and $y = -5$.

b. $x - y + xj = 3 + 2j$.

Solution Equating real parts of both sides of the equation gives

$$x - y = 3. \tag{1}$$

Equating imaginary parts gives

$$x = 2.$$

Substituting this value of x in Eq. 1 gives $2 - y = 3$, from which $y = -1$. The solution is $x = 2$ and $y = -1$.

Example 5

Solve $x^2 + x + 1 = 0$ by the quadratic formula.

Solution Here $a = 1$, $b = 1$, and $c = 1$.

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm j\sqrt{3}}{2}.$$

Thus

$$x = -\frac{1}{2} + \frac{\sqrt{3}}{2}j, -\frac{1}{2} - \frac{\sqrt{3}}{2}j.$$

Note that the solutions are a pair of conjugates. Whenever the discriminant of a quadratic equation is negative (here it is -3), there are two different imaginary solutions (conjugates).

We conclude this section by again remarking that complex numbers have wide applications in electrical theory. For example, the impedance of an ac circuit can be expressed in the form $a + bj$, where the real part represents the resistance of the circuit and the imaginary part represents the reactance of the circuit. Later in this chapter you will see how complex numbers are used in situations of this type.

Problem Set 13.1

In Problems **1–8**, *simplify the expressions.*

1. j^9.
2. j^{14}.
3. j^{15}.
4. j^{488}.
5. $-j^{66}$.
6. j^{243}.
7. j^{324}.
8. $j^2 \cdot j^3 \cdot j^4$.

In Problems **9–20**, *write each number in terms of* j.

9. $\sqrt{-81}$.
10. $\sqrt{-36}$.
11. $-\sqrt{-25}$.
12. $\sqrt{-27}$.
13. $\sqrt{-32}$.
14. $\sqrt{-144}$.
15. $\sqrt{-\frac{1}{4}}$.
16. $-\sqrt{-\frac{1}{100}}$.
17. $\sqrt{-0.09}$.
18. $\sqrt{-0.04}$.
19. $\sqrt{-\pi^6}$.
20. $\sqrt{-4^{10}}$.

In Problems **21–36**, *for each complex number state* (*a*) *the real part*, (*b*) *the imaginary part, and* (*c*) *the conjugate. Simplify first, if necessary.*

21. $-6 + 5j$.
22. -2.
23. $-31j - 2$.
24. $-3 + 5j$.
25. $-3j$.
26. $2 - \sqrt{-9}$.
27. $7 + \frac{3}{5}j$.
28. πj.
29. j^{17}.
30. j^2.
31. 7.
32. $-\sqrt[3]{-64}$.

33. $\sqrt[3]{-8}$.

34. $6 + \sqrt{-3}$.

35. $-\sqrt{-49}$.

36. 0.

In Problems **37–40**, *solve the equations if* x *and* y *are real numbers.*

37. $2x + yj = 4 - 6j$.

38. $x - 2j = 17 + yj$.

39. $(x + y) + (x - y)j = 18 - 14j$.

40. $x + (x - y)j = 4j$.

41. Two ac voltages are given by the expressions $\frac{3}{2} + (y + 1)j$ and $(7 + x) - 3j$, respectively. If the voltages are equal, what are the values of x and y? (Assume that x and y are real numbers.)

42. What can be said about a complex number that is equal to its conjugate?

43. Determine whether the following statements are true or false.
   (a) $\frac{1}{2} + 2j = 2j + 0.5$.
   (b) $3 - \sqrt{-5} = 3 - j\sqrt{5}$.
   (c) $\sqrt[3]{-27} = 3j$.
   (d) $5 - 6j = (7 - 2) + (4 - 10)j$.

44. Suppose $a + bj = 0$. To what must $a^2 + b^2$ be equal?

In Problems **45–60**, *solve the equation.*

45. $x^2 - 4x + 5 = 0$.

46. $x^2 - 2x + 2 = 0$.

47. $x^2 + 2x + 3 = 0$.

48. $2x^2 - 4x + 5 = 0$.

49. $x^2 - 2x + 4 = 0$.

50. $3t^2 - 3t + 1 = 0$.

51. $x^2 + 4 = 0$.

52. $x^2 + 8 = 0$.

53. $6r^2 + 8r + 3 = 0$.

54. $x^2 = x - 1$.

55. $-3r^2 + 5r = 4$.

56. $x(x + 5) = 5(x - 5)$.

57. $1 + x^2 = 0$.

58. $(x + 1)^2 = -4$.

59. $x(4x - 3) = x^2 - 2$.

60. $\dfrac{x^2}{5} + 2 = x$.

61. A Maxwell bridge is an ac bridge circuit that permits the inductance L_x and resistance R_x of a coil to be measured. A theoretical analysis of the circuit yields the equation

$$R_x + j\omega L_x = R_2 R_3\left(\frac{1}{R_1} + j\omega C_1\right).$$

Determine the resistance and inductance of a coil in terms of the circuit constants R_1, R_2, R_3, C_1, and ω.

13.2 OPERATIONS WITH COMPLEX NUMBERS

Algebraic operations with complex numbers are defined in such a way that the usual rules for real numbers apply, such as the commutative, associative, and distributive laws. We begin with addition and subtraction.

Day: Date:

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

$$(a + bj) + (c + dj) = (a + c) + (b + d)j.$$

$$(a + bj) - (c + dj) = (a - c) + (b - d)j.$$

That is, the sum (or difference) of two complex numbers is a complex number whose real part is the sum (or difference) of the real parts of the given numbers and whose imaginary part is the sum (or difference) of their imaginary parts.

Example 1

a. $(2 + 3j) + (4 + j) = (2 + 4) + (3 + 1)j = 6 + 4j.$

b. $(5 + 2j) - (3 + 6j) = (5 - 3) + (2 - 6)j = 2 - 4j.$

Rather than memorize the preceding definitions, you can add or subtract complex numbers in a more direct and efficient way. Simply apply the usual rules for removing parentheses and combining like terms. Consider j as you would any other literal number.

Add or subtract complex numbers as you would ordinary algebraic expressions. Treat j like any other literal number.

Example 2

Perform the indicated operations.

a. $(-1 - 2j) + (1 - 6j) = -1 - 2j + 1 - 6j = -8j.$

b. $(3 - 4j) - (-4j) = 3 - 4j + 4j = 3.$

c. $2 + (8 - 3j) = 2 + 8 - 3j = 10 - 3j.$

d. $5 - 2j - (-6 + j) = 5 - 2j + 6 - j = 11 - 3j.$

e. $$\sqrt{8} + \sqrt{-\frac{3}{2}} - \sqrt{-24} = \sqrt{8} + j\sqrt{\frac{3}{2}} - j\sqrt{24}$$

$$= 2\sqrt{2} + \frac{1}{2}j\sqrt{6} - 2j\sqrt{6}$$

$$= 2\sqrt{2} - \frac{3}{2}j\sqrt{6}.$$

Note that the complex numbers $\sqrt{-\frac{3}{2}}$ and $\sqrt{-24}$ were *first* placed in $a + bj$ form before we proceeded to other operations.

Before discussing multiplication, we give you a **note of caution!** Recall that when radicals were studied, we used the rule $\sqrt{a}\sqrt{b} = \sqrt{ab}$. However, this rule is true provided a and b are nonnegative *real* numbers. The rule does *not* apply when both a and b are negative. Using this rule in the latter case leads to incorrect results. For example, applying it to $\sqrt{-1}\sqrt{-1}$ gives

$$\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1,$$

which is false because

$$\sqrt{-1}\sqrt{-1} = j \cdot j = j^2 = -1.$$

Thus, to help avoid errors in computations, *always express square roots of negative numbers in the bj form* ***before*** *performing any algebraic operations with them.*

The complex numbers $a + bj$ and $c + dj$ are multiplied the same way binomials are. We make use of the distributive law and the fact that $j^2 = -1$.

$$\begin{aligned}(a + bj)(c + dj) &= a(c + dj) + bj(c + dj)\\ &= ac + adj + bcj + bdj^2\\ &= ac + (ad + bc)j + bd(-1),\end{aligned}$$

$$(a + bj)(c + dj) = (ac - bd) + (ad + bc)j.$$

MULTIPLICATION OF COMPLEX NUMBERS

$$(a + bj)(c + dj) = (ac - bd) + (ad + bc)j.$$

More simply:

Multiply complex numbers as you would algebraic expressions. However, all complex numbers should be expressed in the form $a + bj$ before multiplying them, and whenever j^2 occurs, it should be replaced by -1.

Example 3

Perform the indicated operations.

a. $$\begin{aligned}(3 + 2j)(2 - 5j) &= 3(2 - 5j) + 2j(2 - 5j)\\ &= 6 - 15j + 4j - 10j^2\\ &= 6 - 11j - 10(-1)\\ &= 16 - 11j.\end{aligned}$$

b. $j(-2 + 3j) = -2j + 3j^2 = -2j + 3(-1) = -3 - 2j.$

c. $(8j)(4j) = 32j^2 = 32(-1) = -32.$

d. $(-3-4j)(2-j) = -3(2-j) - 4j(2-j)$

$$= -6 + 3j - 8j + 4j^2$$
$$= -6 - 5j + 4(-1)$$
$$= -10 - 5j.$$

e. $(4+5j)^2$.

Solution We treat this as a square of a binomial:

$$(4+5j)^2 = (4)^2 + 2(4)(5j) + (5j)^2$$
$$= 16 + 40j + 25j^2$$
$$= 16 + 40j + 25(-1)$$
$$= -9 + 40j.$$

f. $2\sqrt{-4}(3 - 2\sqrt{-3}) + j = 2(2j)(3 - 2j\sqrt{3}) + j$

$$= (4j)(3 - 2j\sqrt{3}) + j$$
$$= 12j - 8j^2\sqrt{3} + j$$
$$= 13j - 8(-1)\sqrt{3}$$
$$= 8\sqrt{3} + 13j.$$

g. $(2+3j)(2-3j) = (2)^2 - (3j)^2 = 4 - 9j^2 = 4 - 9(-1) = 13$. Here we used the special product $(a+b)(a-b) = a^2 - b^2$.

In Part g of Example 3, the product of a complex number and its conjugate was a real number. This is true in general because

$$(a+bj)(a-bj) = a^2 - (bj)^2 = a^2 - b^2j^2 = a^2 - b^2(-1) = a^2 + b^2,$$

which is *always* a real number.

PRODUCT OF CONJUGATES

$(a+bj)(a-bj) = a^2 + b^2$, a real number.

We emphasize that *the product of conjugates is a* ***sum*** *of squares of real numbers*:

$$(5+2j)(5-2j) = 5^2 + 2^2 = 29.$$

Do *not* write the product as a difference:

$$(5+2j)(5-2j) \neq 5^2 - 2^2 = 21.$$

(Students commonly make this error.)

Since the product of a complex number and its conjugate is a real number, to perform the division $(a + bj)/(c + dj)$ we are motivated to multiply *both* the numerator and denominator by the conjugate of the *denominator*. The resulting denominator is a real number, so the quotient can easily be put in the form $a + bj$. The procedure resembles that of rationalizing the denominator of an algebraic expression.

$$\frac{a+bj}{c+dj} = \frac{a+bj}{c+dj}\cdot\frac{c-dj}{c-dj}, \qquad c+dj \neq 0$$

$$= \frac{ac - adj + bcj - bdj^2}{c^2 + d^2}$$

$$= \frac{(ac + bd) + (bc - ad)j}{c^2 + d^2}$$

$$= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}j.$$

DIVISION OF COMPLEX NUMBERS

$$\frac{a+bj}{c+dj} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}j.$$

More simply:

To find the quotient of two complex numbers, multiply both the numerator and denominator by the conjugate of the denominator and simplify.

Example 4

Find $\dfrac{2-j}{3+2j}$.

Solution We multiply both the numerator and denominator by the conjugate of the denominator, $3 - 2j$:

$$\frac{2-j}{3+2j} = \frac{(2-j)(3-2j)}{(3+2j)(3-2j)}$$

$$= \frac{6 - 4j - 3j + 2j^2}{3^2 + 2^2}$$

$$= \frac{6 - 7j + 2(-1)}{9 + 4}$$

$$= \frac{4 - 7j}{13} = \frac{4}{13} - \frac{7}{13}j.$$

Example 5

a. $$\frac{-3+4j}{4-3j} = \frac{(-3+4j)(4+3j)}{(4-3j)(4+3j)}$$

$$= \frac{-12-9j+16j+12j^2}{4^2+3^2} = \frac{-12+7j+12(-1)}{16+9}$$

$$= \frac{-24+7j}{25} = -\frac{24}{25} + \frac{7}{25}j.$$

b. $$\frac{2}{1-j} = \frac{2(1+j)}{(1-j)(1+j)} = \frac{2+2j}{1^2+1^2}$$

$$= \frac{2+2j}{2} = \frac{2}{2} + \frac{2}{2}j = 1 + j.$$

c. $$\frac{6}{5j} = \frac{6(-5j)}{5j(-5j)} = \frac{-30j}{5^2} = \frac{-30j}{25} = -\frac{6}{5}j.$$

Alternatively, if the denominator of a fraction is a pure imaginary number, as it is in $\frac{6}{5j}$, we can simply multiply the numerator and denominator of the fraction by j. Thus

$$\frac{6}{5j} = \frac{6(j)}{5j(j)} = \frac{6j}{5j^2} = \frac{6j}{-5} = -\frac{6}{5}j.$$

d. $$j^{-7} = \frac{1}{j^7} = \frac{1}{j^3} = \frac{1}{-j} = \frac{1(j)}{(-j)j} = \frac{j}{-j^2} = \frac{j}{1} = j.$$

Example 6

If $f(z) = z^2 + 6z + \frac{1}{z}$, find $f(3j)$.

Solution

$$f(3j) = (3j)^2 + 6(3j) + \frac{1}{3j}$$

$$= 9j^2 + 18j + \frac{1(j)}{(3j)j} = -9 + 18j + \frac{j}{3j^2}$$

$$= -9 + 18j - \frac{1}{3}j = -9 + \frac{53}{3}j.$$

Example 7

When elements of a circuit are connected in *series*, the total impedance Z of the circuit is the *sum* of the individual impedances. Suppose that a resistor, a capacitor, and an inductor are

connected in series. If their impedances, in ohms, are 180, $10 + 60j$, and $22 - 110j$, respectively, find the circuit impedance.

Solution Letting $180 = Z_1$, $10 + 60j = Z_2$, and $22 - 110j = Z_3$, we have

$$\begin{aligned} Z &= Z_1 + Z_2 + Z_3 \\ &= (180) + (10 + 60j) + (22 - 110j) \\ &= 212 - 50j\ \Omega. \end{aligned}$$

Example 8

When two elements of an electrical circuit are connected in *parallel*, the total impedance Z of the circuit is given by

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2},$$

where Z_1 and Z_2 are the impedances of the two elements. Suppose capacitive and inductive elements are connected in parallel. If their impedances (in ohms) are $7 - 12j$ and $3 + 26j$, find the circuit impedance (in ohms).

Solution Letting $7 - 12j = Z_1$ and $3 + 26j = Z_2$, we have

$$\begin{aligned} Z &= \frac{(7 - 12j)(3 + 26j)}{(7 - 12j) + (3 + 26j)} = \frac{21 + 182j - 36j + 312}{10 + 14j} = \frac{333 + 146j}{10 + 14j} \\ &= \frac{333 + 146j}{10 + 14j} \cdot \frac{10 - 14j}{10 - 14j} = \frac{3330 - 4662j + 1460j - 2044j^2}{100 + 196} \\ &= \frac{5374 - 3202j}{296} = \frac{5374}{296} - \frac{3202}{296} j = 18.16 - 10.82j. \end{aligned}$$

Problem Set 13.2

In Problems **1–54**, *perform the indicated operations.*

1. $(3 - 5j) + (-6 + 4j)$.

2. $(-2 + 6j) + (7 + 4j)$.

3. $(7 - 3j) - (9 - 6j)$.

4. $(4 + 3j) - (5j - 6)$.

5. $2j + 3j - 3$.

6. $2j - 6j + 4j$.

7. $(2 - j) + (3 + j) - (4 - j)$.

8. $4(2 + j) - 3(6 + 2j) + 7j$.

9. $(2 + j)(3 + j)$.

10. $(1 + j)(6 - 2j)$.

11. $(4 + 3j)(5 - 2j)$.

12. $(-2 - j)(-3 + j)$.

13. $2(2 - j)(3 + 2j)$.

14. $2(-5 + j)(-5 - j)$.

15. $(2j^2)(3j)^2$.

16. $j(2j)(3j)(4j)$.

17. $3j(2 - 5j)$.

18. $(8j)(2j)^2(1 + j)$.

19. $(3 + 2j)^2$.

20. $(8 - 4j)^2$.

21. $(5 + 2j)(5 - 2j)$.

22. $(-4 - 2j)(-4 + 2j)$.

23. $\dfrac{3 + 4j}{2}$.

24. $\dfrac{10 - 8j}{5}$.

25. $\dfrac{3 - j}{4 + j}$.

26. $\dfrac{1 - j}{1 + j}$.

27. $\dfrac{2 - j}{3 - j}$.

28. $\dfrac{2 - j}{2 + j}$.

29. $\dfrac{2}{1 - 4j}$.

30. j^{-5}.

31. $\dfrac{3}{2j}$.

32. $\dfrac{3}{2(1 - j)}$.

33. $\dfrac{4 - 2j}{-j}$.

34. $\dfrac{1}{j} + \dfrac{2}{j}$.

35. $\dfrac{5 + 3j}{5 - 6j}$.

36. $\dfrac{2 + 3j}{4 - 2j}$.

37. $\sqrt{-4} + 2\sqrt{-8} + 3\sqrt{12}$.

38. $5 - 2\sqrt{-16} + (3 - \sqrt{-2})$.

39. $\sqrt{-2}\sqrt{-20}$.

40. $(-\sqrt{-4})(-\sqrt{-5})(-\sqrt{-20})$

41. $(6 + 2j)(-7 - \sqrt{-4})$.

42. $(2 - \sqrt{-3})(3 + \sqrt{-12})$.

43. $(3 - \sqrt{-25})^2$.

44. $(2j - \sqrt{-64})^2$.

45. $\left(\dfrac{1}{3} - \dfrac{4}{5}j\right)\left(\dfrac{3}{8} + \dfrac{2}{7}j\right)$.

46. $\dfrac{\sqrt{-12}}{\sqrt{-9}}$.

47. $\dfrac{1}{(-4 + 3j)(2 + 4j)}$.

48. $\dfrac{5}{(2 + j)(2 - j)}$.

49. $(2 - j)(3 + j)(1 + j)$.

50. $(1 + j)^2(1 - j)$.

51. $\dfrac{\dfrac{2}{3} - \dfrac{j}{2}}{\dfrac{1}{3} + \dfrac{j}{4}}$.

52. $\dfrac{\dfrac{1}{3} + \dfrac{j}{4}}{\dfrac{2}{3} + \dfrac{j}{2}}$.

53. $\dfrac{(6 + 2j)(4 - 3j)}{(1 + j)^2}$.

54. $5 + 2j + (1 + j)^{-2}$.

55. If $f(z) = z^2 - 2z + 1$, find $f(1 + j)$.

56. If $f(z) = 2z^2 - z + 1$, find $f(2 - j)$.

57. In electrical theory, impedance Z (in ohms) is given by

$$Z = \frac{V}{I},$$

where V is voltage (in volts) and I is current (in amperes).
(a) Find Z if $V = 5 + 5j$ and $I = 3 + 4j$.
(b) Find V if $Z = 1 + j$ and $I = 1 - j$.

58. The voltage V across a circuit element is given by $V = IZ$, where V is in volts, I is in amperes, and Z is in ohms. Find the voltage across an element if $I = 4 - 3j$ and $Z = 8 - 15j$.

59. Figure 13.1 shows part of an electrical circuit. Kirchhoff's laws imply that $I_1 + I_2 = I_3$. Find I_2 if $I_1 = 7 + 2j$ and $I_3 = 9 - 5j$.

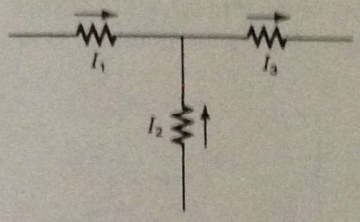


FIGURE 13.1

60. A student said that $-1/j = j^2/j = j$. Is this correct?

61. A resistor, a capacitor, and an inductor are connected in *series*. Their impedances (in ohms) are 8, $3 - 7j$, and $2 + 15j$, respectively. Find the circuit impedance (see Example 7).

62. If the resistor and capacitor in Problem 61 were connected in *parallel*, find the impedance of the combination (see Example 8).

63. If the capacitor and inductor in Problem 61 were connected in *parallel*, find the circuit impedance (see Example 8).

64. The impedance Z of a combination series-parallel circuit is given by

$$Z = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3},$$

where $Z_1 = 1 + 3j$, $Z_2 = 1 + j$, and $Z_3 = 1 + 2j$. Find Z (in ohms).

65. In finding the Thévenin equivalent for a complex circuit, the open-circuit voltage V_0 must be found, where

$$V_0 = V_2 + \frac{Z_2}{Z_1 + Z_2}(V_1 - V_2).$$

Given that $V_1 = 2 + 0j$, $V_2 = 0.707 + 0.707j$, $Z_1 = 1 + 2j$, and $Z_2 = 2 + 3j$, find V_0.

66. The Hay bridge is an ac circuit used in electrical measurements. It is governed by the equation

$$\left(R_1 - j\frac{1}{\omega C_1}\right)(R_x + j\omega L_x) = R_2 R_3.$$

Express R_x and L_x in terms of the other circuit constants.

13.3 POLAR FORM OF A COMPLEX NUMBER

Real numbers can be geometrically represented by points on a *line*, whereas complex numbers are represented by points in a rectangular coordinate *plane*. By one method the complex number $a + bj$ is represented by the point (a, b). The abscissa a corresponds to the real part of $a + bj$, and the ordinate b corresponds to the imaginary part of $a + bj$ [see Fig. 13.2(a)]. For example, in Fig. 13.2(b) the number $2 + 3j$ is represented by the point $(2, 3)$ in the first quadrant; the number $-3 - j$ is

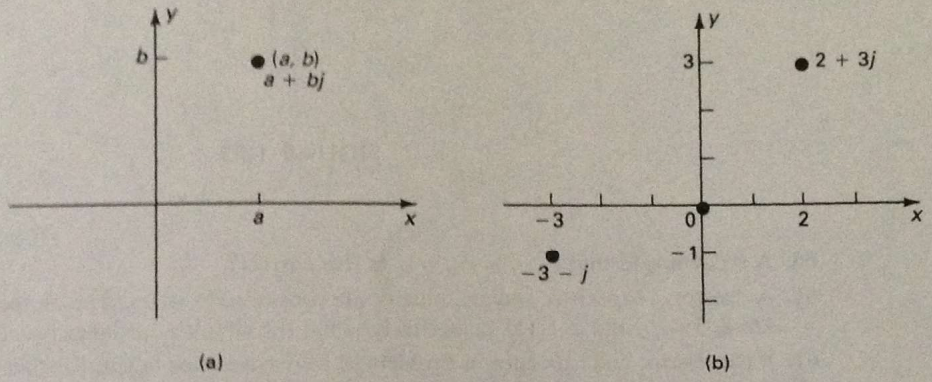


FIGURE 13.2

represented by the point $(-3, -1)$ in the third quadrant. The origin $(0, 0)$ corresponds to 0 or $0 + 0j$. Since the points in the plane are considered here to represent complex numbers, we refer to the plane as the **complex plane** (also called the *Argand plane*).

Observe in Fig. 13.3 that the point $(a, 0)$ on the x-axis corresponds to the real number $a = a + 0j$, whereas the point $(0, b)$ on the y-axis corresponds to the pure imaginary number $bj = 0 + bj$. For these reasons the x-axis is called the **real axis** and the y-axis is called the (pure) **imaginary axis**. The fact that $(0, 0)$ lies both on the real axis *and* the imaginary axis gives a geometric picture of the previous statement in Example 3 of Sec. 13.1 that zero is both a real and a pure imaginary number.

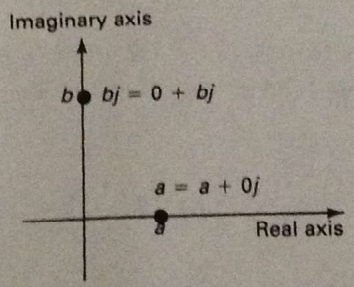


FIGURE 13.3

Since the complex number $a + bj$ can be represented by the point (a, b), we can speak of the *point* $a + bj$ and no confusion should arise. Also, $a + bj$ is called the **rectangular form** of a complex number.

Example 1

Locate the points in the complex plane that correspond to the complex numbers $2 + j$, $3 - 2j$, $-1 - 2j$, $-1 + 3j$, -3, 2, j, and $-3j$.

Solution See Fig. 13.4.

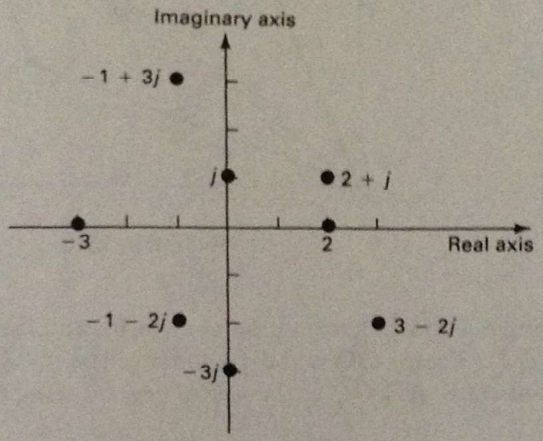


FIGURE 13.4

The complex number $a + bj$ can also be represented in the plane by the vector **OP** from the origin to the point $P(a, b)$, as indicated in Fig. 13.5. Hence we can speak interchangeably of the *complex number* $a + bj$, the *point* $a + bj$, and the *vector* $a + bj$.

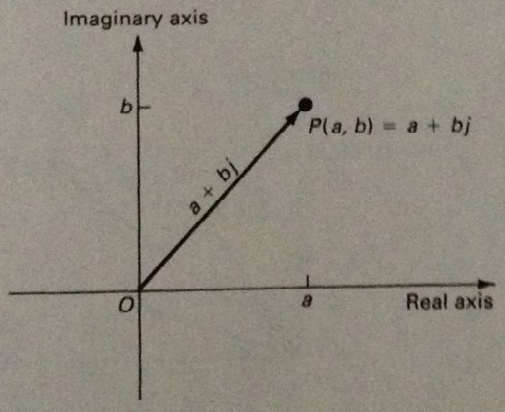


FIGURE 13.5

Addition and subtraction of complex numbers can be interpreted geometrically as addition and subtraction of vectors in standard position. In Fig. 13.6, the vector

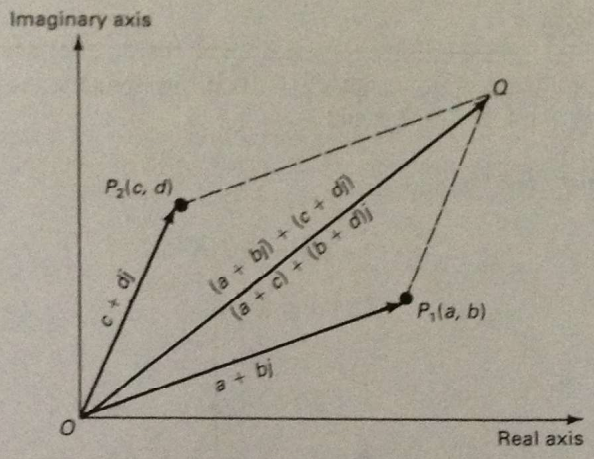


FIGURE 13.6

representing $a + bj$ has terminal point $P_1(a, b)$, and the vector for $c + dj$ has terminal point $P_2(c, d)$. Vector **OQ** is the diagonal of the parallelogram having these vectors for adjacent sides, and **OQ** is the vector representing the sum

$$(a + bj) + (c + dj) = (a + c) + (b + d)j.$$

Example 2

Geometrically find the sum $(2 + 3j) + (3 + j)$.

Solution The sum of $2 + 3j$ and $3 + j$ is represented geometrically by the diagonal vector $5 + 4j$ of the parallelogram with the given vectors for adjacent sides (see Fig. 13.7).

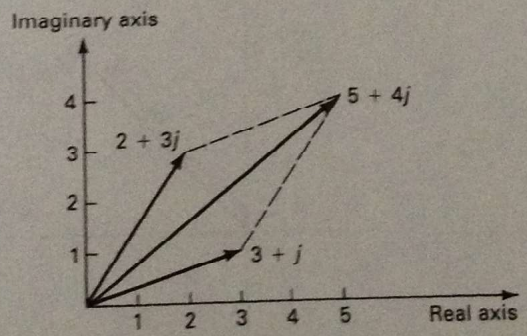


FIGURE 13.7

The subtraction $(a + bj) - (c + dj)$ can be expressed geometrically as the vector sum $(a + bj) + (-c - dj)$, where the vector representing $-c - dj$ is a vector equal in magnitude but opposite in direction to the vector $c + dj$.

Example 3

Geometrically determine the difference $(2 + j) - (3 - 2j)$.

Solution In Fig. 13.8, the difference $(2 + j) - (3 - 2j)$ is represented by the diagonal vector $-1 + 3j$ of the parallelogram which has as adjacent sides the vector for $2 + j$ and the vector which is equal in magnitude but opposite in direction to $3 - 2j$ (that is, $-3 + 2j$).

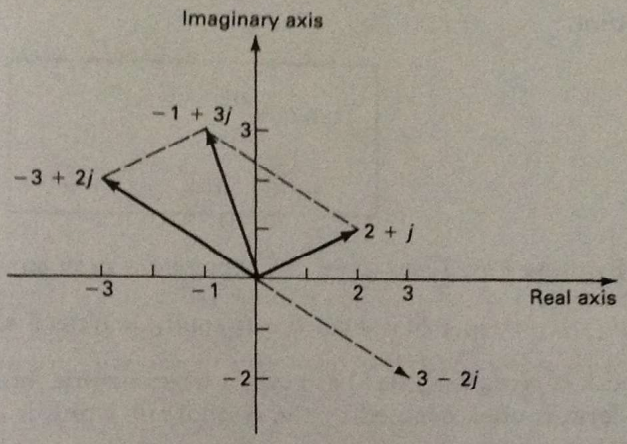


FIGURE 13.8

Figure 13.9 shows a vector **OP** representing the complex number $a + bj$. The angle θ that **OP** makes with the positive real axis is called an **amplitude**, or **argument**,

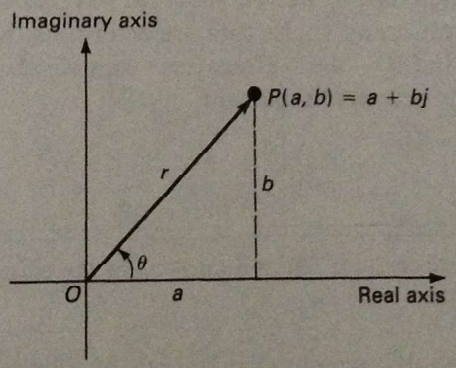


FIGURE 13.9

of $a + bj$. The length r of **OP** is called the **absolute value**, or **modulus**, of $a + bj$ and is always positive or zero. It is also called the **magnitude** of $a + bj$ and may be denoted by $|a + bj|$. From Fig. 13.9 we conclude that $\cos\theta = a/r$, so

$$\boxed{a = r\cos\theta.} \tag{1}$$

Similarly, $\sin\theta = b/r$, or

$$\boxed{b = r\sin\theta.} \tag{2}$$

Moreover,

$$\boxed{\tan\theta = \frac{b}{a},} \tag{3}$$

$$\boxed{r = |a + bj| = \sqrt{a^2 + b^2}.} \tag{4}$$

By using Eqs. 1 and 2, we can express $a + bj$ in terms of r and θ:

$$a + bj = r\cos\theta + (r\sin\theta)j = r(\cos\theta + j\sin\theta).$$

We speak of $\boldsymbol{r(\cos\theta + j\sin\theta)}$ as being a **trigonometric form**, or **polar form**, of $a + bj$. Polar form is often denoted by the symbol $r\angle\theta$, which is read r *at angle* θ. Another abbreviation is $r\,\mathrm{cis}\,\theta$.

POLAR FORM

$$a + bj = r(\cos\theta + j\sin\theta) = r\angle\theta.$$

We remark that a complex number does not have a unique polar form. That is, in Fig. 13.9 there is only one choice for r, whereas the argument of $a + bj$ could be any one of infinitely many angles that are coterminal with the indicated argument θ, such as $\theta + 360°$ and $\theta - 360°$. However, when we choose θ such that $0° \le \theta < 360°$, we may call θ the **principal argument**.

Example 4

Locate the number $2(\cos 300° + j\sin 300°)$ in the complex plane and express the number in rectangular form.

Solution The absolute value r is 2 and the amplitude θ is 300°, so the number is represented by the vector of length 2 (units) that makes an angle of 300° with the positive real axis (see Fig. 13.10). The number can also be written as $2\angle 300°$.

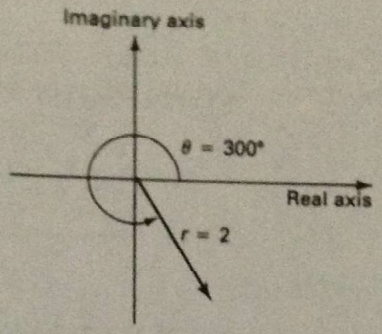


FIGURE 13.10

To convert from polar form to rectangular form, we can use Eqs. 1 and 2.

$$a = r\cos\theta = 2\cos 300^\circ = 2\left(\frac{1}{2}\right) = 1.$$

$$b = r\sin\theta = 2\sin 300^\circ = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}.$$

Thus

$$2(\cos 300^\circ + j\sin 300^\circ) = a + bj = 1 - j\sqrt{3}.$$

More directly, we can simply substitute for the trigonometric values:

$$2(\cos 300^\circ + j\sin 300^\circ) = 2\left[\frac{1}{2} + j\left(-\frac{\sqrt{3}}{2}\right)\right] = 1 - j\sqrt{3}.$$

Example 5

Express each of the following complex numbers in rectangular form.

a. $4(\cos 45^\circ + j\sin 45^\circ)$.

Solution By substitution we have

$$4(\cos 45^\circ + j\sin 45^\circ) = 4\left(\frac{\sqrt{2}}{2} + j\cdot\frac{\sqrt{2}}{2}\right) = 2\sqrt{2} + 2j\sqrt{2}.$$

b. $8(\cos 90^\circ + j\sin 90^\circ)$.

Solution

$$8(\cos 90^\circ + j\sin 90^\circ) = 8(0 + j\cdot 1) = 8j.$$

c. $3\angle 150°$.

Solution

$$3\angle 150° = 3(\cos 150° + j \sin 150°)$$
$$= 3\left(-\frac{\sqrt{3}}{2} + j \cdot \frac{1}{2}\right) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}j.$$

d. $24.3\angle 27.4°$.

Solution

$$24.3\angle 27.4° = 24.3(\cos 27.4° + j \sin 27.4°)$$
$$= 24.3(0.8878 + 0.4602j)$$
$$= 21.6 + 11.2j.$$

Some calculators have the capability to perform conversions from polar form to rectangular form. It involves keying in the values of θ and r (in a particular order). Then the values of a and b in $a + bj$ are displayed sequentially. A reverse procedure is used for conversion from rectangular form to polar form. You should check the manual that came with your calculator to see if your calculator has this capability.

To convert a complex number from rectangular form to polar form, we make use of Eqs. 3 and 4:

$$\tan \theta = \frac{b}{a}, \qquad r = \sqrt{a^2 + b^2}.$$

The next example shows the procedure.

Example 6

The impedance Z of an inductor is given by $Z = R + X_L j$, where the real part R represents the resistance of the inductor and the imaginary part X_L represents the inductive reactance (see Fig. 13.11). (Assume that Z, R, and X_L are measured in ohms.) If $Z = 2 + 2j$, express Z in polar form.

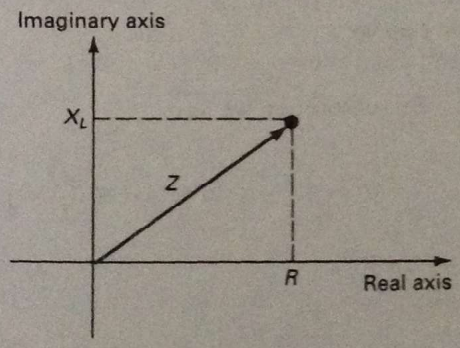


FIGURE 13.11

Solution First, we note that $2 + 2j$ lies in the first quadrant. In Fig. 13.12, $2 + 2j$ is geometrically represented by both the point $(2, 2)$ and the vector from the origin to $(2, 2)$.

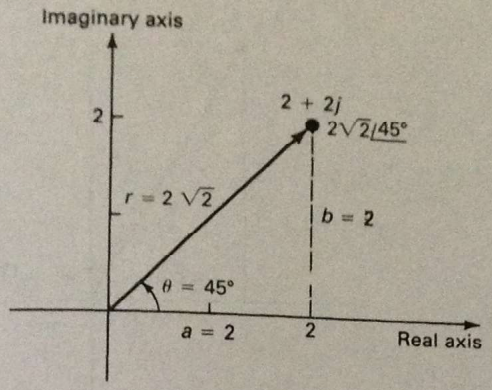


FIGURE 13.12

Second, for the absolute value r, we have $a = 2$ and $b = 2$, so

$$r = \sqrt{a^2 + b^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}\ \Omega$$

Now, to find the amplitude θ we have

$$\tan\theta = \frac{b}{a} = \frac{2}{2} = 1.$$

Thus θ is a first-quadrant angle with a tangent of 1. Using our knowledge of special angles, we conclude that θ could be $45°$, $45° + 360°$, $45° - 360°$, and so on for other coterminal angles. However, in general *we usually choose θ such that* $0° \le \theta < 360°$. Thus we choose $\theta = 45°$. (That is, $45°$ is the principal argument.) We can now write

$$2 + 2j = r(\cos\theta + j\sin\theta)$$

$$= 2\sqrt{2}(\cos 45° + j\sin 45°) \quad \text{or} \quad 2\sqrt{2}/\underline{45°}\ \Omega.$$

Because, in general, there are two choices for θ such that both $\tan\theta = b/a$ and $0° \le \theta < 360°$, the correct choice will be evident by first determining the quadrant in which the point $a + bj$ lies. For our situation above, $\tan 45° = 1$ and $\tan 225° = 1$, but we must choose $\theta = 45°$ because $2 + 2j$ lies in the first quadrant.

Example 7

Express $-3 + 3j\sqrt{3}$ in polar form.

Solution The point $-3 + 3j\sqrt{3}$ lies in the second quadrant (see Fig. 13.13).

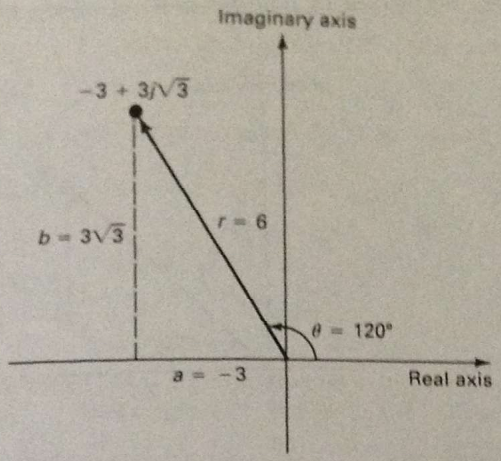


FIGURE 13.13

Since $a = -3$ and $b = 3\sqrt{3}$, we have

$$r = \sqrt{a^2 + b^2} = \sqrt{(-3)^2 + (3\sqrt{3})^2} = \sqrt{9 + 27} = 6$$

and

$$\tan\theta = \frac{b}{a} = \frac{3\sqrt{3}}{-3} = -\sqrt{3}.$$

For $\tan\theta = -\sqrt{3}$, θ may be either 120° or 300° (from our knowledge of special angles). But $-3 + 3j\sqrt{3}$ is in the second quadrant, so the correct choice is 120°. Therefore,

$$-3 + 3j\sqrt{3} = 6(\cos 120° + j \sin 120°).$$

In Example 7, a typical direct calculator result for an angle with a tangent of $-\sqrt{3}$ (or -1.732) is $-60°$. That result is *unacceptable* because the problem requires the amplitude to be a second-quadrant angle. As discussed in Sec. 7.1, we should first find the acute reference angle whose tangent is $\sqrt{3}$ (we omit the negative sign). A calculator result is 60°. Thus $\theta = 180° - 60° = 120°$. We emphasize that due to our preference for θ to lie between 0° and 360°, we choose $6\angle 120°$ as our answer. You should realize, however, that $6\angle -240°$ could be used as a polar form for the given number. Similarly, in Example 6 another polar form for $2 + 2j$ is given by $2\sqrt{2}[\cos(-315°) + j \sin(-315°)]$. The amplitude for the complex number zero can be any angle.

Example 8

The voltage across an inductor in an ac circuit is $3 - 4j$ volts. Express that voltage in polar form.

Solution The vector $3 - 4j$ lies in the fourth quadrant. We have

$$r = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5 \text{ V}$$

and

$$\tan \theta = \frac{b}{a} = \frac{-4}{3} = -\frac{4}{3} = -1.3333.$$

The reference angle with a tangent of 1.3333 is 53.1°. Because θ is a fourth-quadrant angle, we choose θ to be $360° - 53.1° = 306.9°$. We then have

$$3 - 4j = 5\underline{/306.9°} \text{ V}.$$

The polar form of a real number or a pure imaginary number is easy to find, as the following example shows.

Example 9

Express each of the following complex numbers in polar form.

a. 2.

Solution The point 2, or $2 + 0j$, lies on the positive real axis and is two units from the origin [see Fig. 13.14(a)]. Therefore, $\theta = 0°$, $r = 2$, and we have $2 = 2(\cos 0° + j \sin 0°)$.

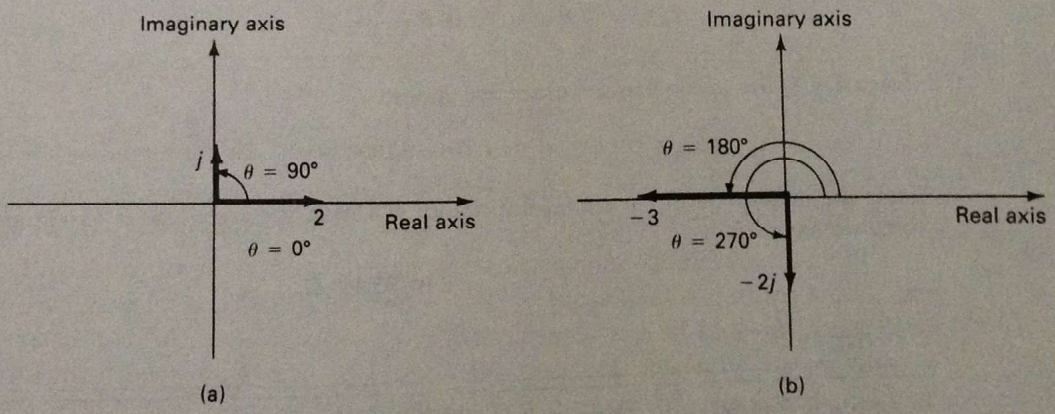


FIGURE 13.14

b. j.

Solution The point j, or $0 + 1j$, lies on the positive imaginary axis and is one unit from the origin [see Fig. 13.14(a)]. Thus $\theta = 90°$ and $r = 1$. Therefore, $j = 1(\cos 90° + j \sin 90°)$.

c. -3.

Solution By inspection of Fig. 13.14(b), clearly $r = 3$ and $\theta = 180°$, so $-3 = 3\underline{/180°}$.

d. $-2j$.

Solution From Fig. 13.14(b) we have $r = 2$ and $\theta = 270°$. Hence $-2j = 2\underline{/270°}$.

Example 10

The total impedance Z (in ohms) of a circuit is given by

$$Z = Z_1 + Z_2,$$

where $Z_1 = 4\underline{/0°}$ and $Z_2 = 8\underline{/90°}$. Find the magnitude (in ohms) and amplitude of the circuit impedance and express the circuit impedance in polar form.

Solution

$$\begin{aligned} Z = Z_1 + Z_2 &= 4\underline{/0°} + 8\underline{/90°} \\ &= 4(\cos 0° + j \sin 0°) + 8(\cos 90° + j \sin 90°) \\ &= 4[1 + j(0)] + 8[0 + j(1)] = 4(1) + 8(j), \\ Z &= 4 + 8j. \end{aligned}$$

The magnitude (r) of Z is given by

$$|4 + 8j| = \sqrt{4^2 + 8^2} = 8.94.$$

To find the amplitude θ, we have

$$\tan \theta = \frac{8}{4} = 2.$$

Since $4 + 8j$ lies in the first quadrant, we choose

$$\theta = 63.43°.$$

Thus the magnitude of the circuit impedance is $8.94\ \Omega$, the amplitude is 63.43°, and in polar form we have

$$Z = 8.94\underline{/63.43°}\ \Omega.$$

Example 11

In Example 7 of Sec. 13.2, the circuit impedance Z (in ohms) is given by

$$Z = 212 - 50j.$$

Find the magnitude (in ohms) of the circuit impedance and express the circuit impedance in polar form.

Solution The magnitude of Z is

$$\sqrt{(212)^2 + (-50)^2} = 217.8.$$

Also,

$$\tan \theta = \frac{-50}{212},$$

from which we find that $\theta = 346.7°$, because Z is in the fourth quadrant. Thus in polar form we have $Z = 217.8/\underline{346.7°}$.

Example 12

In Example 8 of Sec. 13.2, the circuit impedance Z (in ohms) is given by

$$Z = 18.16 - 10.82j.$$

Find the magnitude (in ohms) of the circuit impedance and the *phase angle* θ, which is the amplitude of Z.

Solution The magnitude of the circuit impedance is

$$\sqrt{(18.16)^2 + (-10.82)^2} = 21.1.$$

The phase angle θ is given by

$$\tan \theta = \frac{-10.82}{18.16},$$

from which $\theta = 329.2°$, because Z lies in the fourth quadrant.

Just as complex numbers can be represented by vectors, liberal adaptation permits vector quantities such as force and velocity to be represented by complex numbers. As a result, problems which involve finding the resultant of a number of vector quantities may be solved by means of algebra of complex numbers. That is, the resultant of two or more vectors can be obtained by first expressing each vector as a complex number and then adding these numbers in the usual manner. In the following example, we shall find the resultant of three forces acting at the same point (that is, *concurrent forces*).

Example 13

Use complex algebra to find the resultant of the following system of forces that act at the origin:

$$A = 30 \text{ N}, \quad \theta_A = 44°,$$
$$B = 40 \text{ N}, \quad \theta_B = 121°,$$
$$C = 20 \text{ N}, \quad \theta_C = 209°,$$

where θ is the angle that the given force makes with the positive x-axis. The system is shown in Fig. 13.15.

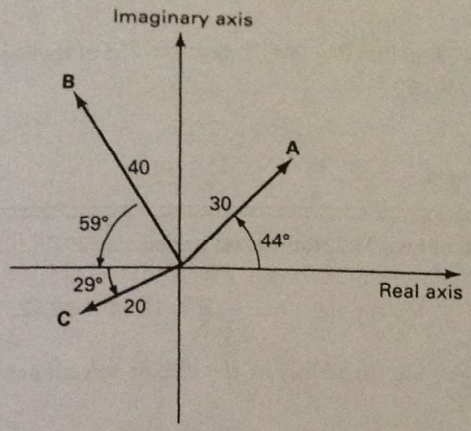


FIGURE 13.15

Solution After first writing each force in polar form, we express them in rectangular form:

$$\mathbf{A} = 30(\cos 44° + j \sin 44°) = 21.58 + 20.84j,$$
$$\mathbf{B} = 40(\cos 121° + j \sin 121°) = -20.60 + 34.29j,$$
$$\mathbf{C} = 20(\cos 209° + j \sin 209°) = -17.49 - 9.70j.$$

We now add the real parts and add the imaginary parts to obtain the resultant.

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = -16.51 + 45.43j = 48.3\underline{/110.0°}.$$

Thus the magnitude of the resultant is 48.3 N and its direction is $\theta = 110.0°$.

Problem Set 13.3

In Problems **1–18**, *express the given complex number in polar form. Do not use a calculator in Problems* **1–16**.

1. $4 + 4j$.

2. $-\sqrt{3} + j$.

3. $2 - 2j\sqrt{3}$.

4. -4.

5. $5j$.

6. $1 + j\sqrt{3}$.

7. $-1-3j$.
8. $-8j$.
9. 1.
10. $\sqrt{2}-j\sqrt{2}$.
11. $-1+j\sqrt{3}$.
12. $-3\sqrt{3}+3j$.
13. $-\sqrt{3}-j$.
14. $6j$.
15. $-5j$.
16. 64.
17. $8+3j$.
18. $5-7j$.

In Problems **19–34**, *locate the given number in the complex plane and express it in rectangular form. Do not use a calculator in Problems* **19–30**.

19. $2(\cos 30^\circ + j \sin 30^\circ)$.
20. $3(\cos 240^\circ + j \sin 240^\circ)$.
21. $6(\cos 120^\circ + j \sin 120^\circ)$.
22. $4(\cos 210^\circ + j \sin 210^\circ)$.
23. $2(\cos 315^\circ + j \sin 315^\circ)$.
24. $3(\cos 150^\circ + j \sin 150^\circ)$.
25. $3(\cos 270^\circ + j \sin 270^\circ)$.
26. $2(\cos 180^\circ + j \sin 180^\circ)$.
27. $4/0^\circ$.
28. $4/240^\circ$.
29. $3/330^\circ$.
30. $\frac{1}{2}/90^\circ$.
31. $2/15^\circ$.
32. $3/40^\circ$.
33. $2/245^\circ$.
34. $4/340^\circ$.
35. If all complex numbers with an absolute value of 1 were plotted in the complex plane, what geometric figure would they form?
36. If $3(\cos 220^\circ + j \sin 220^\circ) = a + bj$, find a and b.
37. If $4(\cos 500^\circ + j \sin 500^\circ) = a + bj$, find a and b.
38. If $2(\cos 0^\circ + j \sin 0^\circ) = a + \left(\frac{b}{c}\right)j$, find a.

In Problems **39–42**, *express the given number in rectangular form.*

39. $3[\cos(-205^\circ) + j \sin(-205^\circ)]$.
40. $7\left(\cos\frac{\pi}{5} + j \sin\frac{\pi}{5}\right)$.
41. $0.75/-42^\circ$.
42. $1/-100.3^\circ$.
43. The impedance of an inductor is $6.2 + 0.8j\ \Omega$. (a) Express the impedance in polar form. Find the (b) resistance and (c) inductive reactance of the inductor (see Example 6).
44. The impedance of a branch of a circuit is $1.5 - 2.3j$ ohms. Express that impedance in polar form.
45. Express a voltage of $23 - 46j$ volts in polar form.
46. Find a polar form of the conjugate of the voltage in Problem 45.
47. The voltage across an inductor in an ac circuit is $11.2/72^\circ$ V. Express the voltage in rectangular form.
48. The electric current in a circuit is $0.026/26^\circ$ A. Express that current in rectangular form.
49. Find the acute angle between the vectors $\mathbf{V} = 2 + 3j$ and $\mathbf{I} = 4 + j$.
50. Arrange the following numbers in order of decreasing absolute value: $4 + 3j$, $-6j$, and $5 - 5j$.

In Problems **51–54**, *geometrically construct the sums and differences.*

51. $(2 - 3j) + (-4 + 3j)$.
52. $(8 - 4j) + (-6 - 2j)$.
53. $(3 + 2j) - (2 + 4j)$.
54. $(3 + 5j) - (5 + 2j)$.

55. Find the value of $20\angle 46^\circ + 34\angle 62^\circ$ and express your answer in polar form.

56. If $Z_1 = 30 + 10j$ and $Z_2 = 20 - 46j$, find the polar form for Z_T if $Z_T = Z_1 + Z_2$.

57. When two elements of an electrical circuit are connected in parallel, the total impedance Z of the circuit is given by

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2},$$

where Z_1 and Z_2 are the impedances of the two elements. If $Z_1 = 2 - 3j$ and $Z_2 = 3 + 4j$, find the (a) magnitude, (b) amplitude, and (c) polar form of the circuit impedance. (Assume that Z, Z_1, and Z_2 are in ohms.)

In Problems **58–61**, *find the resultant of the given vectors in the manner of Example 13. Give your answer in polar form.*

58. $A = 20, \theta_A = 125^\circ$,
$B = 30, \theta_B = 315^\circ$,
$C = 10, \theta_C = 70^\circ$.

59. $A = 100, \theta_A = 310^\circ$,
$B = 120, \theta_B = 35^\circ$,
$C = 50, \theta_C = 40^\circ$.

60. $A = 200, \theta_A = 42^\circ$,
$B = 100, \theta_B = 85^\circ$,
$C = 50, \theta_C = 155^\circ$.

61. $A = 120, \theta_A = 80^\circ$,
$B = 49, \theta_B = 125^\circ$,
$C = 85, \theta_C = 215^\circ$.

13.4 EXPONENTIAL FORM OF A COMPLEX NUMBER

It is not uncommon for an engineer to encounter, and indeed perform, computations involving the *exponential form* of a complex number. This form makes use of a special irrational number that is symbolized by the letter e and is approximated by

$$e \approx 2.71828.$$

The number e plays an important role in mathematics; you will encounter it again in Chapter 15.

A complex number $z = a + bj$ having polar form $r(\cos\theta + j\sin\theta)$ can also be expressed in the **exponential form** defined by

$$\boxed{z = re^{j\theta},}$$

where

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta.}$$

The last equation is called **Euler's formula** and requires that θ be in radians. Our definition of the exponential form $re^{j\theta}$ allows the usual rules for exponents to hold, as will be illustrated in the next section.

Example 1

Express $4 - 4j$ in exponential form.

Solution In polar form we have $4 - 4j = 4\sqrt{2}(\cos 315^\circ + j \sin 315^\circ)$, which in terms of *radians* is $4\sqrt{2}\left(\cos \frac{7\pi}{4} + j \sin \frac{7\pi}{4}\right)$. Thus $r = 4\sqrt{2}$ and $\theta = \frac{7\pi}{4}$, so

$$4 - 4j = re^{j\theta} = 4\sqrt{2}e^{7\pi j/4}.$$

Example 2

Express $3.2(\cos 23.6^\circ + j \sin 23.6^\circ)$ in exponential form.

Solution Here $r = 3.2$ and in terms of radians we have $\theta = 23.6^\circ = 0.412$. Thus

$$3.2(\cos 23.6^\circ + j \sin 23.6^\circ) = 3.2e^{0.412j}.$$

Example 3

Express $3.5e^{1.65j}$ in polar and rectangular forms.

Solution We can immediately write a polar form, which is then used to obtain the rectangular form.

$$\begin{aligned} 3.5e^{1.65j} &= 3.5(\cos 1.65 + j \sin 1.65) && \text{[polar form]} \\ &= 3.5(-0.0791 + 0.9969j) \\ &= -0.28 + 3.49j && \text{[rectangular form].} \end{aligned}$$

Problem Set 13.4

In Problems **1–12**, *express the given complex number in exponential form. If possible, do not use a calculator.*

1. $1 + j$.

2. -6.

3. $-2\sqrt{3} + 2j$.

4. $-3 - 3j$.

5. $-7j$.

6. $1 - j\sqrt{3}$.

7. $-4.6 - 7.8j$.

8. $-0.46 + 1.53j$.

9. $2.8(\cos 14.6^\circ + j \sin 14.6^\circ)$.

10. $7.3(\cos 123.6^\circ + j \sin 123.6^\circ)$.

11. $3.7(\cos 2.46 + j \sin 2.46)$.

12. $3\left(\cos \frac{\pi}{5} + j \sin \frac{\pi}{5}\right)$.

In Problems **13–20**, *express the given complex number in polar and rectangular forms. If possible, do not use a calculator.*

13. $e^{\pi j/2}$.
14. $2e^{(4/3)\pi j}$.
15. $3e^{3\pi j/4}$.
16. $4e^{\pi j}$.
17. $2e^{(5\pi/6)j}$.
18. $e^{(3\pi/2)j}$.
19. $4e^{2.35j}$.
20. $1.6e^{3.01j}$.
21. Express $e^{j\omega t}$ in polar form.

13.5 OPERATIONS IN POLAR FORM

Multiplication or division of complex numbers is easily performed when the numbers are in polar form. We can determine a formula for the product of two complex numbers by considering the product of the exponential forms $r_1e^{j\theta_1}$ and $r_2e^{j\theta_2}$. From rules of exponents, we have

$$(r_1e^{j\theta_1})(r_2e^{j\theta_2}) = r_1r_2e^{j\theta_1+j\theta_2} = r_1r_2e^{j(\theta_1+\theta_2)}.$$

In polar form this gives the following rule:

MULTIPLICATION IN POLAR FORM

$$r_1(\cos\theta_1 + j\sin\theta_1)\cdot r_2(\cos\theta_2 + j\sin\theta_2) = r_1r_2[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)].$$

Therefore, *the product of two complex numbers in polar form is a complex number in which the absolute value is the product of the absolute values of the given numbers and the amplitude is the sum of the amplitudes of the given numbers.* More compactly we have

$$(r_1\angle\theta_1)(r_2\angle\theta_2) = r_1r_2\angle\theta_1+\theta_2.$$

Example 1

Find each of the following products and express the answer in rectangular form.

a. $2(\cos 10^\circ + j\sin 10^\circ)\cdot 4(\cos 20^\circ + j\sin 20^\circ)$.

Solution The product of the absolute values is $2\cdot 4 = 8$ and the sum of the amplitudes is $10^\circ + 20^\circ = 30^\circ$. Thus

$$\begin{aligned} 2(\cos 10^\circ + j\sin 10^\circ)\cdot 4(\cos 20^\circ + j\sin 20^\circ) &= 8(\cos 30^\circ + j\sin 30^\circ) \\ &= 8\left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) \\ &= 4\sqrt{3} + 4j. \end{aligned}$$

b. $(8\angle -35^\circ)(5\angle 160^\circ) = (8\cdot 5)\angle -35^\circ + 160^\circ = 40\angle 125^\circ$

$= 40(\cos 125^\circ + j \sin 125^\circ)$

$= -22.9 + 32.8j.$

c. $(3\angle 9^\circ)(4\angle 30^\circ)(2\angle -6^\circ)$.

Solution The preceding rule can be extended so that it applies to a product of any number of complex numbers. Thus

$$(3\angle 9^\circ)(4\angle 30^\circ)(2\angle -6^\circ) = (3\cdot 4\cdot 2)\angle 9^\circ + 30^\circ - 6^\circ = 24\angle 33^\circ$$

$$= 24(\cos 33^\circ + j \sin 33^\circ)$$

$$= 20.1 + 13.1j.$$

Multiplication of the complex number $a + bj$ by j has special geometrical significance. If $a + bj = r(\cos\theta + j \sin\theta)$ and $j = 1(\cos 90^\circ + j \sin 90^\circ)$, then by the preceding rule we have

$$(a + bj)j = r[\cos(\theta + 90^\circ) + j \sin(\theta + 90^\circ)],$$

which is the vector obtained by rotating $a + bj$ counterclockwise through an angle of 90°. Thus *the effect of multiplying a vector by j is to rotate the vector counterclockwise by 90°*. It is based on this rotational effect that we speak of j as an *operator*.

We now turn our attention to the quotient of two complex numbers. Using laws of exponents gives

$$\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2} e^{j\theta_1 - j\theta_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}.$$

In polar form this means:

DIVISION IN POLAR FORM

$$\frac{r_1(\cos\theta_1 + j \sin\theta_1)}{r_2(\cos\theta_2 + j \sin\theta_2)} = \frac{r_1}{r_2}[(\cos(\theta_1 - \theta_2) + j \sin(\theta_1 - \theta_2)].$$

Therefore, *the quotient of two complex numbers in polar form is a complex number in which the absolute value is the absolute value of the numerator divided by the absolute value of the denominator and the amplitude is the result of subtracting the amplitude of the denominator from that of the numerator.* More compactly we have

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle\theta_1 - \theta_2.$$

Example 2

Perform the indicated operations and express the answer in polar form and rectangular form.

a. $\dfrac{3(\cos 75^\circ + j \sin 75^\circ)}{6(\cos 15^\circ + j \sin 15^\circ)} = \dfrac{3}{6}[\cos(75^\circ - 15^\circ) + j \sin(75^\circ - 15^\circ)]$

$= \dfrac{1}{2}(\cos 60^\circ + j \sin 60^\circ)$ [polar form]

$= \dfrac{1}{2}\left(\dfrac{1}{2} + j\dfrac{\sqrt{3}}{2}\right)$

$= \dfrac{1}{4} + j\dfrac{\sqrt{3}}{4}$ [rectangular form].

b. $\dfrac{8\angle 10^\circ}{3\angle -20^\circ} = \dfrac{8}{3}\angle 10^\circ - (-20^\circ) = \dfrac{8}{3}\angle 30^\circ$ [polar form]

$= \dfrac{8}{3}(\cos 30^\circ + j \sin 30^\circ)$

$= \dfrac{8}{3}\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}j\right) = \dfrac{4\sqrt{3}}{3} + \dfrac{4}{3}j$ [rectangular form].

c. $\dfrac{(3\angle 10^\circ)(5\angle 140^\circ)}{7\angle 230^\circ} = \dfrac{15\angle 150^\circ}{7\angle 230^\circ} = \dfrac{15}{7}\angle 150^\circ - 230^\circ$

$= \dfrac{15}{7}\angle -80^\circ$ [polar form]

$= \dfrac{15}{7}[\cos(-80^\circ) + j \sin(-80^\circ)]$

$= 0.37 - 2.11j$ [rectangular form].

In general, addition (or subtraction) cannot be performed in polar coordinates. For example

$$2\angle 35^\circ + 5\angle 10^\circ \neq 7\angle 45^\circ.$$

However, in the sense of adding like terms, it can be performed when the amplitudes are the same. Thus

$$3\angle 10^\circ + 6\angle 10^\circ = 9\angle 10^\circ.$$

When adding complex numbers in polar form, we convert to rectangular form to perform the addition. Example 3a shows the procedure.

Example 3

Perform the indicated operations and express the answer in polar form.

a. $$\frac{24/10^\circ}{4/100^\circ} + \frac{25/0^\circ}{2.5/-123^\circ} = 6/-90^\circ + 10/123^\circ$$

$$= (0 - 6j) + (-5.45 + 8.39j)$$

$$= -5.45 + 2.39j = 5.95/156.3^\circ.$$

b. $$\frac{1}{(2.3/15.4^\circ)(1.8/13.8^\circ)} = \frac{1/0^\circ}{4.14/29.2^\circ} = 0.24/-29.2^\circ = 0.24/330.8^\circ.$$

Example 4

A capacitive element with impedance $Z_1 = 13.9/-59.7^\circ\ \Omega$ and an inductor with impedance $Z_2 = 26.2/83.4^\circ\ \Omega$ are connected in parallel. Find the circuit impedance Z (in ohms).

Solution Using the formula from Example 8 of Sec. 13.2, we have

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(13.9/-59.7^\circ)(26.2/83.4^\circ)}{13.9/-59.7^\circ + 26.2/83.4^\circ}$$

$$= \frac{364.2/23.7^\circ}{(7.01 - 12.0j) + (3.01 + 26.0j)}$$

$$= \frac{364.2/23.7^\circ}{10.02 + 14.0j} = \frac{364.2/23.7^\circ}{17.2/54.4^\circ}$$

$$= 21.2/-30.7^\circ = 21.2/329.3^\circ.$$

Problem Set 13.5

In Problems **1–14**, *perform the indicated operations and give the answers in polar form.*

1. $2(\cos 17^\circ + j \sin 17^\circ) \cdot 4(\cos 43^\circ + j \sin 43^\circ)$.
2. $4(\cos 125^\circ + j \sin 125^\circ) \cdot 5(\cos 85^\circ + j \sin 85^\circ)$.
3. $3 \cos 105^\circ + j \sin 105^\circ) \cdot 2(\cos 105^\circ + j \sin 105^\circ)$.
4. $5(\cos 324^\circ + j \sin 324^\circ) \cdot 3(\cos 126^\circ + j \sin 126^\circ)$.
5. $\dfrac{30(\cos 145^\circ + j \sin 145^\circ)}{3(\cos 100^\circ + j \sin 100^\circ)}$.
6. $\dfrac{25(\cos 384^\circ + j \sin 384^\circ)}{5(\cos 174^\circ + j \sin 174^\circ)}$.
7. $\dfrac{18(\cos 284^\circ + j \sin 284^\circ)}{3(\cos 59^\circ + j \sin 59^\circ)}$.
8. $\dfrac{32(\cos 125^\circ + j \sin 125^\circ)}{16(\cos 170^\circ + j \sin 170^\circ)}$.
9. $2/44^\circ \cdot 2/11^\circ$
10. $9/39^\circ \cdot 8/28^\circ$.
11. $1/46^\circ \cdot 3/-12^\circ \cdot 7/37^\circ$.
12. $3/-45^\circ \cdot 4/-87^\circ \cdot 0.5/64^\circ$.
13. $\dfrac{2/134^\circ}{4/-250^\circ}$.
14. $\dfrac{4/251^\circ}{5/264^\circ}$.

In Problems **15–22**, *perform the indicated operations and give the answers in rectangular form.*

15. $\dfrac{(3\angle 325^\circ)(12\angle 254^\circ)}{18\angle 9^\circ}$.

16. $\dfrac{4\angle 275^\circ}{(2\angle 44^\circ)(50\angle 19^\circ)}$.

17. $\dfrac{(4\angle 77^\circ)(4\angle 157^\circ)}{(3\angle 200^\circ)(2\angle 50^\circ)}$.

18. $\dfrac{(3\angle 46^\circ)(5\angle 295^\circ)}{(4\angle 200^\circ)(3\angle 23^\circ)}$.

19. $\dfrac{(7\angle -10^\circ)(4\angle 15^\circ)}{(3\angle 41^\circ)(2\angle -63^\circ)}$.

20. $8\angle -10^\circ + 4\angle 15^\circ$.

21. $\dfrac{10\angle 10^\circ}{4\angle 60^\circ} + \dfrac{15\angle 5^\circ}{3\angle 0^\circ}$.

22. $\dfrac{5\angle -4^\circ + 6\angle 5.6^\circ}{(5\angle -4^\circ)(6\angle 5.6^\circ)}$.

23. If a capacitor of impedance $6\angle 275^\circ$ (Ω) and an inductor of impedance $8\angle 40^\circ$ (Ω) are connected in parallel, find the circuit impedance and express the answer in rectangular form. (See Example 4.)

24. The voltage V (in volts) across a circuit with an impedance Z (in ohms) and carrying a current I (in amperes) is given by $V = IZ$. Find V if $I = 1.5\angle 38^\circ$ and $Z = 6\angle 48^\circ$. Express the answer in both polar and rectangular forms.

25. In Problem 24, if the current in the circuit is $4\angle 50^\circ$ and the voltage is $20\angle 30^\circ$, find the impedance of the circuit and express the answer in polar form.

26. Given that the impedance Z of a circuit element is the reciprocal of its admittance Y, find the impedance if $Y = 0.5\angle 30^\circ$. Give your answer in polar form.

27. In the analysis of a mechanical network, the following determinant was encountered:

$$\Delta = \begin{vmatrix} 10 & -0.5 & 0.25\angle 90^\circ \\ 0 & 1.3\angle 20.1^\circ & -0.8\angle 71.2^\circ \\ 0 & -0.8\angle 72.3^\circ & 0.6\angle 62.6^\circ \end{vmatrix}.$$

Find Δ in rectangular form. *Hint*: Expand along the first column.

28. Show that the effect of dividing a complex number $a + bj$ by j is to rotate the vector for $a + bj$ clockwise by 90°.

29. Show that $\dfrac{1}{r\angle \theta} = \dfrac{1}{r}\angle -\theta$.

13.6 DE MOIVRE'S THEOREM

To evaluate an expression such as $(\sqrt{3} + j)^5$, it is obvious that using repeated multiplication would be rather laborious. Fortunately, using polar form reduces the work considerably. To see why, observe that the nth power of $r(\cos\theta + j\sin\theta)$ is given by

$$[r(\cos\theta + j\sin\theta)]^n = (re^{j\theta})^n = r^n e^{n\theta j}$$
$$= r^n(\cos n\theta + j\sin n\theta).$$

That is, **to raise a complex number to a power n, we raise its absolute value to the nth power and multiply its amplitude by n.** This important result, which is true for any integer n, is known as *De Moivre's theorem.*

DE MOIVRE'S THEOREM

$$[r(\cos\theta + j\sin\theta)]^n = r^n(\cos n\theta + j\sin n\theta).$$

For example, to find $(\sqrt{3}+j)^5$, we first write $\sqrt{3}+j$ in polar form.

$$\sqrt{3}+j = 2(\cos 30^\circ + j\sin 30^\circ).$$

Thus

$$\begin{aligned}(\sqrt{3}+j)^5 &= [2(\cos 30^\circ + j\sin 30^\circ)]^5 \\ &= 2^5[\cos(5\cdot 30^\circ) + j\sin(5\cdot 30^\circ)] \qquad \text{[De Moivre's theorem]} \\ &= 32(\cos 150^\circ + j\sin 150^\circ) \\ &= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) \\ &= -16\sqrt{3} + 16j.\end{aligned}$$

Example 1

Evaluate by using De Moivre's theorem and give the answer in rectangular form.

a. $[2(\cos 10^\circ + j\sin 10^\circ)]^6 = 2^6[\cos(6\cdot 10^\circ) + j\sin(6\cdot 10^\circ)]$

$$= 64(\cos 60^\circ + j\sin 60^\circ)$$

$$= 64\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 32 + 32j\sqrt{3}.$$

b. $(3\underline{/30^\circ})^{-3} = 3^{-3}\underline{/-3\cdot 30^\circ} = \frac{1}{27}\underline{/-90^\circ} = \frac{1}{27}[0 + j(-1)] = -\frac{1}{27}j.$

Example 2

Evaluate $(1+j)^{12}$ by using De Moivre's theorem and give the answer in rectangular form.

Solution First we write $1+j$ in polar form.

$$1 + j = \sqrt{2}(\cos 45^\circ + j\sin 45^\circ).$$

Applying De Moivre's theorem gives

$$\begin{aligned}[\sqrt{2}(\cos 45^\circ + j\sin 45^\circ)]^{12} &= (\sqrt{2})^{12}[\cos(12\cdot 45^\circ) + j\sin(12\cdot 45^\circ)] \\ &= (2^{1/2})^{12}(\cos 540^\circ + j\sin 540^\circ) \\ &= 2^6[-1 + j(0)] \\ &= -64.\end{aligned}$$

Example 3

Evaluate by using De Moivre's theorem and give the answer in rectangular form.

a. $(3 + 2j)^4$.

Solution In polar form we have

$$3 + 2j = \sqrt{13}/\underline{33.69°}.$$

Applying De Moivre's theorem gives

$$(3 + 2j)^4 = (\sqrt{13}/\underline{33.69°})^4 = (13^{1/2})^4/\underline{4 \cdot 33.69°}$$

$$= 13^2/\underline{134.76°} = -119.0 + 120.0j.$$

b. $$\frac{(2 + 2j)^{15}}{(\sqrt{3} + j)^{21}} = \frac{(\sqrt{8}/\underline{45°})^{15}}{(2/\underline{30°})^{21}} = \frac{(2^{3/2})^{15}/\underline{15 \cdot 45°}}{2^{21}/\underline{21 \cdot 30°}}$$ [since $\sqrt{8} = (2^3)^{1/2} = 2^{3/2}$]

$$= \frac{2^{45/2}/\underline{675°}}{2^{21}/\underline{630°}} = 2^{3/2}/\underline{45°}$$

$$= 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)$$

$$= 2 + 2j.$$

We conclude this section by using De Moivre's theorem to solve the equation $w^n = z$ for w, where w and z are complex numbers and n is a positive integer. That is, we want to find an nth root of a complex number z. Suppose $z = r(\cos\theta + j\sin\theta)$ and $w = \rho(\cos\varphi + j\sin\varphi)$, where ρ and φ are the Greek letters rho and phi, respectively. Then substituting into $w^n = z$ gives

$$[\rho(\cos\varphi + j\sin\varphi)]^n = r(\cos\theta + j\sin\theta),$$

or, by De Moivre's theorem,

$$\rho^n(\cos n\varphi + j\sin n\varphi) = r(\cos\theta + j\sin\theta).$$

Both sides of this equation represent the same number. But two numbers in polar form represent the same number if, in addition to having the same absolute value, their amplitudes are coterminal angles. In other words, $n\varphi$ can differ from θ by any integral multiple of 360° (or 2π if θ is in radians). Symbolically, this means that

$$n\varphi = \theta + k \cdot 360°$$

$$\varphi = \frac{\theta}{n} + \frac{k \cdot 360°}{n}, \qquad \text{where } k \text{ is an integer.}$$

Also, $\rho^n = r$ is true if $\rho = \sqrt[n]{r}$, the positive nth root of r. Thus the nth roots of z are given by the following formula:

nTH ROOTS OF A COMPLEX NUMBER

The nth roots of $r(\cos\theta + j\sin\theta)$ are denoted by w_k and are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{k\cdot 360^\circ}{n}\right) + j\sin\left(\frac{\theta}{n} + \frac{k\cdot 360^\circ}{n}\right)\right],$$

where $k = 0, 1, 2, \ldots, n-1$.

Each of the n values of k from zero to $n-1$ gives a different root. For $k > n-1$, repetition of a previous root occurs. In summary, there are exactly n different complex nth roots of a nonzero complex number. These roots have the same absolute value.

Example 4

Find the six sixth roots of -64. That is, find the six solutions of the equation $x^6 = -64$.

Solution We first express -64 in polar form:

$$-64 = 64(\cos 180^\circ + j\sin 180^\circ).$$

Thus $r = 64$, $\theta = 180^\circ$, and $n = 6$. Denoting the sixth roots by w_k and substituting in the preceding formula, we get

$$w_k = \sqrt[6]{64}\left[\cos\left(\frac{180^\circ}{6} + \frac{k\cdot 360^\circ}{6}\right) + j\sin\left(\frac{180^\circ}{6} + \frac{k\cdot 360^\circ}{6}\right)\right]$$

$$= 2[\cos(30^\circ + k\cdot 60^\circ) + j\sin(30^\circ + k\cdot 60^\circ)], \qquad k = 0, 1, \ldots, 5.$$

Next we successively replace k by $0, 1, \ldots, 5$. This gives the roots $w_0, w_1, \ldots, w_5$.

If $k = 0$,

$$w_0 = 2(\cos 30^\circ + j\sin 30^\circ) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) = \sqrt{3} + j.$$

If $k = 1$,

$$w_1 = 2(\cos 90^\circ + j\sin 90^\circ) = 2(0 + j) = 2j.$$

If $k = 2$,

$$w_2 = 2(\cos 150^\circ + j\sin 150^\circ) = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}j\right) = -\sqrt{3} + j.$$

If $k = 3$,

$$w_3 = 2(\cos 210^\circ + j \sin 210^\circ) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}j\right) = -\sqrt{3} - j.$$

If $k = 4$,

$$w_4 = 2(\cos 270^\circ + j \sin 270^\circ) = 2(0 - j) = -2j.$$

If $k = 5$,

$$w_5 = 2(\cos 330^\circ + j \sin 330^\circ) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}j\right) = \sqrt{3} - j.$$

You may verify that for $k > 5$, no new roots are obtained. As geometrically shown in Fig. 13.16, the roots correspond to equally spaced points on a circle with center at the origin and radius 2. They appear as vertices of a regular six-sided polygon in the complex plane.

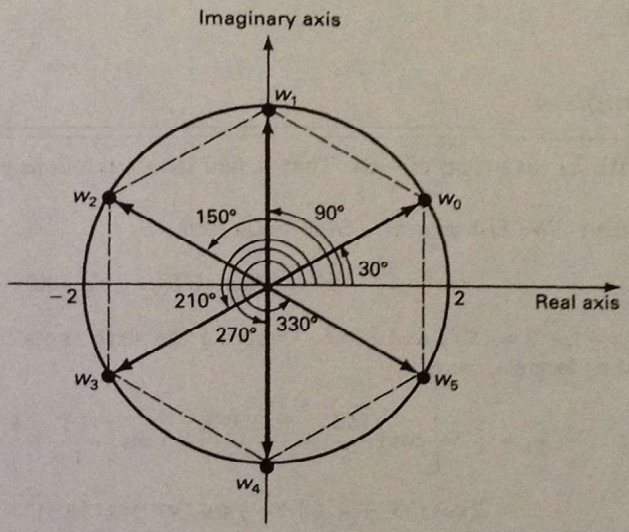


FIGURE 13.16

Example 5

Find the three cube roots of $-1 + j\sqrt{3}$.

Solution We have

$$-1 + j\sqrt{3} = 2(\cos 120^\circ + j \sin 120^\circ).$$

Thus $r = 2$, $\theta = 120^\circ$, and $n = 3$, so

$$w_k = \sqrt[3]{2}\left[\cos\left(\frac{120^\circ}{3} + \frac{k \cdot 360^\circ}{3}\right) + j \sin\left(\frac{120^\circ}{3} + \frac{k \cdot 360^\circ}{3}\right)\right]$$

$$= \sqrt[3]{2}[\cos(40^\circ + k \cdot 120^\circ) + j \sin(40^\circ + k \cdot 120^\circ), \qquad k = 0, 1, 2.$$

Thus

$$\text{If } k = 0, \quad w_0 = \sqrt[3]{2}(\cos 40^\circ + j \sin 40^\circ),$$

$$\text{If } k = 1, \quad w_1 = \sqrt[3]{2}(\cos 160^\circ + j \sin 160^\circ),$$

$$\text{If } k = 2, \quad w_2 = \sqrt[3]{2}(\cos 280^\circ + j \sin 280^\circ).$$

See Fig. 13.17. Note that this problem could have been posed by saying, "Solve the equation $x^3 = -1 + j\sqrt{3}$."

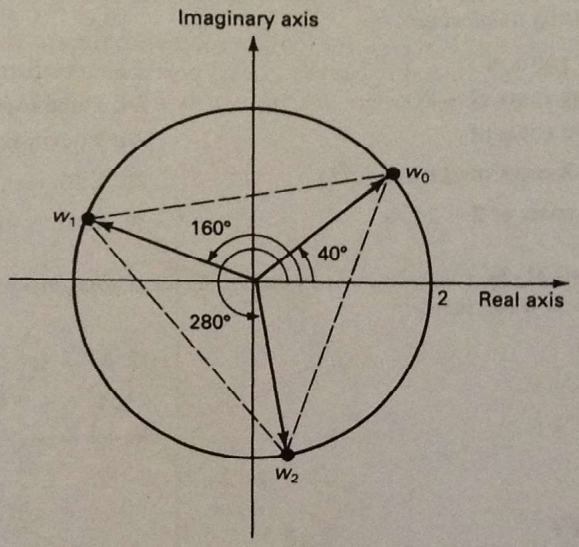


FIGURE 13.17

Problem Set 13.6

In Problems **1–20**, *use De Moivre's theorem to evaluate the expression. Give your answer in both polar and rectangular forms. Do not use a calculator.*

1. $[2(\cos 10^\circ + j \sin 10^\circ)]^3$.

2. $[3(\cos 25^\circ + j \sin 25^\circ)]^6$.

3. $[\sqrt{2}(\cos 15^\circ + j \sin 15^\circ)]^8$.

4. $[2(\cos 150^\circ + j \sin 150^\circ)]^4$.

5. $(\cos 40^\circ + j \sin 40^\circ)^{-6}$.

6. $[2(\cos 45^\circ + j \sin 45^\circ)]^{-3}$.

7. $(\sqrt{2} + j\sqrt{2})^5$.

8. $(-\sqrt{2} - j\sqrt{2})^4$.

9. $(-\sqrt{3} + j)^4$.

10. $\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}j\right)^5$.

11. $(\sqrt{3}+j)^4$.

12. $\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)^4$.

13. $(-\sqrt{2}+j\sqrt{2})^6$.

14. $(1+j)^7$.

15. $(1-j)^{10}$.

16. $(-\sqrt{2}+j\sqrt{2})^4$.

17. $\dfrac{(\sqrt{2}+j\sqrt{2})^{10}}{(\sqrt{3}+j)^{10}}$.

18. $\dfrac{(1-j)^{15}}{(1+j)^{12}}$.

19. $(-2+2j)^{-8}$.

20. $(1-j\sqrt{3})^{-4}$.

In Problems **21–30**, *find the indicated roots. Give the answer in rectangular form if you can do so without using a calculator.*

21. Cube roots of 1.

22. Fourth roots of 16.

23. Square roots of $-8j$.

24. Cube roots of -4.

25. Square roots of j.

26. Fourth roots of $\frac{1}{2}(1-j\sqrt{3})$.

27. Fourth roots of $2(-1+j\sqrt{3})$.

28. Fifth roots of $1+j$.

29. Sixth roots of $2-2j\sqrt{3}$.

30. Sixth roots of $-1-j$.

In Problems **31–36**, *solve the given equation. Express your answer in rectangular form if you can do so without a calculator.*

31. $x^3=1$.

32. $x^4=1$.

33. $x^5-1=0$.

34. $x^6-1=0$.

35. $x^3=1+j$.

36. $x^3=-\sqrt{2}-j\sqrt{2}$.

13.7 REVIEW

Important Terms and Symbols

Section 13.1 j-operator, j, imaginary unit, pure imaginary number, complex number, $a+bj$, real part, imaginary part, imaginary number, (complex) conjugate, equality of complex numbers.

Section 13.3 Complex plane, real axis, imaginary axis, rectangular form, amplitude, argument, absolute value, magnitude, modulus, $|a+bj|$, polar form, $r(\cos\theta+j\sin\theta)$, $r\angle\theta$, principal argument.

Section 13.4 Exponential form, Euler's formula.

Section 13.6 De Moivre's theorem, nth roots of a complex number.

Formula Summary

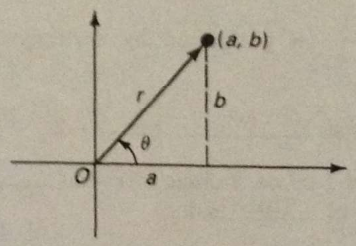
j-OPERATOR

$$j^2=-1, \qquad j=\sqrt{-1}$$

POLAR FORM OF COMPLEX NUMBER

$$a + bj = r(\cos\theta + j\sin\theta) = r\angle\theta,$$

where $r = \sqrt{a^2 + b^2}$ and $\tan\theta = \dfrac{b}{a}$.



OPERATIONS IN POLAR FORM

$$(r_1\angle\theta_1)(r_2\angle\theta_2) = r_1r_2\angle\theta_1 + \theta_2. \qquad \frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1}{r_2}\angle\theta_1 - \theta_2.$$

DE MOIVRE'S THEOREM

$$[r(\cos\theta + j\sin\theta)]^n = r^n(\cos n\theta + j\sin n\theta)$$

EXPONENTIAL FORM

$$z = re^{j\theta},$$

where $e^{j\theta}$ is given by Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

Review Questions

1. The powers j^2, j^3, j^4, j^5, and j^{-2} are equal, respectively, to _____.
2. The real part of $3 - 4j$ is _____(a)_____ and the imaginary part is _____(b)_____.
3. Among the numbers $\sqrt[3]{-8}$, $\sqrt{-2}$, and $\sqrt{(-2)(-2)}$, which are pure imaginary?
4. The number $a + bj$ is the conjugate of what number? _____
5. The product of a complex number and its conjugate is _____(always)(sometimes)_____ a real number.

6. In the representation of a complex number $a + bj$ in the complex plane, the absolute value of $a + bj$ is the distance from $(0, 0)$ to the point __(a)__. This distance is given by the expression __(b)__.

7. Representation of a complex number in polar form __(is)(is not)__ unique.

8. A polar form for 8 is _____.

9. In rectangular form, $3(\cos 270° + j \sin 270°)$ is equal to _____.

10. The division $\dfrac{1}{1-j}$ is performed by multiplying the numerator and denominator by _____.

11. $\sqrt{(-2)(-2)}$ equals __(a)__ but $\sqrt{-2}\sqrt{-2}$ equals __(b)__.

12. The nth roots of 1 lie on a circle of radius __(a)__ and determine the vertices of a regular polygon of __(b)__ sides.

13. The imaginary part of a complex number __(is)(is not)__ a real number.

14. If $1 + zj = 0$, then z equals _____.

15. The square of a pure imaginary number __(is)(is not)__ a real number.

16. In the complex plane, the vector representing the number $-2 + j\sqrt{3}$ has its initial point at __(a)__ and terminates at the point __(b)__.

17. If z is a sixth root of $-2 + j\sqrt{3}$, then z^6 equals _____.

18. In the complex plane, $e^{j\theta}$ is a point on a circle with radius _____.

Answers to Review Questions

1. $-1, -j, 1, j, -1$. 2. (a) 3, (b) -4. 3. $\sqrt{-2}$. 4. $a - bj$. 5. Always.
6. (a) (a, b), (b) $\sqrt{a^2 + b^2}$. 7. Is not. 8. $8(\cos 0° + j \sin 0°)$. 9. $-3j$. 10. $1 + j$.
11. (a) 2, (b) -2. 12. (a) 1, (b) n. 13. Is. 14. j. 15. Is.
16. (a) $(0, 0)$, (b) $(-2, \sqrt{3})$. 17. $-2 + j\sqrt{3}$. 18. 1.

Review Problems

In Problems **1–20**, *perform the operations and simplify.*

1. $(8 - 3j) + 2(6 + 4j)$.
2. $9j^2 - 3j + 2j^3$.
3. $(4 + 3j) - (6 + j) + 10j$.
4. $3j(1 + 2j)$.
5. $(1 + 2j)(3 + 4j)$.
6. $(1 - j)^2$.
7. $(2 + j)(2 - j)$.
8. $(2 - 5j) + j(6j)$.
9. $\dfrac{4 + 2j}{3 + j}$.
10. $\dfrac{3j}{6 - 2j}$.
11. $-\sqrt{-3} + \sqrt{-12} + 2$.
12. $\sqrt{-8}(1 - \sqrt{-2})$.
13. $(1 + \sqrt{-4})(1 + \sqrt{-9})$.
14. $\dfrac{2}{2 + \sqrt{-25}}$.

15. $8 + 2j - j(2 + j) + 5j^3 - \sqrt{-4}$.

16. $(2 + j)(3 - 3j) - 6j(j + 7)$.

17. $\dfrac{7}{j + 2}$.

18. $\dfrac{j - 6}{5 + 2j}$.

19. $\dfrac{2 + j}{(1 + j)(2 - j)}$.

20. $(3 + 2j)(3 - 2j) - (1 + 3j)^2$.

In Problems **21–24**, *solve the equations.*

21. $2x - 5 - 2x^2 = 0$.

22. $3x^2 - 2x + 6 = 0$.

23. $3(t - 1) = 2t^2$.

24. $x^2 + 5 = -1(1 - x)$.

In Problems **25–32**, *express in polar form.*

25. $4\sqrt{2} + 4j\sqrt{2}$.

26. $\dfrac{3}{5} - \dfrac{3\sqrt{3}}{5}j$.

27. $-\dfrac{\sqrt{3}}{2} + \dfrac{j}{2}$.

28. $-27j$.

29. $7 - 8j$.

30. $-0.3 - 0.2j$.

31. $3e^{\pi j/4}$.

32. $2e^{1.5j}$.

In Problems **33–38**, *express in rectangular form.*

33. $\sqrt{3}(\cos 315^\circ + j \sin 315^\circ)$.

34. $\frac{1}{2}(\cos 210^\circ + j \sin 210^\circ)$.

35. $\sqrt{2}(\cos 135^\circ + j \sin 135^\circ)$.

36. $\frac{2}{3}(\cos 60^\circ + j \sin 60^\circ)$.

37. $3(\cos 22.1^\circ + j \sin 22.1^\circ)$.

38. $2.4\angle 320.7^\circ$.

In Problems **39–45**, *perform the indicated operations and give each answer in polar and rectangular forms.*

39. $2(\cos 341^\circ + j \sin 341^\circ) \cdot 3(\cos 109^\circ + j \sin 109^\circ)$.

40. $(1.4\angle 40^\circ)(7.5\angle 75.2^\circ)$.

41. $\dfrac{\sqrt{8}(\cos 284^\circ + j \sin 284^\circ)}{\sqrt{2}(\cos 59^\circ + j \sin 59^\circ)}$.

42. $\dfrac{0.780\angle 42^\circ}{0.156\angle 73^\circ}$.

43. $[2.4(\cos 10^\circ + j \sin 10^\circ)]^5$.

44. $\dfrac{(3\angle 23^\circ)^4}{(2\angle 17^\circ)^5}$.

45. $\dfrac{[2(\cos 22^\circ + j \sin 22^\circ)]^8 \cdot [3(\cos 125^\circ + j \sin 125^\circ)]^5}{[3(\cos 78^\circ + j \sin 78^\circ])^4 \cdot [2(\cos 139^\circ + j \sin 139^\circ)]^2}$.

In Problems **46–48**, *give all answers in polar form.*

46. Find all square roots of $-1 + j\sqrt{3}$.

47. Find all fifth roots of $-\sqrt{2} - j\sqrt{2}$.

48. Find all eighth roots of 1.

In Problems **49–52**, *express in exponential form.*

49. $5j$.

50. $-1 + j\sqrt{3}$.

51. $-1 + j$.

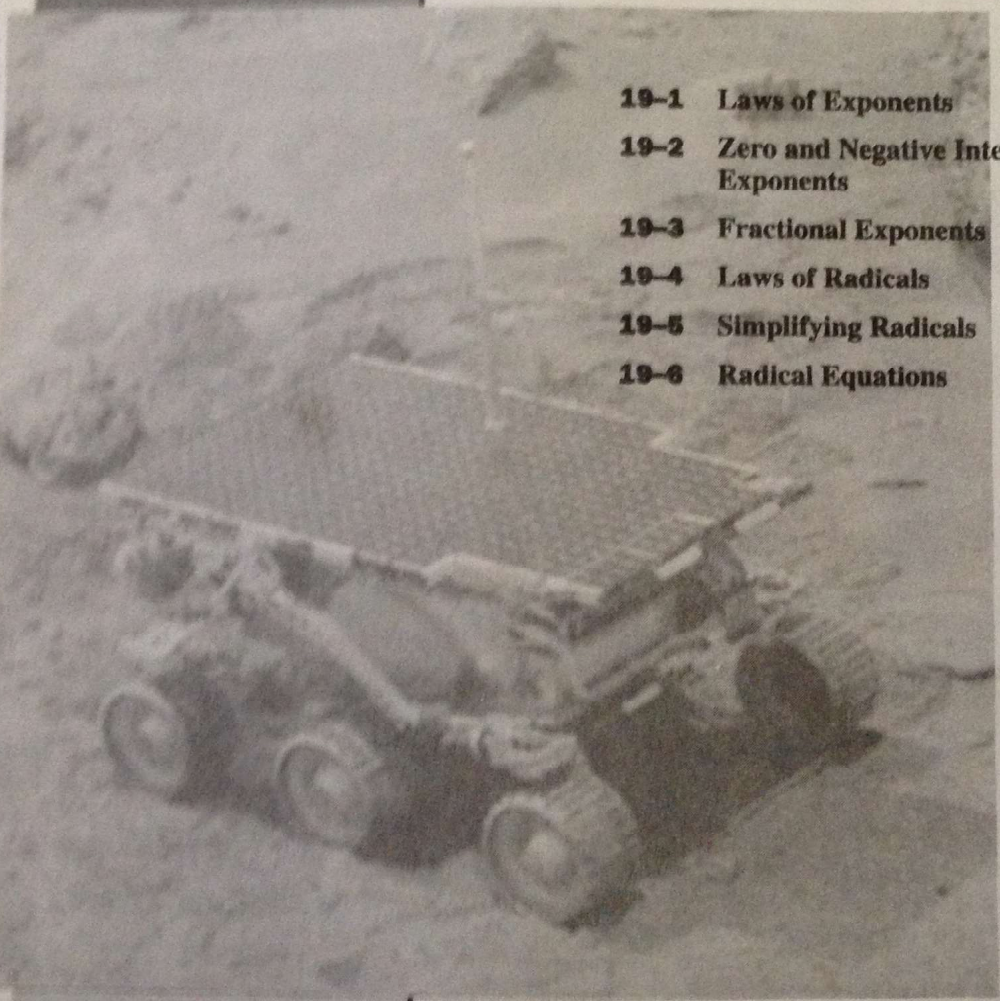
52. 9.

53. In the analysis of a circuit, the current was found to be $100/(4 + 6j)$ amperes. What is the absolute value of the current?

54. In an alternating current series circuit, the admittance Y is expressed as the reciprocal of the impedance Z. If $Z = 16.8 - 22.1j$, find Y.

55. The current I through a particular circuit is given by $I = V/Z$, where I is in amperes, voltage V is in volts, and impedance Z is in ohms.
(a) Find I if $V = 8.1 - 2.0j$ and $Z = 5.6 - 0.3j$.
(b) Find Z if $I = 1 + j$ and $V = 1 - j$.

56. Suppose $R + \left(\omega L - \dfrac{1}{\omega C}\right)j = re^{j\theta}$. Find r and $\tan\theta$.



19–1 Laws of Exponents

19–2 Zero and Negative Integers as Exponents

19–3 Fractional Exponents

19–4 Laws of Radicals

19–5 Simplifying Radicals

19–6 Radical Equations

The Sojourner Rover on the rocky surface of Mars. (Courtesy of NASA/Jet Propulsion Laboratory)

In this chapter the laws of exponents are applied to literal expressions and to expressions with zero and negative integers as exponents. The laws for radicals are used to simplify expressions with radicals. The chapter concludes with the solution of radical equations.

19–1 LAWS OF EXPONENTS

The fundamental laws for working with exponents, along with an example of the application of each, are given below. Exponents follow the five stated laws:

RULE 19–1. Five Laws of Exponents

1. $x^m x^n = x^{m+n}$ (19–1)

   Example: $x^4 x^3 = x^{4+3} = x^7$

2. $(x^m)^n = x^{mn}$ (19–2)

   Example: $(x^7)^3 = x^{7\times 3} = x^{21}$

3. $(xy)^m = x^m y^m$ (19–3)

   Example: $(xy)^3 = x^3 y^3$

4. $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}, y \neq 0$ (19–4)

   Example: $\left(\frac{x}{y}\right)^3 = \frac{x^3}{y^3}$

5. $\frac{x^m}{x^n} = x^{m-n}, x \neq 0$ (19–5)

   Example: $\frac{x^5}{x^3} = x^{5-3} = x^2$

EXERCISE 19–1

Use the five laws of exponents to simplify each of the following.

1. a^3a^3
2. $(x^2)^3$
3. $(2b)^3$
4. $(xy)^2$
5. $\frac{c^3}{c^2}$
6. $b^x b^{5x}$
7. $-(7c)^2$
8. $c^{3a}c^{-2a}$
9. $(y^{2x})^2$
10. $\frac{8x^3}{x^2}$
11. $\left(\frac{2x}{3a}\right)^3$
12. $(x^m y^n)^2$

13. $(a^b)^c$ 14. $\dfrac{24x^5y^4}{-6x^3y^3}$ 15. $-2x(xy)^3$

16. $(x^{3a})^a$ 17. $y^{a+b}y^{2-b}$ 18. $(2ab^2)^3$

19. $\left(\dfrac{2x^5y}{x^2}\right)^3$ 20. $\left(\dfrac{2a^2}{b^3}\right)^3$ 21. $\left(\dfrac{5S^2}{T}\right)^2$

19–2 ZERO AND NEGATIVE INTEGERS AS EXPONENTS

Zero Exponent

When evaluating the expression a^m/a^m, the solution is 1. Applying the laws of exponents to this expression results in $a^m/a^m = a^{m-m} = a^0$. However, we already know that $a^m/a^m = 1$, so a^0 must also be 1. Rule 19–2 states this.

RULE 19–2. Zero Exponent

Any number, except zero, raised to the zero power is 1. Thus $a^0 = 1$ $(a \neq 0)$.

Negative Exponents

What does the expression a^{-3} mean? Let's investigate the meaning of this result: $a^4/a^7 = a^{4-7} = a^{-3}$. However, a^4/a^7 and a^7/a^4 are reciprocals. $a^4/a^7 = a^{-3}$, and $a^7/a^4 = a^3$. Then a^{-3} and a^3 are also reciprocals. Therefore, $a^3 = 1/a^{-3}$, and $a^{-3} = 1/a^3$. We see that a^{-3} is the same as $1/a^3$.

RULE 19–3. Negative Exponents

Numbers written with negative exponents may be written with positive exponents by transferring the factor from one term of the fraction to the other term. *Examples:* a^{-m} may be written as $1/a^m$ *and* $1/b^{-n}$ may be written as b^n.

EXAMPLE 19–1 Express $\dfrac{3}{2x^{-2}}$ with positive exponents.

Solution

$$\frac{3}{2x^{-2}} = \frac{3x^2}{2}$$

EXAMPLE 19–2 Express $\dfrac{x^{-5}}{3^{-1}x}$ with positive exponents and simplify.

Solution
$$\frac{x^{-5}}{3^{-1}x} = \frac{3}{x^5 \cdot x} = \frac{3}{x^6}$$

EXERCISE 19–2

Write the value of each of the following.

1. 5^0 **2.** $3^0(2)$ **3.** $4a^0$
4. $(4a)^0$ **5.** $(-4)^0$ **6.** $6/(3x^0)$
7. $(2 + 2a^0)/4$ **8.** 2^{-1} **9.** 2^{-2}
10. $(-3)^{-2}$ **11.** $1/3^{-2}$ **12.** 5×3^{-1}
13. $32(4^{-2})$ **14.** $1/(2 + a^0)^{-2}$ **15.** $1/(2^0 + a^0)$

Express each of the following with positive exponents.

16. a^{-3} **17.** xy^{-2} **18.** $-5b^{-2}$
19. $\dfrac{2a^{-3}}{b}$ **20.** $\dfrac{a^{-1}b^{-1}}{c^{-1}}$ **21.** $\dfrac{3a}{b^{-2}}$
22. $(x^{-1}y)^{-2}$ **23.** $(x^2y^{-3})^{-1}$ **24.** $(x^{-2}/y^{-3})^{-2}$
25. $(x/y)^{-2}$ **26.** $\left(\dfrac{2ab}{a^2}\right)^{-2}$ **27.** $(3b^2c)^{-3}$

19–3 FRACTIONAL EXPONENTS

Fractional exponents follow the laws of exponents where the numerator indicates the power of the base and the denominator indicates the root of the base. Thus, $a^{m/n} = \sqrt[n]{a^m}$, read as "the nth root of a to the mth power."

RULE 19–4. Fractional Exponents

Fractional exponents follow the laws of integer exponents:

1. The numerator of the fraction is the power of the base.
2. The denominator of the fraction is the root of the base.

Fractional exponents are used to simplify radicals, as is shown in the following examples. They are the basis of the system of logarithms studied in Chapter 20.

Day: Date:

EXAMPLE 19–3 Write $8^{2/3}$ with a radical and evaluate.

Solution

$$8^{2/3} = \sqrt[3]{8^2}$$

Square 8:

$$8^{2/3} = \sqrt[3]{64}$$

Extract root: $\sqrt[3]{64} = 4$

$\therefore$ $8^{2/3} = \sqrt[3]{8^2} = 4$

EXAMPLE 19–4 Evaluate $\sqrt[4]{3^6}$ with your calculator.

Solution Express $\sqrt[4]{3^6}$ with fractional exponents:

$$3^{6/4}$$

Express the fraction as a decimal:

$$3^{6/4} = 3^{1.5}$$

Evaluate $3^{1.5}$ using $\boxed{y^x}$ on your calculator:

$$3 \boxed{y^x} 1.5 \Rightarrow 5.196$$

$\therefore$ $\sqrt[4]{3^6} = 5.2$

EXAMPLE 19–5 Simplify $\left(\dfrac{a^2b^3}{a^3}\right)^{1/6}$.

Solution Distribute the 1/6 to the other exponents:

$$\left(\frac{a^2b^3}{a^3}\right)^{1/6} = \frac{a^{2/6}b^{3/6}}{a^{3/6}}$$

Clear the denominator:

$$\frac{a^{2/6}b^{3/6}}{a^{3/6}} = a^{2/6}a^{-3/6}b^{3/6}$$

Simplify:

$$a^{-1/6}b^{1/2} = b^{1/2}/a^{1/6}$$

$\therefore$ $\left(\dfrac{a^2b^3}{a^3}\right)^{1/6} = b^{1/2}/a^{1/6}$

EXAMPLE 19–6 Simplify $(\sqrt[3]{x^2})(\sqrt[2]{x})^3$.

Solution Express with fractional exponents:

$$x^{2/3}(x^{1/2})^3 = x^{2/3}x^{3/2}$$

Find the common denominator and add exponents:

$$x^{2/3}x^{3/2} = x^{4/6}x^{9/6} = x^{13/6}$$

Simplify:

$$x^{13/6} = x^2 \cdot x^{1/6} = x^2\sqrt[6]{x}$$

$$\therefore \quad (\sqrt[3]{x^2})(\sqrt[2]{x})^3 = x^2\sqrt[6]{x}$$

EXERCISE 19–3

Calculator Drill

Evaluate each of the following.

1. $25^{1/2}$	**2.** $36^{1/2}$	**3.** $144^{1/2}$
4. $27^{1/3}$	**5.** $64^{1/3}$	**6.** $32^{1/5}$
7. $4^{3/2}$	**8.** $125^{2/3}$	**9.** $32^{2/5}$
10. $48^{5/6}$	**11.** $96^{3/8}$	**12.** $50^{7/5}$
13. $\left(\frac{1}{16}\right)^{1/2}$	**14.** $\left(\frac{4}{9}\right)^{3/2}$	**15.** $(0.062)^{1/3}$
16. $(0.125)^{3/2}$	**17.** $(6^2)^{1/4}$	**18.** $(5^4)^{3/4}$
19. $(7^3)^{3/10}$	**20.** $(3^5)^{2/5}$	**21.** $(4)^{5/6}$

Written Exercise

Express each of the following as a radical.

22. $5^{1/2}$	**23.** $a^{1/3}$	**24.** $b^{2/3}$
25. $x^{5/2}$	**26.** $5a^{1/2}$	**27.** $7a^{4/3}$
28. $-(2a^3)^{1/2}$	**29.** $-(7b^5)^{1/3}$	**30.** $(5c^2)^{3/2}$

Express the following without a radical sign.

31. $\sqrt{2}$	**32.** $\sqrt{5}$	**33.** $-2\sqrt{a}$
34. $\sqrt[3]{b^2}$	**35.** $-5b\sqrt[5]{c^2d}$	**36.** $-7c\sqrt{a+b}$
37. $\sqrt[3]{T+s}$	**38.** $\sqrt[4]{4\pi+\theta}$	**39.** $-\sqrt[5]{R-t}$

Simplify.

40. $\left(\frac{a^{1/3}}{b^{2/3}}\right)^6$	**41.** $(a^{2/3})^{1/2}$	**42.** $\left(\frac{a^{1/2}b^{4/3}}{a^{-2}b}\right)^3$
43. $\sqrt[7]{x^4}(\sqrt[7]{x})^3$	**44.** $\frac{(\sqrt[3]{y})^5}{\sqrt[3]{y^2}}$	**45.** $\frac{\sqrt[3]{a^4}}{\sqrt[4]{a^3}}$

19–4 LAWS OF RADICALS

The following five basic laws for radicals are used when operating on expressions containing radicals.

RULE 19–5. Five Laws for Radicals

1. $(\sqrt[n]{x})^n = x$ (19–6)

   Example: $(\sqrt[5]{x})^5 = (x^{1/5})^5 = x^{5/5} = x$

2. $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$ (19–7)

   Example: $\sqrt[3]{48} = \sqrt[3]{8 \cdot 6} = \sqrt[3]{8}\,\sqrt[3]{6} = 2\sqrt[3]{6}$

3. $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$ (19–8)

   Example: $\sqrt[3]{\dfrac{27}{8}} = \dfrac{\sqrt[3]{27}}{\sqrt[3]{8}} = \dfrac{3}{2}$

4. $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$ (19–9)

   Example: $\sqrt[5]{\sqrt[3]{x}} = \sqrt[15]{x} = x^{1/15}$

5. $\sqrt[m]{x^n} = x^{n/m}$ (19–10)

   Example: $\sqrt[3]{x^5} = x^{5/3}$

EXERCISE 19–4

Apply the indicated law for radicals to the following.

Equation 19–6:

1. $(\sqrt[3]{7})^3$
2. $(\sqrt[3]{bc})^3$
3. $-(\sqrt[3]{6^3})^3$
4. $-(\sqrt{a^3})^2$
5. $(\sqrt[7]{x^{14}})^7$
6. $-(\sqrt[3]{y^9})^3$

Equation 19–7:

7. $\sqrt[3]{ab}$
8. $\sqrt[5]{5x}$
9. $-\sqrt[4]{cd}$
10. $-\sqrt{13xy}$
11. $\sqrt[7]{4mn}$
12. $-\sqrt[3]{11bc}$

Equation 19–8:

13. $\sqrt{\dfrac{2}{3}}$
14. $\sqrt{\dfrac{7}{13}}$
15. $\sqrt{\dfrac{a}{b}}$
16. $-\sqrt{\dfrac{2b}{c}}$
17. $-\sqrt{\dfrac{3x}{5y}}$
18. $-\sqrt{\dfrac{ax}{by}}$

Equation 19–9:

19. $\sqrt{\sqrt[3]{7}}$ 20. $\sqrt{\sqrt[5]{5}}$ 21. $\sqrt[3]{\sqrt{a}}$
22. $\sqrt[3]{\sqrt[4]{ax}}$ 23. $-\sqrt[5]{\sqrt{aby}}$ 24. $-\sqrt[3]{\sqrt[7]{3x}}$

Equation 19–10:

25. $\sqrt[3]{3^2}$ 26. $\sqrt{2^3}$ 27. $\sqrt{6^5}$
28. $\sqrt[5]{c^3}$ 29. $-\sqrt[9]{a^7}$ 30. $-\sqrt[7]{b^3}$

19–5 SIMPLIFYING RADICALS

A radical may be simplified by the following means: (1) removing factors from the radicand, (2) lowering the index of the radical, or (3) rationalizing the denominator. Each of these means of simplifying a radical is investigated separately.

Simplifying by Removing Factors

Remove factors from the radicand by expressing the radicand as a product of factors using Equation 19–7. Select the factors so that one factor contains all the perfect nth powers, where n is the index of the radical. This is demonstrated in the following examples.

EXAMPLE 19–7 Simplify $\sqrt{27a^3b}$.

Solution Factor into two radicals, one containing perfect squares:

$$\sqrt{27a^3b} = \sqrt{9a^2}\,\sqrt{3ab}$$

$$\therefore \quad \sqrt{27a^3b} = 3a\sqrt{3ab}$$

EXAMPLE 19–8 Simplify $\sqrt[3]{-x^7}$.

Solution Factor into two radicals, one containing perfect cubes:

$$\sqrt[3]{-x^7} = \sqrt[3]{(-1)^3x^6}\,\sqrt[3]{x}$$

$$\therefore \quad \sqrt[3]{-x^7} = -x^2\sqrt[3]{x}$$

Simplifying by Lowering the Index

Lower the index by first expressing the radicand as a power of some quantity, and then apply Equation 19–10. This procedure is shown in Example 19–9.

EXAMPLE 19–9 Simplify $\sqrt[6]{27}$.

Solution Express 27 as a power of 3:

$$\sqrt[6]{27} = \sqrt[6]{3^3}$$

Express with fractional exponents:

$$\sqrt[6]{3^3} = 3^{3/6} = 3^{1/2}$$

$$\therefore \quad \sqrt[6]{27} = 3^{1/2} = \sqrt{3}$$

Simplifying by Rationalizing the Denominator

The process of removing the radical from the denominator of a fraction is called **rationalizing the denominator.** When the denominator is rationalized, it is changed from an irrational to a rational form. The procedure for rationalizing the denominator is given in Rule 19–6 and demonstrated in the following examples.

RULE 19–6. Rationalizing the Denominator

To rationalize the denominator, multiply the numerator and denominator of the radicand by a factor that will make the denominator a perfect nth power, where n is the index of the radical.

EXAMPLE 19–10 Rationalize $\sqrt{\frac{1}{3}}$.

Solution Multiply numerator and denominator by 3, thus making the denominator a perfect square:

$$\sqrt{\frac{1 \cdot 3}{3 \cdot 3}} = \sqrt{\frac{3}{9}}$$

Apply Equation 19–8:

$$\sqrt{\frac{3}{9}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3} = \frac{1}{3}\sqrt{3}$$

$$\therefore \quad \sqrt{\frac{1}{3}} = \frac{1}{3}\sqrt{3}$$

EXAMPLE 19–11 Rationalize $\sqrt[3]{\frac{3}{2x}}$.

Solution Multiply numerator and denominator by $4x^2$ to make the denominator a perfect cube:

$$\sqrt[3]{\frac{3}{2x}} = \sqrt[3]{\frac{12x^2}{8x^3}}$$

Apply Equation 19–8:

$$\sqrt[3]{\frac{12x^2}{8x^3}} = \frac{\sqrt[3]{12x^2}}{\sqrt[3]{8x^3}} = \frac{\sqrt[3]{12x^2}}{2x}$$

$$\therefore \quad \sqrt[3]{\frac{3}{2x}} = \frac{1}{2x}\sqrt[3]{12x^2}$$

To rationalize the denominator when the radicals are of different *orders*, multiply the numerator and denominator by a factor that will make the denominator a rational form.

EXAMPLE 19–12 Rationalize $\frac{\sqrt[3]{3}}{\sqrt{2}}$.

Solution Multiply numerator and denominator by $\sqrt{2}$:

$$\frac{\sqrt[3]{3}}{\sqrt{2}} = \frac{\sqrt[3]{3}\sqrt{2}}{\sqrt{2}\sqrt{2}} = \frac{\sqrt[3]{3}\sqrt{2}}{2}$$

Combine the radicals; express them using fractional exponents:

$$\frac{\sqrt[3]{3}\sqrt{2}}{2} = \frac{3^{1/3} \cdot 2^{1/2}}{2}$$

Express the exponents with common denominators:

$$\frac{3^{2/6} \cdot 2^{3/6}}{2} = \frac{(3^2 \cdot 2^3)^{1/6}}{2}$$

Simplify and write as a radical:

$$\frac{(3^2 \cdot 2^3)^{1/6}}{2} = \frac{\sqrt[6]{72}}{2}$$

$$\therefore \quad \frac{\sqrt[3]{3}}{\sqrt{2}} = \frac{\sqrt[6]{72}}{2}$$

EXERCISE 19–5

Simplify each of the following radicals, expressing the answer as a radical.

1. $\sqrt{64x^3y^5}$
2. $\sqrt[3]{8a^2y^6}$
3. $\sqrt[3]{54x^{11}y^8}$
4. $\sqrt{32x^4y^3}$
5. $-\sqrt{18a^6b^8}$
6. $\sqrt{\dfrac{16x^7}{a^2y^6}}$
7. $\sqrt{\dfrac{12a^5}{9b^{10}}}$
8. $\sqrt[8]{16}$
9. $\sqrt[9]{64}$
10. $\sqrt[6]{125}$
11. $\sqrt[10]{32}$
12. $\sqrt[6]{25x^4}$
13. $\sqrt[8]{81a^2}$
14. $\sqrt[10]{x^4y^8}$
15. $\sqrt{\dfrac{3}{5}}$
16. $\sqrt{\dfrac{3}{7}}$
17. $\sqrt{\dfrac{5}{3x^2}}$
18. $\dfrac{\sqrt[3]{4}}{\sqrt{2}}$
19. $\dfrac{\sqrt[3]{2a}}{\sqrt{3a}}$
20. $\dfrac{-\sqrt{6x^2}}{\sqrt[3]{9}}$
21. $\sqrt{\dfrac{1-\frac{1}{2}}{2}}$
22. $\sqrt{2-\left(\frac{1}{5}\right)^2}$
23. $\sqrt{1-\left(\dfrac{\sqrt{3}}{2}\right)^2}$
24. $\sqrt{1-\left(\frac{3}{5}\right)^2}$
25. $\sqrt{4y^2+36}$
26. $\sqrt[3]{\dfrac{16x^7}{y^4}}$
27. $\sqrt{\left(\dfrac{b^{2/3}c^{1/2}}{2^{-1}a^{-2}}\right)^6}$
28. $\sqrt[3]{\left(\dfrac{a^2b}{125a^{-3}}\right)^{-1}}$
29. $\sqrt{2x+6+\dfrac{9}{2x}}$
30. $\sqrt{\sqrt[5]{32}\cdot\sqrt{4}}$
31. $\sqrt[3]{\dfrac{a}{8}+\dfrac{-3}{y^3}}$
32. $\sqrt{\dfrac{4b^2}{9}-\dfrac{16}{x^2}}$

19–8 RADICAL EQUATIONS

An equation in which a variable appears as part of the radicand (within the radical) or has a fractional exponent is called a **radical equation.** Thus, $\sqrt{y} = 2$ and $\sqrt{x} + 1 = 3$ are radical equations. The equation $x + \sqrt{3} = 0$ is not a radical equation because x is not part of the radicand.

RULE 19–7. Solving Radical Equations

To solve a radical equation:

1. Isolate the radical term in one member of the equation.
2. Raise each member of the equation to a power equal to the index of the radical.
3. Solve the resulting equation for the root.
4. Check the root, as the preceding process may introduce roots into the derived equation that the original equation did not have.

EXAMPLE 19–13 Solve $5 + \sqrt{x} = 10$ for x.

Solution To solve, follow the steps of Rule 19–7.

Step 1: Isolate $\sqrt{x}$ in the left member:

$$\sqrt{x} = 5$$

Step 2: Square both members:

$$(\sqrt{x})^2 = 5^2$$

Step 3: Solve:

$$x = 25$$

Step 4: Check.

Check Does $5 + \sqrt{x} = 10$ when $x = 25$?

$$5 + \sqrt{25} = 10$$
$$5 + 5 = 10 \quad \text{Yes.}$$

$\therefore$ $x = 25$ is the root.

EXAMPLE 19–14 Solve $2\sqrt{y - 2} + 4 = 8$.

Solution Use the steps of Rule 19–7.

Step 1: Isolate $\sqrt{y-2}$ in the left member:

$$\sqrt{y-2} = \frac{8-4}{2} = 2$$

Step 2: Square both members:

$$y - 2 = 4$$

Step 3: Solve:

$$y = 6$$

Check Does $2\sqrt{y-2} + 4 = 8$ when $y = 6$?

$$\begin{aligned} 2\sqrt{6-2} + 4 &= 8 \\ 2\sqrt{4} + 4 &= 8 \\ 4 + 4 &= 8 \\ 8 &= 8 \quad \text{Yes.} \end{aligned}$$

$\therefore$ $y = 6$ is the solution.

EXAMPLE 19–15 Solve $-4 = x - \sqrt{3x + 10}$ for x.

Solution Follow the steps of Rule 19–7.

Step 1: Isolate $\sqrt{3x + 10}$ in the left member:

$$\sqrt{3x + 10} = x + 4$$

Step 2: Square both members:

$$\begin{aligned} (\sqrt{3x + 10})^2 &= (x + 4)^2 \\ 3x + 10 &= x^2 + 8x + 16 \end{aligned}$$

Step 3: Combine like terms and equate to zero:

$$x^2 + 5x + 6 = 0$$

Solve the quadratic by factoring:

$$\begin{aligned} (x + 3)(x + 2) &= 0 \\ x + 3 &= 0 \qquad x + 2 = 0 \\ x &= -3 \qquad x = -2 \end{aligned}$$

Step 4: There are two possible roots; check both.

Check Does $-4 = x - \sqrt{3x + 10}$ when $x = -3$?

$$\begin{aligned} -4 &= -3 - \sqrt{-9 + 10} \\ -4 &= -4 \quad \text{Yes.} \end{aligned}$$

Check Does $-4 = x - \sqrt{3x + 10}$ when $x = -2$?

$$-4 = -2 - \sqrt{-6 + 10}$$
$$-4 = -2 - 2$$
$$-4 = -4 \quad \text{Yes.}$$

$\therefore$ $x = -2$ and $x = -3$ are both roots.

EXERCISE 19–6

Solve and check each of the following.

1. $\sqrt{x} = 7$
2. $\sqrt{x} = 5$
3. $\sqrt{4x} = 8$
4. $\sqrt{3a} = 6$
5. $\sqrt{6x + 1} = 5$
6. $\sqrt{b - 1} = 2$
7. $\sqrt{a + 2} = 8$
8. $5 = \sqrt{1 + 4x}$
9. $2\sqrt{x + 3} = 6$
10. $4\sqrt{2a} = 8$
11. $5\sqrt{3y} = 15$
12. $3\sqrt{x + 1} = 9$
13. $2\sqrt{a} - 1 = 2$
14. $\sqrt{2y} + 3 = 1$
15. $\sqrt{6a} - 2 = 5$
16. $3 - \sqrt{2x - 7} = 0$
17. $\sqrt{5c - 2} + 3 = 6$
18. $\sqrt{x^2 - 8} + 4 = x$
19. $\sqrt{5 + a^2} - 5 = -a$
20. $x + 3\sqrt{x} = 10$
21. $\sqrt{y + 4} = 2 + y$
22. $4 + \sqrt{x + 2} = x$
23. $3 - \sqrt{7x - 3} = 2x$
24. $\sqrt{\dfrac{6 + 2x}{5}} = 4$
25. $\sqrt{\dfrac{4y}{3}} - 6 = 2$
26. Solve $4a = c\sqrt{3}$ for c.
27. Solve $t = \sqrt{\dfrac{2S}{g}}$ for g.
28. Solve $t = \pi\sqrt{\dfrac{l}{g}}$ for l.
29. Solve $r = \sqrt[3]{\dfrac{3V}{4\pi}}$ for V.
30. Solve $E = \sqrt{V^2 + 2IR}$ for R.

SELECTED TERMS

fractional exponent An exponent that is a fraction; an exponent of $1/n$ indicates the nth root; thus $a^{1/n}$ is the nth root of a.

15.1 EXPONENTIAL FUNCTIONS

It is not unusual for a physical quantity, such as the charge on a discharging capacitor, the temperature of a cooling body, and the amount of a radioactive substance, to be described mathematically by a function involving a constant raised to a variable power, like $f(x) = 2^x$. Such functions are called *exponential functions*. Do not confuse the exponential function $f(x) = 2^x$ with the *power function* $g(x) = x^2$, which has a variable base and a constant exponent.

> The **exponential function** with base b is defined by
>
> $$f(x) = b^x,$$
>
> where $b > 0$, $b \neq 1$, and the exponent x is any real number.

The reason we excluded $b = 1$ is because $1^x = 1$ for all x.

Since the exponent in b^x can be any real number, you may wonder how we assign a value to an irrational power, like $6^{\sqrt{2}}$. Stated simply, we use approximations. First, $6^{\sqrt{2}}$ is approximately $6^{1.4} = 6^{7/5} = \sqrt[5]{6^7}$, which *is* defined. Better approximations are $6^{1.41} = 6^{141/100} = \sqrt[100]{6^{141}}$, and so on. In this way the meaning of $6^{\sqrt{2}}$ becomes clear.

Some functions that do not appear to have the exponential form b^x can be put in that form by applying rules for exponents. For example, $2^{-x} = 1/(2^x) = (\frac{1}{2})^x$ and $3^{2x} = (3^2)^x = 9^x$.

Figure 15.1 shows the graphs of the exponential functions $y = 2^x$, $y = 3^x$, and $y = (\frac{1}{2})^x$. Notice the following:

1. The domain of an exponential function is all real numbers.
2. The range is all positive real numbers. Thus b^x is positive for all x.
3. Since $b^0 = 1$ for every base b, each graph has y-intercept $(0, 1)$. There is no x-intercept. Each graph always lies above the x-axis.

x	2^x	3^x	$\left(\frac{1}{2}\right)^x$
-2	$\frac{1}{4}$	$\frac{1}{9}$	4
-1	$\frac{1}{2}$	$\frac{1}{3}$	2
0	1	1	1
1	2	3	$\frac{1}{2}$
2	4	9	$\frac{1}{4}$
3	8	27	$\frac{1}{8}$

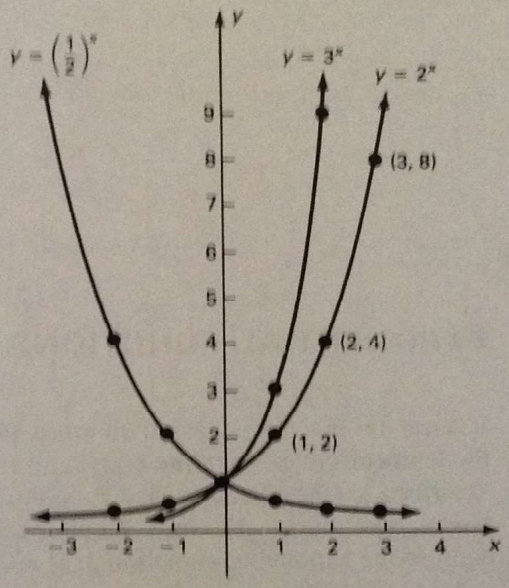


FIGURE 15.1

We also see in Fig. 15.1 that the graph of $y = b^x$ has one of two basic shapes, depending on whether $b > 1$ or $0 < b < 1$:

4. If $b > 1$, as in $y = 2^x$ and $y = 3^x$, then the graph of $y = b^x$ *rises* from left to right. That is, as x increases, y also increases. But y can also take on values very close to zero. (In quadrant II observe how the graph of $y = 3^x$ gets close to the x-axis. For example, if $x = -100$, then $3^x = 3^{-100} = 1/3^{100}$ is a number close to zero.) Notice also that in quadrant I, the greater the value of b, the more quickly the graph rises. (Compare the graphs of $y = 2^x$ and $y = 3^x$.)
5. If $0 < b < 1$, as in $y = (\frac{1}{2})^x$, then the graph of $y = b^x$ falls from left to right. As x increases, then y *decreases* and gets closer to 0.

One of the numbers most useful for a base in an exponential function is a certain irrational number denoted by the letter e, in honor of the Swiss mathematician and physicist Leonhard Euler (1707–1783):

e is approximately 2.71828

Although e may seem to be a strange number to use as a base for an exponential function, it arises quite naturally in applied mathematics, science, and engineering technology. It occurs in many problems involving natural growth or decay.

Values of e^x can be obtained conveniently by using the $\boxed{e^x}$ key on a calculator (or $\boxed{\text{INV}}$ $\boxed{\text{LN}}$ on some calculators). For example, to find $e^{2.45}$, we key in 2.45 and press the $\boxed{e^x}$ key. You should verify that the (rounded) display is 11.588. The $\boxed{y^x}$ key can also be used to find values for an exponential function. (See Sec. A.5 in Appendix A for a further discussion.) The graph of $y = e^x$ is shown in Fig. 15.2.

x	e^x
−2	0.1353
−1	0.3679
0	1
1	2.7183

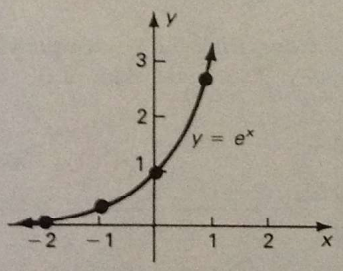


FIGURE 15.2

There are many physical quantities for which the rate of growth or decay at any time is proportional to the amount of the quantity present at that time. Examples are bacteria growth, decay of a radioactive element, and the decay of a current in some electrical circuits. Such growth or decay can be described in terms of a function of the form

$$Q = Q(t) = Q_0 e^{kt}, \tag{1}$$

where Q is the amount of the quantity at time t, and Q_0 and k are constants. Due to the form of Eq. 1, we say that the quantity has **exponential growth** if $k > 0$ and **exponential decay** if $k < 0$.* *If* $t = 0$ in Eq. 1, then

$$Q = Q_0 e^0 = Q_0(1) = Q_0.$$

Thus Q_0 represents the amount of the quantity present when $t = 0$; that is, Q_0 is the initial amount.

* Note that $e^{kt} = (e^k)^t = b^t$, where $b = e^k$. If $k > 0$, then $b > 1$ and so e^{kt} increases as t increases. If $k < 0$, then $0 < b < 1$ and e^{kt} decreases as t increases.

With exponential decay, to stress that k is negative it is customary to replace k by $-\lambda$, where λ (lambda) is a positive constant called the **decay constant**. Thus Eq. 1 becomes

$$\boxed{Q = Q(t) = Q_0 e^{-\lambda t}.} \tag{2}$$

Example 1

Suppose that the number Q of milligrams of a radioactive substance present after t years is given by the exponential decay function

$$Q = Q(t) = 100e^{-0.035t}.$$

How much of the substance is present initially? How many milligrams are present after 20 years?

Solution Comparing the given equation to Eq. 2 gives $Q_0 = 100$. Thus there are 100 mg present initially. That is, the value of $Q(t)$ is 100 when $t = 0$. After 20 years the amount present is $Q(20)$:

$$Q(20) = 100e^{-0.035(20)} = 49.66 \text{ mg}.$$

The time required for one-half of a radioactive substance to decay is called the **half-life** of the substance. It is the value of t for which $Q = Q_0/2$ in Eq. 2. For example, the half-life of radioactive carbon 14 (^{14}C) is approximately 5600 years. Figure 15.3 shows the graph of radioactive decay.

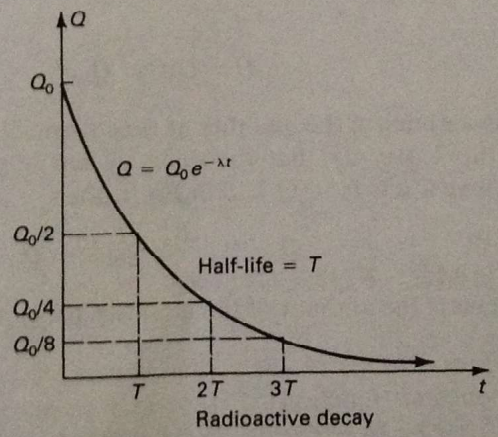


FIGURE 15.3

Problem Set 15.1

In Problems **1–4**, *graph the function.*

1. $y = 4^x$.
2. $y = (\frac{1}{3})^x$.
3. $f(t) = (\frac{1}{4})^t$.
4. $f(t) = 3^{t/2}$.

In Problems **5–8**, *use a calculator to find the value of the given number to four decimal places.*

5. $e^{2.4}$.
6. $e^{0.06}$.
7. $e^{-4.2}$.
8. $e^{-3.67}e^{2.87}$.

In Problems **9–14**, *find the function values. A calculator is not needed except for the last part of each problem, for which you should give your answer to four decimal places.*

9. $f(x) = 9^x$; $f(2)$, $f(-2)$, $f(\frac{1}{2})$, $f(2.8)$.
10. $g(x) = 4(3)^x$; $g(0)$, $g(2)$, $g(-1)$, $g(e)$.
11. $h(t) = 3(16)^{t/2}$; $h(1)$, $h(\frac{1}{2})$, $h(-\frac{1}{2})$, $h(-3.2)$.
12. $f(t) = 5 - (\frac{1}{2})^t$; $f(2)$, $f(3)$, $f(-3)$, $f(-4.3)$.
13. $g(x) = 1 + 2(\frac{1}{8})^{1-x}$; $g(1)$, $g(-1)$, $g(\frac{1}{3})$, $g(-e)$.
14. $h(x) = \dfrac{6(0.25)^{(4-x)/2}}{5}$; $h(0)$, $h(2)$, $h(3)$, $h(-0.4)$.
15. The number Q of milligrams of a radioactive substance present after t years is given by $Q = 75e^{-0.0244t}$.
    (a) How much of the substance is present initially?
    (b) How many milligrams are present after 14 years?
16. Show that the exponential decay function in Problem 15 can be expressed as $Q = 75(0.976)^t$.
17. The decaying current i, in amperes, in an ac circuit is given by $i = 10e^{-2t}\sin(100t)$, where time t is in seconds and the angle $100t$ is in radians. Find the current when $t = 1.2$ s.
18. The charge q on a discharging capacitor is given by $q = 8e^{-2.8t}$, where t is time in seconds and q is in microcoulombs. Find q when $t = 0.3$ s.
19. A function used in statistics is

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

Find $f(0)$, $f(-1)$, and $f(1)$. Give your answers to three decimal places.

15.2 LOGARITHMIC FUNCTIONS

The next functions of interest to us in this chapter are *logarithmic functions*, which are related to exponential functions. Figure 15.4 shows the graph of the exponential function $s = f(t) = 2^t$. Here f sends an input number t into a *positive* output number s. For example, f sends 2 into 4: $f(2) = 4$.

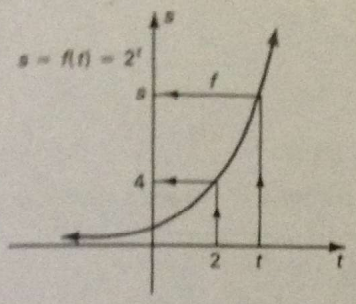


FIGURE 15.4

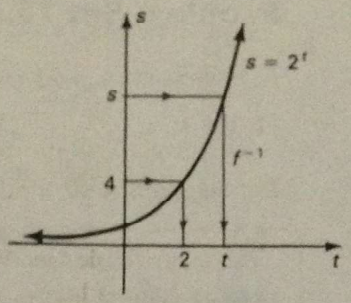


FIGURE 15.5

Now look at the same curve in Fig. 15.5. The direction of the small arrows conveys a different functional relationship. With each positive number s on the vertical axis, we can associate exactly one value of t. For example, with $s = 4$ we associate $t = 2$. By thinking of s as an input and t as an output, we have a function that sends values of s into values of t. We shall denote this function by f^{-1} (read f inverse). Thus

$$f^{-1}(4) = 2.$$

We caution you that the -1 in f^{-1} is not treated as an exponent, so f^{-1} does not mean $1/f$.

The functions f and f^{-1} are related. Figure 15.6 shows that f^{-1} *reverses* the

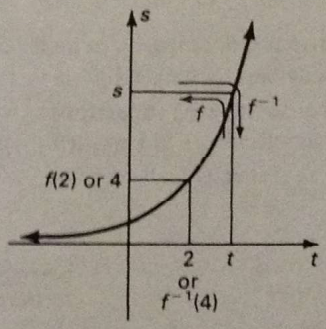


FIGURE 15.6

action of f, and vice versa. For example,

$$f \text{ sends 2 into 4} \quad \text{and} \quad f^{-1} \text{ sends 4 into 2.}$$

More generally, for all possible values of s and t, we have $f(t) = s$ and $f^{-1}(s) = t$. The domain of f^{-1} is the range of f (all positive real numbers), and the range of f^{-1} is the domain of f (all real numbers).

We give a special name to f^{-1}. It is called the **logarithmic function with base 2** and is written as $\log_2$ [read *logarithm* (or log) *base* 2]. Thus $f^{-1}(4) = \log_2 4 = 2$, and we say that the *logarithm* base 2 of 4 is 2.

In summary,

$$\text{If } s = 2^t, \quad \text{then} \quad t = \log_2 s. \tag{1}$$

We now generalize our discussion to other bases. In Eq. 1, replacing 2 by b, s by x, and t by y gives the following definition.

> The **logarithmic function** with base b, where $b > 0$ and $b \neq 1$, is denoted by $\log_b$ and is defined by
>
> $$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$
>
> The domain of $\log_b$ is all positive numbers, and the range is all real numbers.

A logarithmic function reverses the action of the corresponding exponential function, and vice versa. Because of this, each logarithmic function is called the *inverse* of its corresponding exponential function, and that exponential function is the inverse of its corresponding logarithmic function.

Always remember: When we say that the log base b of x is y, we mean that b raised to the y power is x.

$$\boxed{\log_b x = y \quad \textbf{means} \quad b^y = x.}$$

In this sense, *a logarithm of a number is an exponent.* It is the power to which we must raise the base to get the number. For example,

$$\log_2 8 = 3 \quad \text{because} \quad 2^3 = 8.$$

We say that $\log_2 8 = 3$ is the **logarithmic form** of the **exponential form** $2^3 = 8$.

Example 1

	Exponential form		*Logarithmic form*
a. Since	$5^2 = 25$,	then	$\log_5 25 = 2$.
b. Since	$3^4 = 81$,	then	$\log_3 81 = 4$.
c. Since	$10^0 = 1$,	then	$\log_{10} 1 = 0$.

Example 2

Logarithmic form		*Exponential form*
a. $\log_{10} 1000 = 3$	means	$10^3 = 1000$.
b. $\log_{64} 8 = \frac{1}{2}$	means	$64^{1/2} = 8$.
c. $\log_2 \frac{1}{16} = -4$	means	$2^{-4} = \frac{1}{16}$.

Example 3

Graph the function $y = \log_2 x$.

Solution It can be awkward to substitute values of x and then find corresponding values of y. For example, if $x = 3$, then $y = \log_2 3$, which is not easily determined. An easier way to plot points is to use the equivalent exponential form $x = 2^y$. We choose values of y and find the corresponding values of x. For example, if $y = 0$, then $x = 1$. This gives the point $(1, 0)$. When $y = 2$, then $x = 2^2 = 4$, and when $y = -1$, we have $x = 2^{-1} = \frac{1}{2}$. This gives the points $(4, 2)$ and $(\frac{1}{2}, -1)$. Other points are shown in Fig. 15.7. From the graph you can see that the domain is all positive real numbers. Thus *negative numbers and* 0 *do not have logarithms.* The range is all real numbers. Observe that as x increases, y increases. Numbers between 0 and 1 have negative logarithms, and the closer such a number is to 0, the more negative is its logarithm. The log of 1 is 0, which corresponds to the intercept $(1, 0)$. There is no y-intercept. This graph is typical for a logarithmic function with $b > 1$.

x	y
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3

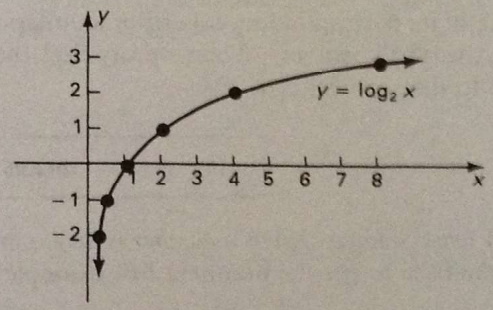


FIGURE 15.7

For many functions f, if $f(m) = f(n)$, this does not imply that $m = n$. For example, if $f(x) = x^2$ and $m = 2$ and $n = -2$, then $f(m) = f(n)$, but $m \neq n$. This is not the case for a logarithmic function. Because the graph in Fig. 15.7 is always rising, it is obvious that if two numbers are different, then their logarithms (y-values) are different. This means that if $\log_2 m = \log_2 n$, then $m = n$. Generalizing to base b, we have the following property:

$$\text{If } \log_b m = \log_b n, \quad \text{then} \quad m = n. \tag{1}$$

There is a similar property for the exponential function:

$$\text{If } b^m = b^n, \quad \text{then} \quad m = n. \tag{2}$$

Logarithms with base 10 are called **common logarithms**. The subscript 10 is usually omitted from the notation:

$$\log x \quad \text{means} \quad \log_{10} x.$$

Logarithms with base e are called **natural logarithms**. We use the notation ln for such logarithms:

$$\ln x \quad \text{means} \quad \log_e x.$$

Common and natural logarithms of numbers can be found with a calculator by using the log and ln keys, respectively. For example, you should verify that $\ln 2 \approx 0.69315$. This means that $e^{0.69315} \approx 2$. A discussion of these keys is found in Sec. A.7.

Figure 15.8 shows the graphs of $y = e^x$ and $y = \ln x$ on the same set of axes. If a mirror were placed along the line $y = x$, then each graph would be the mirror image of the other graph. The reason for this is that $y = e^x$ and $y = \ln x$ are inverses of each other, so the corresponding values for x and y are reversed. For example, $y = e^x$ crosses the y-axis at $(0, 1)$, so $y = \ln x$ crosses the x-axis at $(1, 0)$.

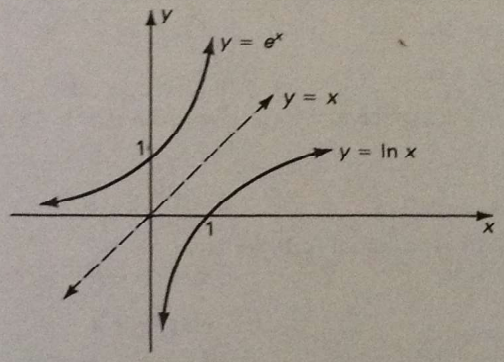


FIGURE 15.8

Example 4

Find each of the following without the aid of a calculator.

a. log 100.

Solution Here the base is 10. Thus log 100 is the power to which we must raise 10 to get 100. Since $10^2 = 100$, $\log 100 = 2$.

b. ln 1.

Solution Here the base is e. Because $e^0 = 1$, $\ln 1 = 0$.

c. $\log 0.1$.

Solution Since $0.1 = \frac{1}{10} = 10^{-1}$, $\log 0.1 = -1$.

d. $\ln e^{-1}$.

Solution Since $\ln e^{-1}$ is the power to which e must be raised to get e^{-1}, clearly we have $\ln e^{-1} = -1$.

e. $\log_{36} 6$.

Solution Because $36^{1/2}$ (or $\sqrt{36}$) is 6, $\log_{36} 6 = \frac{1}{2}$.

Example 5

Solve each equation for x.

a. $\log_3 x = 4$.

Solution In exponential form, $3^4 = x$. Thus $x = 81$.

b. $\ln (x + 1) = 7$.

Solution The exponential form gives $e^7 = x + 1$. Thus $x = e^7 - 1$.

c. $\log_x 49 = 2$.

Solution In exponential form we have

$$\log_x 49 = 2$$
$$x^2 = 49$$
$$x = \pm 7.$$

A negative number cannot be a base for a logarithm, so we disregard $x = -7$ and choose $x = 7$.

d. $e^{0.2x} = 6$.

Solution In logarithmic form,

$$0.2x = \ln 6$$
$$x = \frac{\ln 6}{0.2} = 8.959.$$

e. $x + 1 = \log_4 16$.

Solution Because $\log_4 16$ is 2, we have $x + 1 = 2$, from which $x = 1$. If you did not recognize that $\log_4 16 = 2$, then you could solve the equation in the following way.

$$x + 1 = \log_4 16$$
$$4^{x+1} = 16 \qquad \text{[converting to exponential form]}$$
$$4^{x+1} = 4^2.$$

From Property (2), $x + 1 = 2$, so $x = 1$.

When the logarithm of a number N is known, the process of finding N is referred to as finding the **antilogarithm**. Because this involves reversing the action of the logarithmic function, we can make use of the 10^x, e^x, or y^x exponential keys on a calculator, as Example 6 shows.

Example 6

a. If $\log N = 4.86052$, find N. That is, find antilog 4.86052.

Solution Because $\log N = 4.86052$, in exponential form we have

$$N = 10^{4.86052} = 72{,}530.$$

That is, antilog $4.86052 = 72{,}530$.

b. If $\ln N = -5.37737$, find N. That is, find $\text{antilog}_e(-5.37737)$.

Solution

$$\text{antilog}_e(-5.37737) = e^{-5.37737} = 4.620 \times 10^{-3}.$$

Thus $N = 4.620 \times 10^{-3}$.

In the previous section the exponential decay function $Q = Q_0 e^{-\lambda t}$ was discussed. Recall that the value of t when $Q = Q_0/2$ is the half-life of a radioactive substance. We wish to find a formula for this value of t.

$$\frac{Q_0}{2} = Q_0 e^{-\lambda t}$$

$$\frac{1}{2} = e^{-\lambda t} \qquad \text{[dividing both sides by } Q_0\text{]}.$$

In logarithmic form we have

$$-\lambda t = \ln \frac{1}{2} = -0.69315,$$

which means

$$\boxed{t = \frac{0.69315}{\lambda}.} \tag{3}$$

Note that the half-life depends on λ. That is, it depends on the substance and not on the amount of the substance.

Example 7

If 60% of a radioactive element remains after 50 d, find the decay constant and the half-life of the element.

Solution We use the function $Q = Q_0 e^{-\lambda t}$, where Q_0 is the amount of the element present at $t = 0$ and λ is the decay constant. Because $Q = 0.6Q_0$ when $t = 50$, we have

$$\begin{aligned} 0.6Q_0 &= Q_0 e^{-50\lambda} \\ 0.6 &= e^{-50\lambda} \\ -50\lambda &= \ln 0.6 \qquad \text{[logarithmic form]} \\ \lambda &= -\frac{1}{50}\ln 0.6 = -\frac{1}{50}(-0.51083) \\ &= 0.01022. \end{aligned}$$

Thus the decay constant is 0.01022. The half-life, from Eq. 3, is

$$\frac{0.69315}{\lambda} = \frac{0.69315}{0.01022} = 67.82 \text{ d}.$$

Note that the unit of λ is the unit of 1/time. Here the unit is $1/\text{day} = (\text{day})^{-1}$.

Problem Set 15.2

In Problems **1–16**, *write each exponential form logarithmically and each logarithmic form exponentially.*

1. $4^3 = 64$.
2. $2 = \log_{12} 144$.
3. $10^5 = 100{,}000$.
4. $\log_9 3 = \frac{1}{2}$.
5. $\log_2 64 = 6$.
6. $4^4 = 256$.
7. $\log_8 2 = \frac{1}{3}$.
8. $8^{2/3} = 4$.
9. $6^0 = 1$.
10. $\log_{1/2} 4 = -2$.
11. $\log_2 x = 14$.
12. $10^{0.48302} = 3.041$.
13. $e^2 = 7.3891$.
14. $e^{0.33647} = 1.4$.
15. $\ln 3 = 1.0986$.
16. $\log 5 = 0.6990$.

In Problems **17** *and* **18**, *graph the function.*

17. $y = f(x) = \log_3 x$.

18. $y = f(x) = \log_{1/2} x$.

In Problems **19–22**, *use a calculator to find the value of the given number to four decimal places.*

19. ln 7.34.

20. ln 3.1.

21. log 0.2538.

22. log 5988.

In Problems **23–36**, *find the value of each number without the aid of a calculator.*

23. $\log_6 36$.

24. $\log_2 32$.

25. $\log_3 27$.

26. $\log_{27} 3$.

27. $\log_{16} 4$.

28. $\log_7 7$.

29. log 10.

30. log 10,000.

31. log 0.01.

32. $\ln e^4$.

33. $\log_5 1$.

34. $\log_2 \sqrt{2}$.

35. $\log_2 \frac{1}{8}$.

36. $\log_5 \frac{1}{25}$.

In Problems **37–60**, *find x without the aid of a calculator.*

37. $\log_3 x = 2$.

38. $\log_2 x = 4$.

39. $\log_5 x = 3$.

40. $\log_4 x = 0$.

41. $\log x = -1$.

42. $\ln x = 1$.

43. $\ln x = 2$.

44. $\log_x 100 = 2$.

45. $\log_x 8 = 3$.

46. $\log_x 3 = \frac{1}{2}$.

47. $\log_x \frac{1}{6} = -1$.

48. $\log_x y = 1$.

49. $\log_4 16 = x$.

50. $\log_3 1 = x$.

51. $\log 10^{-7} = x$.

52. $\log_2 \frac{1}{16} = x$.

53. $\log_{25} 5 = x$.

54. $\log_9 9 = x$.

55. $\log_3 x = -4$.

56. $\log_x (2x - 3) = 1$.

57. $\log_x (6 - x) = 2$.

58. $\log_8 64 = x - 1$.

59. $2 + \log_2 4 = 3x - 1$.

60. $\log_3 (x + 2) = -2$.

In Problems **61–68**, *find x to three decimal places.*

61. $e^{3x} = 2$.

62. $e^{2x-5} + 1 = 4$.

63. $0.1e^{0.1x} = 0.5$.

64. $3e^{2x} - 1 = \frac{1}{2}$.

65. $\log x = 2.35$.

66. $\log x = -0.78$.

67. $\ln x = -0.03$.

68. $\ln x = 4.295$.

69. Evaluate $\ln 54 - (\ln 17.2)^2$. Give your answer to four decimal places.

70. Evaluate $\log 87 + 3 \log 0.834$. Give your answer to four decimal places.

71. In chemistry, the acidity or basicity of an aqueous solution at room temperature is determined by finding the pH of the solution. If the hydrogen-ion concentration (in moles per liter) is denoted by $[H^+]$, then the pH is given by

$$\text{pH} = -\log [H^+].$$

If $\text{pH} < 7$, the solution is acidic. If $\text{pH} > 7$, it is basic. If $\text{pH} = 7$, the solution is said to be neutral.

(a) What is the pH of vinegar with $[H^+]$ equal to 3×10^{-4}?

(b) An ammonia cleaning solution has a pH of 7.85. What is the $[H^+]$ of this solution?

72. If $\ln \frac{V}{V_0} = -\frac{t}{RC}$, solve for V.

73. The intensity level β of a sound wave of intensity I is defined by

$$\beta = 10 \log \frac{I}{I_0},$$

where I_0 is a standard reference intensity taken to be 10^{-12} W/m^2, which approximately corresponds to the faintest audible sound. The intensity level is measured in decibels (db). If a second source has an intensity of 3×10^{-10} W/m^2, what is its intensity level to the nearest tenth of a decibel?

74. The work, in joules, done by a 1-kg sample of nitrogen gas as its volume changes from an initial value V_i to a final value V_f during an isothermal (constant-temperature) process is given by

$$W = 8.1 \times 10^4 \ln \frac{V_f}{V_i}.$$

If such a sample expands isothermally from a volume of 3 L to a volume of 7 L, determine the work done by the gas to the nearest hundred joules.

75. Atmospheric pressure p varies with the altitude h above the earth's surface. For altitudes up to about 10 km, the pressure p (in millimeters of mercury) is given approximately by

$$p = 760e^{-0.125h},$$

where h is in kilometers.
(a) Find p at an altitude of 7.3 km.
(b) At what altitude will the pressure be 400 mm of mercury?

76. In studies of electric fields, one may encounter a device called a cylindrical capacitor, two concentric cylindrical conductors. The capacitance per unit length of such a device, in microfarads per meter, can be shown to be *inversely* proportional to

$$\ln \frac{r_0}{r_i},$$

where r_0 and r_i are the radii of the outer and inner cylinders, respectively. If the capacitance per unit length of a particular cylindrical capacitor is 10 μF/m when the outer radius is twice the inner radius, find the capacitance per unit length when it is three times the inner radius.

77. If 30% of the initial amount of a radioactive sample *remains* after 100 s, find the decay constant and the half-life of the element. (Refer to Example 7.)

78. If 30% of the initial amount of a radioactive sample *has decayed* after 100 s, find the decay constant and the half-life of the element. (Refer to Example 7.)

79. Radon has a half-life of 3.82 d.
(a) Find the decay constant.
(b) What fraction of the original amount of it remains after 7.64 d? *Hint*: The fraction of the original amount at any time is given by Q/Q_0.

15.3 PROPERTIES OF LOGARITHMS

Because logarithms are exponents, their properties can be obtained by using properties of exponents. For example, we shall consider the logarithm of a product, such as $\log_b (mn)$. If we let $x = \log_b m$ and $y = \log_b n$, then $b^x = m$ and $b^y = n$. Therefore,

$$mn = b^x b^y = b^{x+y}.$$

Thus $mn = b^{x+y}$. Converting this to logarithmic form gives $\log_b (mn) = x + y$. But, from above, x is $\log_b m$ and y is $\log_b n$. Thus we have our first property.

PROPERTY 1

$$\log_b (mn) = \log_b m + \log_b n.$$

That is, *the logarithm of the product of two numbers is the sum of the logarithms of the numbers.* For example, $\log (3 \cdot 5) = \log 3 + \log 5$.

In some of the examples and exercises that follow, we shall use Table 15.1, which gives the values of a few common logarithms to four decimal places. This table, which you should verify with a calculator, is useful in illustrating the many properties of logarithms.

TABLE 15.1 Common Logarithms

x	$\log x$	x	$\log x$
2	0.3010	7	0.8451
3	0.4771	8	0.9031
4	0.6021	9	0.9542
5	0.6990	10	1.0000
6	0.7782	e	0.4343

Example 1

Use Table 15.1 to find each of the following. Check the answers with a calculator.

a. log 15.

Solution Log 15 is not in Table 15.1. But we can write 15 as the product $3 \cdot 5$. Thus

$$\begin{aligned} \log 15 &= \log (3 \cdot 5) \\ &= \log 3 + \log 5 \qquad \text{[Property 1]} \\ &= 0.4771 + 0.6990 \qquad \text{[Table 15.1]} \end{aligned}$$

$$\log 15 = 1.1761.$$

A calculator value for log 15 is the same.

b. log 56.

Solution

$$\log 56 = \log (8 \cdot 7) = \log 8 + \log 7 = 0.9031 + 0.8451 = 1.7482,$$

which agrees with a calculator value for log 56.

The next two properties can be proved in the same way as Property 1.

PROPERTY 2

$$\log_b \frac{m}{n} = \log_b m - \log_b n.$$

That is, *the logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.* For example, $\log \frac{2}{3} = \log 2 - \log 3$.

PROPERTY 3

$$\log_b m^n = n \log_b m.$$

That is, *the logarithm of the nth power of a number is equal to n times the logarithm of the number.* For example, $\log 2^8 = 8 \log 2$. Note that $\log 2^8 \neq (\log 2)^8$.

Make sure that you clearly understand Properties 1–3. They do *not* apply to the log of a sum $[\log_b (m + n)]$, log of a difference $[\log_b (m - n)]$, or quotient of logs $[\log_b m / \log_b n]$. For example, the logarithm of a sum is *not* the sum of logarithms. In general,

$$\log_b (m + n) \neq \log_b m + \log_b n,$$

$$\log_b (m - n) \neq \log_b m - \log_b n,$$

$$\frac{\log_b m}{\log_b n} \neq \log_b (m - n),$$

and

$$\frac{\log_b m}{\log_b n} \neq \log_b \left(\frac{m}{n}\right).$$

Example 2

Use Table 15.1 to find each of the following. Check the answers with a calculator.

a. $\log \frac{9}{2}$.

Solution

$$\log \frac{9}{2} = \log 9 - \log 2 \qquad \text{[Property 2]}$$

$$= 0.9542 - 0.3010 \qquad \text{[Table 15.1]},$$

$$\log \frac{9}{2} = 0.6532.$$

b. $\log 64$.

Solution

$$\log 64 = \log (8^2)$$

$$= 2 \log 8 \qquad \text{[Property 3]}$$

$$= 2(0.9031) \qquad \text{[Table 15.1]},$$

$$\log 64 = 1.8062.$$

c. $\log \sqrt{5}$.

Solution

$$\log \sqrt{5} = \log (5^{1/2}) = \frac{1}{2} \log 5 = \frac{1}{2}(0.6990) = 0.3495.$$

d. $\log \frac{16}{21}$.

Solution

$$\log \frac{16}{21} = \log 16 - \log 21$$

$$= \log (4^2) - \log (3 \cdot 7)$$

$$= 2 \log 4 - [\log 3 + \log 7]$$

$$= 2(0.6021) - [0.4771 + 0.8451] = -0.1180.$$

Note the use of brackets in the third line. It is wrong to write $2 \log 4 - \log 3 + \log 7$. A calculator value for $\log \frac{16}{21}$ is -0.1181. The difference in the results is due to rounding in Table 15.1.

Example 3

a. $\log[(a)(b)(c)] = \log[(ab)(c)]$

$$= \log (ab) + \log c$$

$$= \log a + \log b + \log c.$$

This technique can, of course, be extended to the logarithm of any number of factors.

Do not confuse $\ln x^2$ with $(\ln x)^2$. We have

$$\ln x^2 = \ln (x \cdot x),$$

$$\text{but} \qquad (\ln x)^2 = (\ln x)(\ln x),$$

which can be written as $\ln^2 x$. Thus in $\ln x^2$ we square x; in $(\ln x)^2$, or $\ln^2 x$, we square $\ln x$.

c. $\log [a(b + c)] = \log a + \log (b + c)$. Note that $\log (b + c)$ cannot be simplified.

Example 4

a. Write $\log \dfrac{1}{x^2}$ in terms of $\log x$.

Solution

$$\log \frac{1}{x^2} = \log x^{-2} = -2 \log x \qquad \text{[Property 3].}$$

b. Write $\log x - \log (x + 3)$ as a single logarithm.

Solution

$$\log x - \log (x + 3) = \log \frac{x}{x + 3} \qquad \text{[Property 2].}$$

c. Write $3 \log_2 10 + \log_2 15$ as a single logarithm.

Solution

$$\begin{aligned} 3 \log_2 10 + \log_2 15 &= \log_2 (10^3) + \log_2 15 && \text{[Property 3]} \\ &= \log_2 [(10^3)15] && \text{[Property 1]} \\ &= \log_2 15{,}000. \end{aligned}$$

d. Write $\ln 3 + \ln 7 - \ln 2 - 2 \ln 4$ as a single logarithm.

Solution

$$\begin{aligned} &\ln 3 + \ln 7 - \ln 2 - 2 \ln 4 \\ &= \ln 3 + \ln 7 - \ln 2 - \ln (4^2) \\ &= \ln 3 + \ln 7 - [\ln 2 + \ln (4^2)] \\ &= \ln (3 \cdot 7) - \ln (2 \cdot 4^2) \\ &= \ln 21 - \ln 32 \\ &= \ln \frac{21}{32}. \end{aligned}$$

e. Write $-\log \frac{x}{2}$ without using a minus sign.

Solution

$$-\log \frac{x}{2} = (-1) \log \frac{x}{2}$$

$$= \log \left(\frac{x}{2}\right)^{-1} \qquad \text{[Property 3]}$$

$$= \log \frac{2}{x}.$$

Example 5

a. Write $\ln \frac{x^5 y}{zw}$ in terms of $\ln x$, $\ln y$, $\ln z$, and $\ln w$.

Solution

$$\ln \frac{x^5 y}{zw} = \ln (x^5 y) - \ln (zw)$$

$$= \ln (x^5) + \ln y - (\ln z + \ln w)$$

$$= 5 \ln x + \ln y - \ln z - \ln w.$$

b. Write $\log \sqrt[3]{\frac{x^2}{x-4}}$ in terms of $\log x$ and $\log (x-4)$.

Solution

$$\log \sqrt[3]{\frac{x^2}{x-4}} = \log \left(\frac{x^2}{x-4}\right)^{1/3} = \frac{1}{3} \log \frac{x^2}{x-4}$$

$$= \frac{1}{3} [\log (x^2) - \log (x-4)]$$

$$= \frac{1}{3} [2 \log x - \log (x-4)].$$

Because $b^0 = 1$ and $b^1 = b$, by converting to logarithmic forms we have two more properties of logarithms:

PROPERTY 4

$\log_b 1 = 0.$

PROPERTY 5

$\log_b b = 1.$

Example 6

Find each of the following.

a. $\log 1$.

Solution

$$\log 1 = \log_{10} 1 = 0 \qquad \text{[Property 4]}.$$

b. $\ln e$.

Solution

$$\ln e = \log_e e = 1 \qquad \text{[Property 5]}.$$

c. $\log 1000$.

Solution

$$\begin{aligned} \log 1000 &= \log_{10} 10^3 = 3 \log_{10} 10 \\ &= 3 \cdot 1 \qquad \text{[Property 5]} \\ &= 3. \end{aligned}$$

d. $\log (10^n)$.

Solution

$$\log (10^n) = n \log_{10} 10 = n \cdot 1 = n.$$

We can generalize the result of Example 6(d):

PROPERTY 6

$$\log_b b^n = n.$$

For example, $\log 10^9 = 9$ and $\ln e^8 = 8$.

Example 7

a. Find $\log_7 \sqrt[9]{7^8}$.

Solution

$$\log_7 \sqrt[9]{7^8} = \log_7 (7^{8/9}) = \frac{8}{9} \qquad \text{[Property 6]}.$$

b. Find $\log_3 \frac{27}{81}$.

Solution

$$\log_3 \frac{27}{81} = \log_3 \frac{3^3}{3^4} = \log_3 (3^{-1}) = -1 \qquad \text{[Property 6]}.$$

c. Find $\ln (e^2) + \log \frac{1}{10}$.

Solution

$$\ln (e^2) + \log \frac{1}{10} = 2 + \log 10^{-1} = 2 + (-1) = 1.$$

d. Find $\log \frac{200}{21}$ by using Table 15.1.

Solution

$$\begin{aligned} \log \frac{200}{21} &= \log 200 - \log 21 = \log (2 \cdot 10^2) - \log (3 \cdot 7) \\ &= \log 2 + \log (10^2) - (\log 3 + \log 7) \\ &= \log 2 + 2 - \log 3 - \log 7 \\ &= 0.3010 + 2 - 0.4771 - 0.8451 \\ &= 0.9788. \end{aligned}$$

Our last property is:

PROPERTY 7

$b^{\log_b n} = n$ and, in particular, $10^{\log x} = x$ and $e^{\ln x} = x$.

Property 7 is true because it states, in logarithmic form, that $\log_b n = \log_b n$.

Example 8

a. $e^{\ln 2x} = 2x$, by Property 7.

b. $10^{\log x^2} = x^2$, by Property 7.

Problem Set 15.3

In Problems **1–48**, *use Table* 15.1 *and properties of logarithms to find each value.*

1. $\log 5$.
2. $\log 3$.
3. $\log 21$.
4. $\log 14$.
5. $\log 35$.
6. $\log 12$.
7. $\log 25$.
8. $\log 49$.
9. $\log \frac{9}{4}$.
10. $\log \frac{7}{3}$.
11. $\log \frac{1}{2}$.
12. $\log \frac{7}{9}$.
13. $\log (8^5)$.
14. $\log (4^{-3})$.
15. $\log \frac{1}{5^4}$.
16. $\log 2^{0.01}$.
17. $\log \sqrt{2}$.
18. $\log \sqrt[3]{4}$.
19. $\log \sqrt[3]{6^2}$.
20. $\log \frac{1}{\sqrt[3]{5}}$.
21. $\log \frac{1}{5^4}$.
22. $\log \frac{3}{35}$.
23. $\log \frac{15}{28}$.
24. $\log \frac{81}{25}$.
25. $\log 10{,}000$.
26. $\log 10^{10}$.
27. $\log 0.01$.
28. $\log (0.1)^3$.
29. $\log 300$.
30. $\log \sqrt[3]{400}$.
31. $\log \frac{100}{9}$.
32. $\log \frac{1}{600}$.
33. $\log \frac{27}{5000}$.
34. $\log \sqrt{\frac{8}{3}}$.
35. $\log_7 7^{48}$.
36. $\log_4 \sqrt[5]{4}$.
37. $\log_5 \sqrt[4]{5^3}$.
38. $\log_2 \frac{1}{\sqrt{2}}$.
39. $\ln e^4$.
40. $\ln \frac{1}{e}$.
41. $\log_2 \frac{2^6}{2^{10}}$.
42. $\log_3 (3^5 \cdot 3^4)^6$.
43. $\log [(10\sqrt{10})^3]$.
44. $\log_5 \frac{25}{\sqrt{5}}$.
45. $\log_7 \frac{\sqrt[3]{49}}{7}$.
46. $\log_8 (64\sqrt[4]{8})$.
47. $\log 10 + \ln (e^2)$.
48. $(\log 100)(\ln \sqrt{e})$.

In Problems **49–62**, *write each expression as a single logarithm.*

49. $\log 7 + \log 4$.
50. $\log_3 10 - \log_3 5$.
51. $\log_2 (x + 2) - \log_2 (x + 1)$.
52. $\log (x^2) + \log x$.
53. $2 \ln 5 - 3 \ln 4$.
54. $3 \log_7 2 + 2 \log_7 3$.
55. $\frac{1}{2} \log_4 2 + 3 \log_4 3$.
56. $\frac{1}{3} \ln x - 5 \ln (x - 2)$.
57. $\log_3 x + \log_3 y - \log_3 z$.
58. $\ln x + \ln (x - 1) - 2 \ln y$.
59. $\log_2 5 + 2 \log_2 3 - \log_2 7$.
60. $\frac{1}{2}(\log_3 x + \log_3 y)$.
61. $2 \ln x + 3 \ln y - 4 \ln z - 2 \ln w$.
62. $\log_6 5 - 2 \log_6 \frac{1}{5} - 2 \log_6 4$.

In Problems **63–72**, *write the given expression in terms of log x, log y, and log z.*

63. $\log (xy)$.
64. $\log (x^2yz)$.
65. $\log \frac{x}{z^2}$.
66. $\log x^6$.
67. $\log \sqrt{x}$.
68. $\log \frac{xy^2}{z^3}$.

69. $\log (xz^2)^{\frac{1}{2}}$

70. $\log \frac{x^{\frac{1}{3}}\sqrt{x}}{z^2}$

71. $\log \sqrt[5]{\frac{x^2y^3}{z^4}}$

72. $\log \frac{1}{\sqrt[3]{z}}$

In Problems 73–78, write each expression in terms of $\log x$, $\log (x + 2)$, *and* $\log (x - 3)$.

73. $\log [x(x + 2)(x - 3)]$

74. $\log \frac{x^2(x + 2)}{x - 3}$

75. $\log \frac{\sqrt{x}}{(x + 2)(x - 3)^2}$

76. $\log [(x - 3)\sqrt{x(x + 2)}]$

77. $\log \sqrt{\frac{x^2(x - 3)^3}{x + 2}}$

78. $\log \frac{1}{x(x - 3)^2(x + 2)^3}$

For an aqueous solution at room temperature, the product of the hydrogen-ion concentration, $[H^+]$, *and the hydroxide-ion concentration,* $[OH^-]$, *is* 10^{-14} *(where the concentrations are in moles per liter):*

$$[H^+][OH^-] = 10^{-14}.$$

In Problems 79 and 80, find the pH of a solution (see explanation in Problem 71 of Problem Set 15.3) with the given $[OH^-]$ *if* $pH = -\log [H^+]$.

79. $[OH^-] = 10^{-4}$.

80. $[OH^-] = 3 \times 10^{-2}$.

81. The intensity level β (beta) of a sound wave of intensity I is given by

$$\beta = 10 \log \frac{I}{I_0},$$

where I_0 *is a standard reference intensity taken to be* 10^{-12} W/m^2, which approximately corresponds to the faintest sound that you can hear. The intensity level is measured in decibels (db). For example, the intensity level of an ordinary conversation is 40 db, and that for a subway train is 100 db. Find the intensity level of the sound of rustling leaves, which has an intensity of 10^{-11}.

82. In Problem 81, suppose that two sound sources have intensities I_1 and I_2. Show that the difference in the intensity levels of the sounds is given by

$$\beta_2 - \beta_1 = 10 \log \frac{I_2}{I_1}.$$

83. According to Richter,[*] the magnitude M of an earthquake occurring 100 km from a certain type of seismometer is given by $M = 3 + \log A$, where A is the recorded trace amplitude (in millimeters) of the quake.
(a) Find the magnitude of an earthquake that records a trace amplitude of 1 mm.
(b) If a particular earthquake has amplitude A_1 and magnitude M_1, determine the magnitude of a quake with amplitude $100A_1$. Express your answer in terms of M_1.

[*] C. F. Richter, *Elementary Seismology* (San Francisco: W. H. Freeman, 1958).

15.4 CHANGE OF BASE

Our next property of logarithms allows us to convert logarithms from one base to another. Let

$$x = \log_b N.$$

Then, in exponential form,

$$b^x = N.$$

Taking the logarithm (base a) of both sides, we have

$$\log_a (b^x) = \log_a N$$

$$x \log_a b = \log_a N$$

$$x = \frac{\log_a N}{\log_a b}.$$

However, from before, $x = \log_b N$. Thus we have the *change of base formula*:

CHANGE OF BASE FORMULA

$$\log_b N = \frac{\log_a N}{\log_a b}.$$

This formula allows us to find a logarithm with base b in terms of logarithms with base a. In particular, from the change of base formula with $a = e$, we have

$$\log_b N = \frac{\ln N}{\ln b} \tag{1}$$

With this formula we can find logarithms with base b by using natural logarithms.

Letting $b = 10$ in Eq. 1 gives

$$\log N = \frac{\ln N}{\ln 10}.$$

Thus

$$\ln N = (\ln 10)(\log N),$$

or

$$\ln N = 2.30259 \log N. \tag{2}$$

It also follows that

$$\log N = 0.43429 \ln N. \tag{3}$$

Example 1

Find $\log_5 8$ by using common logarithms.

Solution Using the change of base formula with $b = 5$, $N = 8$, and $a = 10$ gives

$$\log_5 8 = \frac{\log 8}{\log 5} = 1.2920.$$

We remind you that $\dfrac{\log 8}{\log 5} \neq \log (8 - 5)$ and $\dfrac{\log 8}{\log 5} \neq \log \dfrac{8}{5}$.

Example 2

Find $\log_4 20$ by using natural logarithms.

Solution Using Eq. 1 with $b = 4$ and $N = 20$, we have

$$\log_4 20 = \frac{\ln 20}{\ln 4} = 2.1610.$$

Problem Set 15.4

In Problems **1–6**, *find the values of the logarithms.*

1. $\log_5 27$.
2. $\log_2 15$.
3. $\log_7 0.67$.
4. $\log_{100} 25$.
5. $\log_6 0.00032$.
6. $\log_5 0.5$.

15.5 EXPONENTIAL AND LOGARITHMIC EQUATIONS

An equation in which the unknown appears in an exponent is called an **exponential equation**. An exponential equation can generally be solved by first taking logarithms of both sides and using properties of logarithms. Although we may use logarithms to any base, we usually use either common or natural logarithms because these can be obtained with a calculator.

Example 1

Solve $6^x = 8.2$.

Solution Taking natural logarithms of both sides, we have

$$\ln 6^x = \ln 8.2.$$

Using Property 3,

$$x \ln 6 = \ln 8.2$$

$$x = \frac{\ln 8.2}{\ln 6} = 1.1743.$$

Example 2

Solve $5^{2x+1} = 36$.

Solution Taking common logarithms of both sides, we have

$$\log 5^{2x+1} = \log 36$$

$$(2x + 1)(\log 5) = \log 36$$

$$2x + 1 = \frac{\log 36}{\log 5}$$

$$2x = \frac{\log 36}{\log 5} - 1$$

$$x = \frac{\log 36}{2 \log 5} - \frac{1}{2}$$

$$x = 0.61328.$$

Example 3

A 6-μF capacitor is charged by connecting it in series with a 2-MΩ resistor and a 100-V battery. In the charging circuit the current i (in amperes) follows an exponential decay given by

$$i = \frac{\mathcal{E}}{R} e^{-t/(RC)}, \tag{1}$$

where C is the capacitance (in farads), R is the resistance (in ohms), and $\mathcal{E}$ is the emf (in volts). In how many seconds after the connection is made will the current in the circuit be 20 μA?

Solution Substituting the given values into Eq. 1 gives

$$20 \times 10^{-6} = \frac{100}{2 \times 10^{6}} e^{-t/[(2 \times 10^{6})(6 \times 10^{-6})]}$$

$$0.4 = e^{-t/12}. \tag{2}$$

Now we solve $e^{-t/12} = 0.4$. Taking natural logarithms of both sides, we have

$$\ln e^{-t/12} = \ln 0.4$$

$$-\frac{t}{12} = \ln 0.4 \qquad \text{[Property 6]}$$

$$t = -12 \ln 0.4$$

$$t = 11.0 \text{ s}.$$

Equation 2 can also be solved by first expressing it in an equivalent logarithmic form:

$$-\frac{t}{12} = \ln 0.4$$

$$t = -12 \ln 0.4 = 11.0 \text{ s}.$$

An equation involving a logarithm of an expression containing the unknown is called a **logarithmic equation**. A logarithmic equation can generally be solved by rewriting it in exponential form, as the following examples show.

Example 4

Solve $\log_2 x + \log_2 (x + 4) = 5$.

Solution We first write the left side as a single logarithm by using Property 1:

$$\log_2 [x(x + 4)] = 5.$$

In exponential form we have

$$x(x + 4) = 2^5$$

$$x^2 + 4x = 32$$

$$x^2 + 4x - 32 = 0 \qquad \text{[quadratic equation]}$$

$$(x - 4)(x + 8) = 0$$

$$x = 4 \quad \text{or} \quad x = -8.$$

We must reject $x = -8$ because the given equation is not defined for negative x (the logarithm of a negative number is not defined). Thus only 4 is a solution.

Example 5

Solve $\log (x + 1) - \log (x - 1) = \log x$.

Solution Simplifying the left side gives

$$\log \frac{x+1}{x-1} = \log x.$$

Recall that if the logarithms of two numbers are equal, then the numbers are equal. Thus

$$\frac{x+1}{x-1} = x$$

$$x + 1 = x(x - 1)$$

$$x + 1 = x^2 - x$$

$$0 = x^2 - 2x - 1.$$

Using the quadratic formula gives $x = 1 \pm \sqrt{2}$. However, the given equation is not defined for $x = 1 - \sqrt{2}$. The solution is $1 + \sqrt{2}$.

Problem Set 15.5

In Problems **1–18**, *solve the equation.*

1. $12^x = 17$.

2. $2^{8x} = 9$.

3. $3^{x-2} = 14$.

4. $5^{2x-4} = 96$.

5. $2^{-t} = 9$.

6. $3^{t/2} = 0.056$.

7. $e^{3t+1} = 35$.

8. $e^{1-t} = 354$.

9. $x^{2.56} = 26$.

10. $t^{2.5} = 24.71$.

11. $\log (x - 3) = 3$.

12. $\log_2 (x + 1) = 4$.

13. $\log (2x + 1) = \log (x + 6)$.

14. $\log x + \log 3 = \log 5$.

15. $\log x - \log (x - 1) = \log 4$.

16. $\log_2 x + 3 \log_2 2 = \log_2 \left(\frac{2}{x}\right)$.

17. $\log (3x - 1) - \log (x - 3) = 2$.

18. $\log_3 (2x + 3) = 4 - \log_3 (x + 6)$.

19. The magnitude of the force F (in newtons) driving a piston varies with the distance x (in meters) of the piston from its central position according to the equation

$$F = 135x^{-7/5}.$$

(a) Find the force acting when the piston is 8.0 cm from its central position.
(b) How far is the piston from its central position when $F = 1000$ N?

20. A sound source has an intensity level of 100 db. Find its intensity in watts per square meter. Refer to the formula in Problem 73 in Problem Set 15.2.

21. A 2-μF capacitor is charged by connecting it in series with a 4-MΩ resistor and a 200-V source.
    (a) Find the current in the circuit at $t = 0$, which is the instant the connections are made. (Refer to Example 3.)
    (b) For what value of t will the current in the circuit be 10 μA?

22. Suppose the number Q of milligrams of a radioactive substance present after t years is given by

$$Q = 100e^{-0.035t}.$$

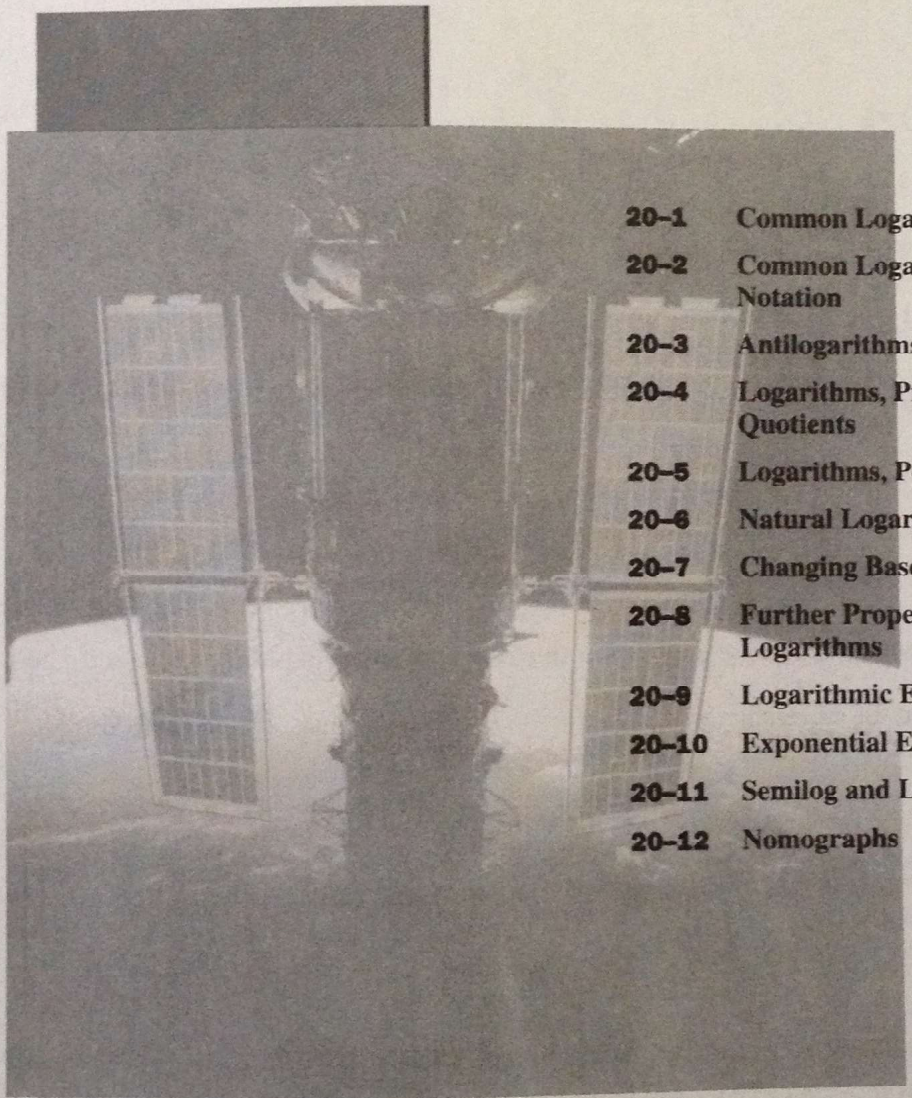
    (a) How many milligrams are present after 0 years?
    (b) After how many years will there be 50 mg present?

23. In one city the population P grows at the rate of 2% per year. The equation $P = 1{,}000{,}000(1.02)^t$ gives the population t years after 1983. Find the value of t for which the population is 1,500,000.

24. A capacitance C is charged by connecting it in series with a resistance R and an emf $\mathcal{E}$.
    (a) Show that the initial current in the circuit (at $t = 0$) is $\mathcal{E}/R$. (See Example 3.)
    (b) The *capacitive time constant* τ_C is given by $\tau_C = RC$. Show that when $t = \tau_C$, the current has decayed to $1/e$ of its initial value.
    (c) In Part (b), approximately what percentage of the initial current does that represent?

25. The water in a midwestern lake contains sediment, and the presence of the sediment reduces the transmission of light through the water. Experiments indicate that the intensity of light is reduced by 10% by passage through 20 cm of water. Suppose that the lake is uniform with respect to the amount of sediment contained by the water. A measuring instrument can detect light at the intensity of 0.17% of full sunlight. This measuring instrument is lowered into the lake. At what depth will it first cease to record the presence of light?

26. A translucent material has the property that although light passes through the material, its intensity is reduced. A particular translucent plastic has the property that a sheet 1 mm thick reduces the intensity of light by 10%. How many such sheets are necessary to reduce the intensity of a beam of light to about 50% of its original value?



20–1 Common Logarithms

20–2 Common Logarithms and Scientific Notation

20–3 Antilogarithms

20–4 Logarithms, Products, and Quotients

20–5 Logarithms, Powers, and Radicals

20–6 Natural Logarithms

20–7 Changing Base

20–8 Further Properties of Natural Logarithms

20–9 Logarithmic Equations

20–10 Exponential Equations

20–11 Semilog and Log–Log Plots

20–12 Nomographs

The Hubble Space Telescope (with a new set of solar panels) sits in the cargo bay of the Space Shuttle Endeavour while in orbit above the horizon of Earth after the historic capture of the malfunctioning instrument for repairs. (Courtesy of NASA/ Johnson Space Center)

21. $12^{x} = 17$

22. $3^{x-2} = 14$

23. $2^{-t} = 9$

24. $e^{3t+1} = 35$

25. $x^{2.56} = 26$

26. $\log (x - 3) = 3$

27. $\log (2x + 1) = \log (x + 6)$

28. $\log x - \log (x - 1) = \log 4$

29. $\log (3x - 1) - \log (x - 3) = 2$

30. The magnitude of the force F (in newtons) driving a piston varies with the distance x (in meters) of the piston from its central position according to the equation

$$F = 135x^{-7/5}$$

(a) Find the force acting when the piston is 8.0 cm from its central position.

(b) How far is the piston from its central position when $F = 1000$ N?

31. A 2- μF capacitor is charged by connecting it in series with a 4-MΩ resistor and a 200-V source.

(a) Find the current in the circuit at t = 0, which is the instant the connections are made. (Refer to Example 3.)

(b) For what value of t will the current in the circuit be 10 μA?

32. Suppose the number Q of milligrams of a radioactive substance present after t years is given by

$$Q = 100e^{-0.035t}$$

(a) How many milligrams are present after 0 years?

(b) After how many years will there be 50 mg present?

33. In one city the population P grows at the rate of 2% per year. The equation $P = 1{,}000{,}000\ (1.02)^t$ gives the population t years after 1983. Find the value of t for which the population is 1,500,000.

A capacitance C is charged by connecting it in series with a resistance R and an emf ε.

(a) Show that the initial current in the circuit (at $t = 0$) is ε/R. (See Example 3.)

(b) The *capacitive time constant* t_C is given by $t_C = RC$. Show that when $t = t_C$, the current has decayed to $1/e$ if its initial value.

(c) In Part (b), approximately what percentage of the initial current does that represent?

34. The water in a Midwestern lake contains sediment, and the presence of the sediment reduces the transmission of light through the water. Experiments indicate that the intensity of light is reduced by 10% by passage through 20 cm of water. Suppose that the lake is uniform with respect to the amount of sediment contained by the water. A measuring instrument can detect light at the intensity of 0.17% of full sunlight. This measuring instrument is lowered into the lake. At what depth will it first cease to record the presence of light?

35. A translucent material has the property that although light passes through the material, its intensity is reduced. A particular translucent plastic has the property that a sheet 1 mm thick reduces the intensity of light by 10%. How many such sheets are necessary to reduce the intensity of a beam of light to about 50% of its original value?

In the preceding chapters, functions involving addition, subtraction, multiplication, division, and raising variables to constant powers have been discussed. In this chapter, we look at raising constants to *variable* powers. We have examined $y = x^2$; now we are going to examine $y = 10^x$.

The following symbols are new to this chapter:

log	common logarithm	10^x	common antilog
ln	natural logarithm	e^x	natural antilog

20–1 COMMON LOGARITHMS

Suppose we know that $a = 10^b$, and we want to solve for b. What do we do? Read $a = 10^b$ as "b is the exponent to which 10 must be raised to yield a." In mathematics, we define common logarithm so that the following statement means the same as the preceding: b is the common logarithm of a. This can be written in the following manner:

$$\textbf{If } a = 10^b \textbf{, then } b = \log(a)$$

The symbol log (a) is read "log of a" and stands for the common logarithm function of a. Notice that the parentheses here indicate "function of" and not multiplication. The parentheses are frequently omitted.

The **common logarithm** of a number is defined as the exponent to which 10 must be raised to yield the number. Thus, 10 is the base of the common logarithms. In general, a **logarithm** is the exponent to which the base must be raised to equal the original number. In the following sections you will see that logarithms follow the rules of exponents.

EXAMPLE 20–1 Find the common logarithm of 1, 10, and 100.

Observation $1 = 10^0$, $10 = 10^1$, and $100 = 10^2$.

Solution Use the definition of common logarithms and the observation:

$$\log(1) = \log(10^0) = 0$$
$$\log(10) = \log(10^1) = 1$$
$$\log(100) = \log(10^2) = 2$$

$\therefore$ Log (1) = 0, log (10) = 1, and log (100) = 2.

Now, turn to your calculator and your owner's guide. Locate the common logarithm function in the guide and the function key $\boxed{\log x}$ or $\boxed{\log}$ on your calculator. This function will calculate the common logarithm of any positive number for you. Use your calculator to work the following examples.

EXAMPLE 20–2 Find the common logarithm of 5, 6.5, and 2.98.

Observation When writing logarithms, we generally write four digits to the right of the decimal point. This corresponds to three place accuracy in the number.

Solution Use your calculator:

5 [log] $\Rightarrow 0.6990$

6.5 [log] $\Rightarrow 0.8129$

2.98 [log] $\Rightarrow 0.4742$

EXAMPLE 20–3 Find the common logarithm of $10^{0.5}$.

Solution 1 Use the definition of common logarithm:

$$\log (10^{0.5}) = 0.5$$

Observation Recall that $10^{0.5}$ means the square root of 10.

Solution 2 Use your calculator:

$$\log (10^{0.5}) = \log (\sqrt{10})$$

[$\sqrt{x}$] $\log (10^{0.5}) = \log (\mathbf{3.162})$

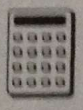
[log] $\log (10^{0.5}) = \mathbf{0.5}$

$\therefore$ Log $(10^{0.5}) = 0.5$ by definition and by calculator.

EXERCISE 20–1

Use the definition to find the common logarithms of the following numbers.

1. 10^3	**2.** 10^1	**3.** 10^4
4. $10^{1.5}$	**5.** $10^{2.5}$	**6.** $10^{0.25}$



Calculator Drill

Use your calculator to find the common logarithms of the following numbers.

7. 2	**8.** 4	**9.** 8
10. 16	**11.** 32	**12.** 2.1
13. 3.1	**14.** 4.1	**15.** 5.1
16. 6.1	**17.** 7.1	**18.** 6.2
19. 3.41	**20.** 3.78	**21.** 5.92
22. 1.05	**23.** 2.95	**24.** 8.01
25. 7.62	**26.** 9.15	**27.** 4.61

20–2 COMMON LOGARITHMS AND SCIENTIFIC NOTATION

In the preceding section we learned the definition of common logarithms. We also learned how to use our calculators to find the common logarithm of a number. We are going to perform an experiment with our calculators on a series of numbers in scientific notation.

EXAMPLE 20–4 Find the common logarithms of 3×10^0, 3×10^1, 3×10^2, and 3×10^9.

Solution Use your calculator:

3 [EE] 0 [log] ⇒ 0.4771
3 [EE] 1 [log] ⇒ 1.4771
3 [EE] 2 [log] ⇒ 2.4771
3 [EE] 9 [log] ⇒ 9.4771

Observation Notice that the fractional part of each answer is the same number (.4771) and that the integer part of each answer is the same as the exponent of 10.

In Example 20–4, notice that common logarithms of the numbers 3×10^0, 3×10^1, 3×10^2, and 3×10^9 all had the same fractional part. It is a fact that the fractional part of a logarithm depends only on the digits of a number and not on the location of the decimal point nor on the exponent of 10. The fractional part of a logarithm is called the **mantissa**. Notice also that the integer part of the logarithm depends only on the exponent of 10. The integer part of the logarithm is called the **characteristic**. If we have a table of common logarithms for numbers between 1 and 10, or if we have a calculator that will find the common logarithm for a number, we can use Rule 20–1 to find the common logarithm of any positive number.

RULE 20–1. Finding the Common Logarithm of a Number

To find the logarithm of a number:

1. Write the number in scientific notation.
2. Determine the mantissa as the common logarithm of the decimal coefficient. (This will be a fraction between zero and one.)
3. Determine the characteristic as the exponent of 10.
4. Form the common logarithm by adding the characteristic and the mantissa.

EXAMPLE 20–5 Find the mantissa, the characteristic, and the common logarithm of 654, 7.83, and 0.0491.

Solution 1 Use the steps of Rule 20–1 and your calculator.

Step 1: Write the number in scientific notation:

SN: $654 = 6.54 \times 10^2$

Step 2: Determine the mantissa:

log mantissa = log (6.54) = **0.8156**

Step 3: Determine the characteristic:

characteristic = 2

Step 4: Form the common logarithm:

$\therefore$ log (654) = 2.8156

Solution 2 Use Rule 20–1 and your calculator.

Step 1: Write the number in scientific notation:

SN: $7.83 = 7.83 \times 10^0$

Step 2: Determine the mantissa:

log mantissa = log (7.83) = **0.8938**

Step 3: Determine the characteristic:

characteristic = 0

Step 4: Form the common logarithm:

$\therefore$ log (7.83) = 0.8938

Solution 3 Use Rule 20–1 and your calculator.

Step 1: Write the number in scientific notation:

SN: $0.0491 = 4.91 \times 10^{-2}$

Step 2: Determine the mantissa:

log mantissa = log (4.91) = **0.6911**

Step 3: Determine the characteristic:

characteristic = -2

Step 4: Form the common logarithm:

$\therefore$ log (0.0491) = $-2 + 0.6911 = -1.3089$

Summary

$$\log (654) = 2 + 0.8156 = 2.8156$$
$$\log (7.83) = 0 + 0.8938 = 0.8938$$
$$\log (0.0491) = -2 + 0.6911 = -1.3089$$

Although the mantissa is positive, the characteristic may be positive or negative. The characteristic is always negative for fractions. In the case of fractions, we combine the characteristic and the mantissa by subtraction. The integer and fractional parts of the resulting logarithm are both negative. This is the answer your calculator will give.

The graph of Figure 20–1 is the plot of the common logarithm function, which reveals several properties of the common logarithm function. For $x > 1$, $\log (x) > 0$; for $1 > x > 0$, $\log (x) < 0$; and for $x = 1$, $\log (x) = 0$.

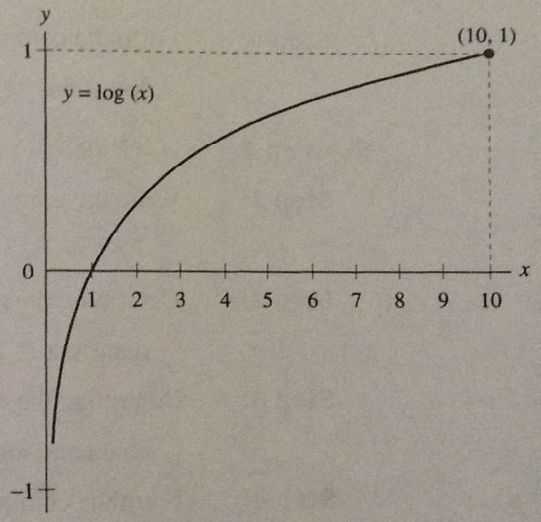


FIGURE 20–1 Plot of the common logarithm function.

EXERCISE 20–2

Use Rule 20–1 and Example 20–5 to find the characteristics, mantissas, and common logarithms of the following numbers.

1. **(a)** 7	**(b)** 70	**(c)** 700	**(d)** 7000
2. **(a)** 2	**(b)** 20	**(c)** 200	**(d)** 2000
3. **(a)** 0.075	**(b)** 0.75	**(c)** 7.5	**(d)** 75
4. **(a)** 0.008	**(b)** 0.8	**(c)** 80	**(d)** 8000
5. **(a)** 1.62	**(b)** 16.2	**(c)** 162	**(d)** 1620
6. **(a)** 99.1	**(b)** 9.91	**(c)** 0.991	**(d)** 0.0991
7. **(a)** 5.55	**(b)** 55.5	**(c)** 555	**(d)** 5550
8. **(a)** 3.61	**(b)** 0.361	**(c)** 0.0361	**(d)** 0.003 61
9. **(a)** 0.001 05	**(b)** 0.0105	**(c)** 0.105	**(d)** 1.05

20–3 ANTILOGARITHMS

The common logarithm of a number is the exponent to which 10 must be raised to give the original number. Knowing this definition, how can we find the original number if we know the common logarithm? By raising 10 to the common logarithm power! This operation is the definition of the **common antilogarithm** function, which is the **inverse** of the common logarithm function. All of this can be written in symbols:

$$\textbf{If } b = \log(a), \textbf{ then } a = \text{antilog}(b) = 10^b$$

Check your calculator and your owner's guide to find out how to evaluate the antilogarithm of a number. Your calculator may use $\boxed{10^x}$ or $\boxed{y^x}$ where you supply the 10 for y. Or your calculator may use two keystrokes, as in $\boxed{\text{INV}}$ $\boxed{\log}$ or $\boxed{f^{-1}}$ $\boxed{\log}$.

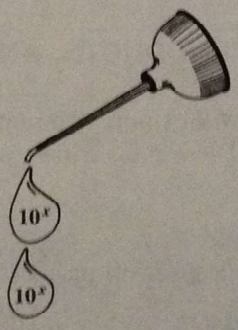
EXAMPLE 20–6 If the common logarithm of A is 2, what is the value of A?

Solution Use the antilogarithm to find A:

$$2 = \log(A)$$

Apply antilog; see Figure 20–2:

FIGURE 20–2 Effects of antilog.



$b = \log(a) \Longrightarrow a = \text{antilog}(b) = 10^b$

$$A = \text{antilog}(2) = 10^2$$

$$\therefore \quad A = 100$$

EXAMPLE 20–7 If the common logarithm of Z is 1.3324, what is the value of Z?

Solution Use the antilogarithm and your calculator:

$$1.3324 = \log(Z)$$

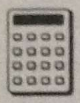
Apply antilog:

$$Z = \text{antilog}\,(1.3324) = 10^{1.3324}$$

$$1.3324\ \boxed{10^x} \Rightarrow 21.5$$

$$\therefore \quad Z = 21.5$$

EXERCISE 20–3



Calculator Drill

Find the antilogarithm of the following numbers.

1. 0.3522
2. 0.9706
3. 0.1592
4. 0.7041
5. 0.5223
6. −0.3871
7. −1.0526
8. 1.3952
9. 1.4671
10. 2.8352
11. 0.3010, 1.3010, 2.3010
12. 0.4771, 1.4771, 2.4771
13. 0.6021, 2.6021, 4.6021
14. 0.6990, 2.6990, 4.6990
15. −0.1549, −1.1549, −2.1549

20–4 LOGARITHMS, PRODUCTS, AND QUOTIENTS

In this section we will explore two important properties of logarithms. They will each be stated as rules. These rules will be useful in solving logarithmic equations.

RULE 20–2. Logarithm of a Product

The logarithm of a product is equal to the sum of the logarithms of the factors.

$$\log\,(AB) = \log\,(A) + \log\,(B)$$

EXAMPLE 20–8 Check the equation $\log\,(20 \cdot 15) = \log\,(20) + \log\,(15)$.

Solution Evaluate the left and right members separately:
Left member:

$\boxed{\times}$
$\boxed{\log}$

$$\log\,(20 \cdot 15) = \log\,(\mathbf{300})$$
$$= 2.4771$$

Right member:

log
+
$\log (20) + \log (15) = \mathbf{1.3010} + \mathbf{1.1761}$
$= 2.4771$

$\therefore \quad \log (20 \cdot 15) = \log (20) + \log (15)$

EXAMPLE 20–9 Solve $P = 3 \times 6$ using logarithms and Rule 20–2.

Solution

$P = 3 \times 6$

Take the log of each member.

$\log (P) = \log (3 \times 6)$

Apply Rule 20–2:

$\log (P) = \log (3) + \log (6)$

log
+
$= 0.4771 + 0.7782$
$= 1.2553$

$\log (P) = 1.2553$

Take the antilog of each member:

10^x
$P = \text{antilog } (1.2553) = \mathbf{10^{1.2553}} = 18$

$\therefore \quad P = 18$

RULE 20–3. Logarithm of a Quotient

The logarithm of a quotient is equal to the difference between the logarithm of the dividend and the logarithm of the divisor.

$$\log (A/B) = \log (A) - \log (B)$$

EXAMPLE 20–10 Check the equation $\log (\frac{75}{15}) = \log 75 - \log 15$.

Solution Evaluate each member of the equation separately.
Left member:

÷
log
$\log (\frac{75}{15}) = \log (\mathbf{5})$
$= 0.6990$

Right member:

log
−
$\log (75) - \log (15) = \mathbf{1.8751} - \mathbf{1.1761}$
$= 0.6990$

$\therefore \quad \log (\frac{75}{15}) = \log (75) - \log (15)$

EXAMPLE 20–11 Solve $Q = \frac{20}{4}$ using logarithms and Rule 20–3.

Solution

$$Q = \frac{20}{4}$$

Take the log of each member:

$$\log (Q) = \log (\tfrac{20}{4})$$

Apply Rule 20–3:

$$\log (Q) = \log (20) - \log (4)$$

log
−
$= 1.3010 - 0.6021$
$= 0.6989$

$$\log (Q) = 0.6990$$

Take the antilog of each member:

10^x
$Q = \text{antilog } (0.6990) = \mathbf{10^{0.6990}} = 5$

$\therefore \quad Q = 5$

EXERCISE 20–4

Check the following equations as in Examples 20–8 and 20–10:

1. $\log (3 \times 5) = \log (3) + \log (5)$
2. $\log (2 \times 7) = \log (2) + \log (7)$
3. $\log (17 \times 3.5) = \log (17) + \log (3.5)$
4. $\log (\frac{625}{25}) = \log (625) - \log (25)$
5. $\log (\frac{18}{2}) = \log (18) - \log (2)$

Determine whether the following statements are true or false:

6. $\log (75) = \log (5) + \log (15)$
7. $\log (14) = \log (7 + 2)$
8. $\log (7) = \log (35) - 5$
9. $\log (20) = \log (4) \log (5)$
10. $\log (9) = \log (72) - \log (8)$

Use logarithms to solve the following as in Examples 20–9 and 20–11:

11. $P = 7 \times 4$
12. $P = 23.4 \times 0.0574$
13. $Q = \frac{16}{4}$
14. $Q = 0.213/0.0790$
15. $Q = 1220/203$

20–5 LOGARITHMS, POWERS, AND RADICALS

In the preceding section we learned that through logarithms multiplication becomes addition and division becomes subtraction. In a similar fashion, raising to powers becomes multiplication.

RULE 20–4. Logarithm of a Power

The logarithm of a power is equal to the product of the exponent and the logarithm of the base.

$$\log (a^n) = n \log (a)$$

EXAMPLE 20–12 Check the equation $\log (5^4) = 4 \log (5)$.

Solution Evaluate each member of the equation.
Left member:

$\boxed{y^x}$ $\log (5^4) = \log (625)$

$\boxed{\log}$ $= 2.7959$

Right member:

$\boxed{\log}$ $4 \log (5) = 4 (0.6990)$

$\boxed{\times}$ $= 2.7959$

$\therefore$ $\log (5^4) = 4 \log (5)$

Square roots can be indicated with a radical sign and with an exponent of one-half. This combined with Rule 20–4 gives a new rule for the logarithm of the square root of a number.

RULE 20–5. Logarithm of the Square Root of a Number

The logarithm of the square root of a number is equal to one-half the logarithm of the number.

$$\log(\sqrt{A}) = \frac{1}{2}\log(A)$$

EXAMPLE 20–13 Check the equation $\log(\sqrt{49}) = \frac{1}{2}\log(49)$.

Solution Evaluate each member of the equation.
Left member:

$\boxed{\sqrt{x}}$ $\boxed{\log}$ $\log(\sqrt{49}) = \log(7)$
$= 0.8451$

Right member:

$\boxed{\log}$ $\boxed{\div}$ $\frac{1}{2}\log(49) = \frac{1}{2}(\mathbf{1.6902})$
$= 0.8451$

$\therefore$ $\log(\sqrt{49}) = \frac{1}{2}\log(49)$

EXAMPLE 20–14 Calculate $\log(\sqrt{72.5})$ without using square root.

Solution Use Rule 20–5.

$$\log(\sqrt{72.5}) = \frac{1}{2}\log(72.5)$$

$\boxed{\log}$ $= \frac{1}{2}(\mathbf{1.8603}) = 0.9302$

$\therefore$ $\log(\sqrt{72.5}) = 0.9302$

In the following example we use Rule 20–5 to evaluate the square root of a number. The same steps can be used with Rule 20–4 to evaluate any power of a number.

EXAMPLE 20–15 Calculate $\sqrt{172}$ without using square root.

Solution Use Rule 20–5 and logarithms:

$$\log(\sqrt{172}) = \frac{1}{2}\log(172)$$

$\boxed{\log}$ $= \frac{1}{2}(\mathbf{2.2355})$

$\boxed{\div}$ $= 1.1178$

Take the antilog of each member:

$\boxed{10^x}$ $\sqrt{172} = \text{antilog } (1.1178) = \mathbf{10^{1.1178}} = 13.1$

$\therefore$ $\sqrt{172} = 13.1$

EXAMPLE 20–16 Calculate $17^{3.49}$.

Solution Use Rule 20–4:

$$\log (17^{3.49}) = 3.49 \log (17)$$

$\boxed{\log}$ $= 3.49(\mathbf{1.2304})$

$\boxed{\times}$ $= 4.2943$

Take the antilog of each member:

$\boxed{10^x}$ $17^{3.49} = \text{antilog } (4.2943) = \mathbf{10^{4.2943}} = 19\ 700$

$\therefore$ $17^{3.49} = 19\ 700$

EXAMPLE 20–17 Solve $N = 4^3$ using logarithms and Rule 20–4.

Solution Take the log of each member:

$$\log (N) = \log (4^3)$$

Apply Rule 20–4:

$$\log (N) = 3 \log (4)$$

$\boxed{\log}$ $= 3(\mathbf{0.6021})$

$\boxed{\times}$ $= 1.8062$

Take the antilog of each member:

$\boxed{10^x}$ $N = \text{antilog } (1.8062) = \mathbf{10^{1.8062}} = 64$

$\therefore$ $N = 64$

EXERCISE 20–5

Check the following equations as in Examples 20–12 and 20–13.

1. $\log (3^2) = 2 \log (3)$
2. $\log (12^2) = 2 \log (12)$
3. $\log (29^2) = 2 \log (29)$
4. $\log (5^3) = 3 \log (5)$
5. $\log (\sqrt{81}) = \frac{1}{2} \log (81)$

6. $\log(\sqrt{961}) = \frac{1}{2}\log(961)$

7. $\log(\sqrt{2^3}) = \frac{3}{2}\log(2)$

Use Rules 20–4 and 20–5 in evaluating the following.

8. $\sqrt{39}$	**9.** $4^{7.5}$	**10.** $\sqrt{409}$
11. $\sqrt{3.5}$	**12.** $\sqrt{11}$	**13.** $(5.2)^{0.28}$
14. $(1.01)^{120}$	**15.** $(87)^{0.314}$	**16.** $\sqrt[3]{6.01 \times 10^{-5}}$

20–6 NATURAL LOGARITHMS

In electronics there are two important logarithm functions. The first is the common logarithm based on the number 10. The second is the natural logarithm based on the number e. The value of e to five digits is 2.7183. Natural logarithms are also called Napierian logarithms for John Napier, the Scottish mathematician who invented logarithms.

To distinguish these two logarithm functions, we will continue to use log ( ) for common logarithms. And we introduce a new symbol ln ( ) for natural logarithms. The following is the definition of the **natural logarithm** function:

If $A = e^B$, then B = natural logarithm of A = ln (A)

The symbol ln (A) is read "ell-en of a" or "lin of a." It may also be read "natural log of a." As with common logarithms, the parentheses are frequently omitted. Remember that they are used here to denote *function*, not multiplication.

Check your calculator and operating instructions for natural logarithms. Locate the function key [ln x] or [ln]. This function will calculate the natural logarithm of any positive number for you.

EXAMPLE 20–18 Find the natural logarithms of 5, 10, and 15.

Solution Use your calculator:

5 [ln] ⇒ 1.6094
10 [ln] ⇒ 2.3026
15 [ln] ⇒ 2.7081

Figure 20–3 presents a graph of both logarithm functions. Notice that both graphs pass through the point (1, 0). Also, both have the same *general* behavior. There is no graph for $x \leq 0$. For $0 < x < 1$, both graphs are below the x-axis. This means both functions are negative in this region. For $x > 1$, both graphs are shown above the x-axis and both functions are positive. Finally, as x increases, both graphs increase.

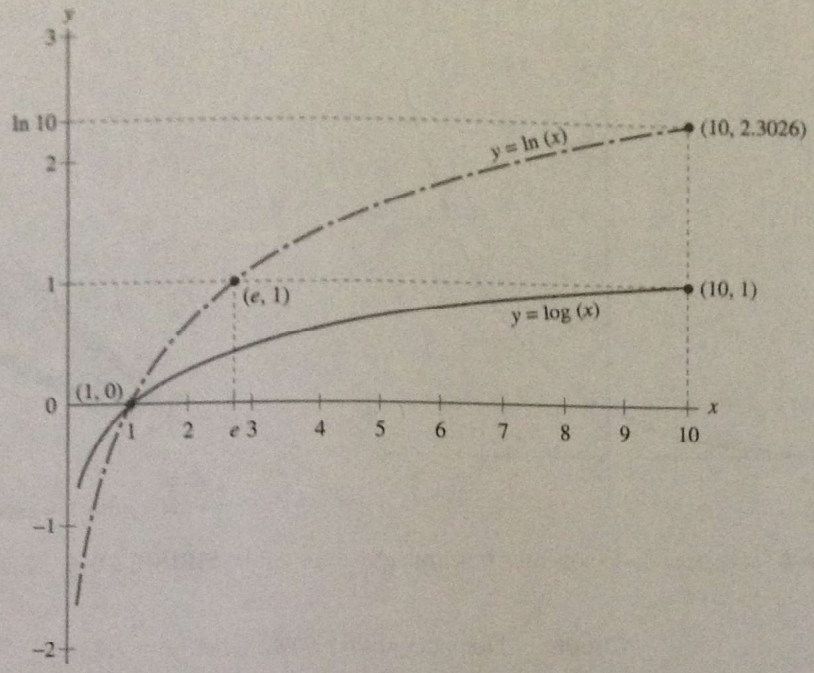


FIGURE 20–3 Comparison of log (x) and ln (x).

Antilogarithm

Recall the definition of the natural logarithm of a number as the exponent to which e must be raised to give the number. Knowing the natural logarithm of a number, how can we find the number? By raising e to the natural logarithm power. This is the definition of the **natural antilogarithm** function, which is the *inverse* of the natural logarithm function. The following symbols express this definition:

$$\textbf{If } a = \ln(b), \textbf{ then } b = \text{antiln}(a) = e^a$$

The antiln function is also known as the *exponential function.* The graph of the exponential function is shown in Figure 20–4.

Check your calculator and owner's guide on how to calculate antiln (x). It may use $\boxed{e^x}$ or a two-keystroke sequence, as in $\boxed{f^{-1}}$ $\boxed{\ln}$ or $\boxed{\text{INV}}$ $\boxed{\ln}$. Figure 20–5 shows the effects of the antiln function.

EXAMPLE 20–19 Find y if ln $(y) = 1.9782$.

Solution Use antiln and your calculator:

$$\ln(y) = 1.9782$$

$$y = \text{antiln}(1.9782) = e^{1.9782}$$

$\boxed{e^x}$

$$y = 7.23$$

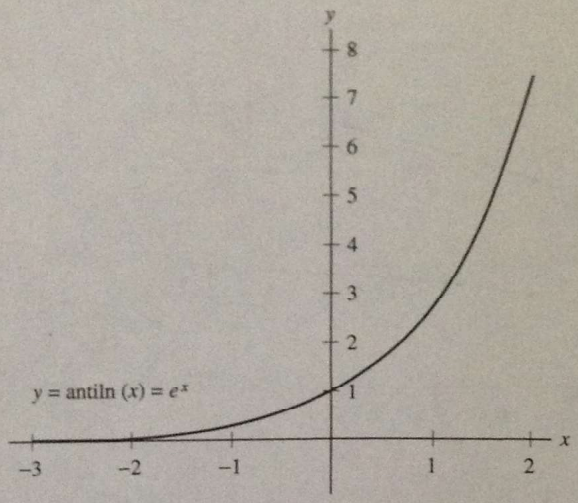


FIGURE 20–4 Graph of the exponential function, e^x.

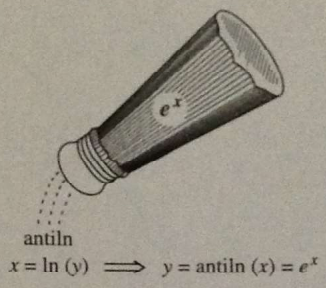


FIGURE 20–5 Effects of antiln.

Check Does ln (y) = 1.9782 when $y = 7.23$?

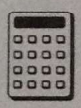
ln ln (7.23) = 1.9782 Yes!

$\therefore$ $y = 7.23$

EXERCISE 20–6

Use the definition to find the natural logarithm of the following terms.

1. $e^{1.2}$	**2.** $e^{3.0}$	**3.** $e^{0.75}$	**4.** $e^{0.12}$	**5.** $e^{-4.1}$
6. $e^{-0.51}$	**7.** e^{a}	**8.** e^{-Rt}	**9.** e^{-z}	**10.** e^{2n}



Calculator Drill

Calculate the natural logarithms of the following numbers.

11. 1.25	**12.** 3.16
13. 4.0	**14.** 5.0
15. 6.0	**16.** 7.0
17. 0.156	**18.** 0.029
19. 0.015	**20.** 18.4
21. 927	**22.** 7040
23. 143.2	**24.** 40.94
25. 6.35×10^4	**26.** 1045
27. 0.831	**28.** 0.0392
29. 79.3	**30.** 2001

Find the natural antilogarithms of the following numbers.

31. 1.0	**32.** 0.0
33. 2.3026	**34.** 2.9444
35. 2.1983	**36.** −5.2785
37. −3.6045	**38.** −1.3863
39. 0.5241	**40.** −99.0919
41. −0.0590	**42.** 4.7205
43. −7.0646	**44.** 0.8751
45. −4.6702	**46.** −3.0011
47. 14.3285	**48.** −4.4273
49. −16.2345	**50.** 0.5623

20–7 CHANGING BASE

Sometimes the question arises of converting a logarithm from one base to another. With the aid of a calculator this is a simple task when the value of the logarithm is known. Take the antilogarithm and then the second logarithm of the number. A more direct method is provided by the following rule:

RULE 20–6. Changing Base

1. To change a common logarithm to a natural logarithm, multiply by 2.3026, the ln of 10:

$$\ln (x) = 2.3026 \log (x)$$

2. To change a natural logarithm to a common logarithm, divide by 2.3026, the ln of 10:

$$\log (x) = \ln (x)/2.3026$$

EXAMPLE 20–20 Check Rule 20–6 when $x = 15$.

Solution Substitute for x; then evaluate each member of the equation:

$$\ln (15) = 2.3026 \log (15)$$

Left member:

ln

$$\ln (15) = 2.7081$$

Right member:

log
× $\quad 2.3026 \log (15) = 2.3026 \times \mathbf{1.1761}$
$\qquad = 2.7081$

$\therefore \quad \ln (15) = 2.3026 \log (15)$

The number 2.3026 may be remembered as ln (10) or 1/log (e). In fact, for any positive number n, where $n \neq 1$, ln (n)/log (n) = 2.3026.

EXERCISE 20–7

Use common logarithms to calculate the following.

1. ln (4) **2.** ln (5.1) **3.** ln (0.49)
4. ln (8.2) **5.** ln (0.79) **6.** ln (10.9)

Use natural logarithms to calculate the following.

7. log (60) **8.** log (21) **9.** log (5.7)
10. log (0.015) **11.** log (0.29) **12.** log (0.091)

20–8 FURTHER PROPERTIES OF NATURAL LOGARITHMS

As we noted earlier, the natural logarithm function and the common logarithm function have the same general behavior. In fact, the rules for logarithms and products, quotients, powers, and radicals are the same for both logarithm functions.

EXAMPLE 20–21 Check Rule 20–2 for natural logarithms:
ln (AB) = ln (A) + ln (B) when $A = 16$ and $B = 3.5$.

Solution Substitute for A and B; then evaluate each member of the equation:

$$\ln (16 \times 3.5) = \ln (16) + \ln (3.5)$$

Left member:

×
ln $\quad \ln (16 \times 3.5) = \ln (\mathbf{56})$
$\qquad = 4.0254$

Right member:

ln
+ $\quad \ln (16) + \ln (3.5) = \mathbf{2.7726} + \mathbf{1.2528}$
$\qquad = 4.0254$

$\therefore \quad \ln(16 \times 3.5) = \ln (16) + \ln (3.5)$

EXAMPLE 20–22 Check Rule 20–3 for natural logarithms: $\ln(A/B) = \ln(A) - \ln(B)$ when $A = 54$ and $B = 60$.

Solution Substitute for A and B; then evaluate each member of the equation:

$$\ln\left(\tfrac{54}{60}\right) = \ln(54) - \ln(60)$$

Left member:

$\boxed{\div}$ $\boxed{\ln}$ $\quad \ln\left(\tfrac{54}{60}\right) = \ln(\mathbf{0.9}) = -0.1054$

Right member (use chain operations):

$\boxed{\ln}$ $\boxed{-}$ $\quad \ln(54) - \ln(60) = \mathbf{3.9890} - \mathbf{4.0943} = -0.1054$

$\therefore \quad \ln\left(\tfrac{54}{60}\right) = \ln(54) - \ln(60)$

EXAMPLE 20–23 Check Rule 20–4 for natural logarithms: $\ln(a^n) = n \ln(a)$ when $a = 1.67$ and $n = 3$.

Solution Substitute for a and n and then evaluate each member of the equation:

$$\ln(1.67^3) = 3 \ln(1.67)$$

Left member (perform chain operations):

$\boxed{y^x}$ $\boxed{\ln}$ $\quad \ln(1.67^3) = \ln(\mathbf{4.66}) = 1.5385$

Right member:

$\boxed{\ln}$ $\boxed{\times}$ $\quad 3 \ln(1.67) = 3(\mathbf{0.5128}) = 1.5385$

$\therefore \quad \ln(1.67^3) = 3 \ln(1.67)$

EXAMPLE 20–24 Check Rule 20–5 for natural logarithms: $\ln(\sqrt{A}) = \frac{1}{2} \ln(A)$ when $A = 79.6$ (use chain operations).

Solution Substitute for A and then evaluate each member of the equation:

$$\ln(\sqrt{79.6}) = \tfrac{1}{2} \ln(79.6)$$

Left member:

79.6 $\boxed{\sqrt{x}}$ $\boxed{\ln}$ $\Rightarrow 2.1885$

Right member:

$$79.6 \boxed{\ln} 2 \boxed{\div} \Rightarrow 2.1885$$

$$\therefore \quad \ln(\sqrt{79.6}) = \tfrac{1}{2}\ln(79.6)$$

In Table 20–1 you will find a summary of the properties of logarithms that we have studied. This table will help in solving logarithmic equations.

TABLE 20–1 Properties of Logarithms

Property	Common	Natural
Base	10	$e = 2.7183$
Logarithm of 1	$\log(1) = 0$	$\ln(1) = 0$
Logarithm of e	$\log(e) = 0.4343$	$\ln(e) = 1$
Logarithm of 10	$\log(10) = 1$	$\ln(10) = 2.3026$
Product	$\log(A \cdot B) = \log(A) + \log(B)$	$\ln(A \cdot B) = \ln(A) + \ln(B)$
Quotient	$\log(A/B) = \log(A) - \log(B)$	$\ln(A/B) = \ln(A) - \ln(B)$
Power	$\log(A^n) = n\log(A)$	$\ln(A^n) = n\ln(A)$
Radical	$\log(\sqrt{A}) = \frac{1}{2}\log(A)$	$\ln(\sqrt{A}) = \frac{1}{2}\ln(A)$
Antilogarithm	$\text{antilog}(x) = 10^x$	$\text{antiln}(x) = e^x$
Change of base	$\log(x) = \ln(x)/2.3026$	$\ln(x) = 2.3026\log(x)$

EXERCISE 20–8

Check the following equations:

1. $\ln(17 \cdot 3) = \ln(17) + \ln(3)$
2. $\ln(9 \cdot 5) = \ln(9) + \ln(5)$
3. $\ln(\frac{65}{4}) = \ln(65) - \ln(4)$
4. $\ln\left(\dfrac{0.695}{0.421}\right) = \ln(0.695) - \ln(0.421)$
5. $\ln(15^2) = 2\ln(15)$
6. $\ln(0.71^2) = 2\ln(0.71)$
7. $\ln(\sqrt{64}) = \frac{1}{2}\ln(64)$
8. $\ln(\sqrt{3.06}) = \frac{1}{2}\ln(3.06)$

Use natural logarithms, Rules 20–4 and 20–5, and Example 20–15 to calculate:

9. $\sqrt{5.41}$ **10.** $\sqrt{0.0792}$ **11.** 17.1^2 **12.** $(6.45)^{0.295}$

13. $(18.7)^{1/3}$ **14.** $(1.51)^{17}$ **15.** $\sqrt{71.3}$ **16.** $(1.01)^{144}$

20–9 LOGARITHMIC EQUATIONS

An equation involving the logarithm of the unknown is referred to as a **logarithmic equation.** Thus, $16 = 4 \log (x)$ is a logarithmic equation. On the other hand, $25x = x \log (14)$ is not a logarithmic equation. The key to solving a logarithmic equation is to treat the logarithm of the unknown as a new variable. Solve for this new variable and apply the antilogarithm to find the unknown.

EXAMPLE 20–25 Solve $16 = 4 \log (x) + 8$.

Solution First solve for $\log (x)$:

$$8 = 4 \log (x)$$
$$\log (x) = 2$$

Apply antilog to find x:

$$x = \text{antilog } (2) = 10^2$$
$$\therefore \quad x = 100$$

EXAMPLE 20–26 Solve $4.5 + \ln (x^2) = \ln (x)$.

Solution Use Table 20–1 to simplify $\ln (x^2)$:

$$4.5 + 2 \ln (x) = \ln (x)$$

Solve for $\ln (x)$:

$$\ln (x) = -4.5$$

Apply antiln to find x:

$$x = \text{antiln } (-4.5) = e^{-4.5} \quad \boxed{e^x}$$
$$x = 1.11 \times 10^{-2}$$
$$\therefore \quad x = 1.11 \times 10^{-2}$$

EXAMPLE 20–27 Solve $\ln (x) - 1 = \log (x)$.

Solution Convert $\ln (x)$ to $\log (x)$; then proceed:

$$2.3026 \log (x) - 1 = \log (x)$$
$$1.3026 \log (x) = 1$$
$$\log (x) = 1/1.3026 \quad \boxed{1/x}$$
$$\log (x) = 0.7677$$

Apply antilog to find x:

$$x = \text{antilog } (0.7677) = 10^{0.7677}$$

10^x $\quad x = 5.86$

$\therefore \quad x = 5.86$

EXERCISE 20–9

Solve the following equations.

1. $1.05 = \log (x^2)$
2. $17 + 2 \log (x) = 6$
3. $5 \log (x) - 3 = 2$
4. $7 \ln (x) = 2 \ln (x^3) + 2.71$
5. $4 \ln (\sqrt{x}) = -1.53$
6. $\ln (x/6) = 0.15$
7. $\log (4x) + \log (x) = 2.459$
8. $4 \log (x) - 3 = \ln (x^3)$
9. $5 \log (\sqrt{x}) = \ln (x) + 1.59$
10. $\log (x/4) + \log (x/7) = 0.361$
11. $\ln (17/V) = 4.5$
12. $\ln (13z) + \ln (z^2) = 9.021$
13. $\ln (V^2) - 3.29 = \ln (V)$
14. $\ln (Y^3) - 2 \ln (Y) = 2.3026$
15. $2.3026 \log (Z) = 1$
16. $\ln (t^2) + \ln (1/t) = 4.39$
17. The strength of the signal to the strength of the noise measured in decibels is defined by the equation $N_{dB} = 10 \log (P_S/P_N)$. Solve for P_N.
18. In a particular circuit, the signal-to-noise ratio was found to be 15. How many decibels is this? (See problem 17.)
19. The number of octaves between two frequencies is given by the formula $N = \log (f_2/f_1)/\log (2)$. Solve for f_2.
20. Find the frequency (f_2) 4.5 octaves above 60 Hz (see problem 19).

20–10 EXPONENTIAL EQUATIONS

An equation with the unknown appearing in an exponent is called an **exponential equation.** Thus, $90 = 120\, e^{-0.01/t}$ is an exponential equation. Exponential equations may be solved by first solving for the term containing the unknown exponent, and then using logarithms to find the unknown.

EXAMPLE 20–28 Solve $10^{2x} - 4 = 16$ for x.

Solution Solve for 10^{2x}; then use common logarithms:

$$10^{2x} = 20$$

Take the common log of both members of the equation:

$$2x = \log (20)$$

$$x = \log (20)/2$$

$$x = 1.3010/2$$

$$\therefore \quad x = 0.6505$$

Observation Since x is like a logarithm, we write it with four digits to the right of the decimal point.

EXAMPLE 20–29 Solve $90 = 120\, e^{-0.01/t}$ for t.

Solution Solve for $e^{-0.01/t}$; then use natural logarithms:

$$90 = 120\, e^{-0.01/t}$$

$$e^{-0.01/t} = \frac{90}{120} = 0.75$$

Take the natural logarithm of both members of the equation:

$$-0.01/t = \ln (0.75)$$

Solve for t:

$$t = -0.01/\ln (0.75)$$

$$t = -0.01/(-0.2877)$$

$$\therefore \quad t = 0.03476$$

EXAMPLE 20–30 Solve $f_2 - f_1 2^n = 0$ for n.

Solution Solve for 2^n; then use common logarithms to find n:

$$f_1 2^n = f_2$$

$$2^n = f_2/f_1$$

$$\log (2^n) = \log (f_2/f_1)$$

Simplify the left member:

$$n \log (2) = \log (f_2/f_1)$$

$$\therefore \quad n = \log (f_2/f_1)/\log (2)$$

Observation A similar solution could have been arrived at using natural logarithms. Both equations would give the same numerical value for n.

EXAMPLE 20–31 Evaluate n in the following equations when $f_1 = 330$ Hz and $f_2 = 2700$ Hz.

Solution Use common logarithms

$$n = \log (f_2/f_1)/\log (2)$$

$$n = \log (2700/300)/\log (2)$$

$$n = \log (8.18)/\log (2)$$

$$n = 0.9128/0.3010$$

$$n = 3.032$$

Use natural logarithms

$$n = \ln (f_2/f_1)/\ln (2)$$

$$n = \ln (2700/330)/\ln (2)$$

$$n = \ln (8.18)/\ln (2)$$

$$n = 2.1017/0.6931$$

$$n = 3.032$$

$\therefore$ The two equations give the same answer: $n = 3.032$

EXERCISE 20–10

Solve the following equations.

1. $14 = 10^x$
2. $3.5 = 10^x - 1$
3. $81 = 0.9\,(10^y)$
4. $0.92 = 3(10^\alpha) + 0.14$
5. $3(10^s) = 11 - 2(10^s)$
6. $18 = e^T$
7. $e^x - 4 = 0$
8. $1.52\,e^{0.1/y} = 40.3$
9. $3.4\,e^x = 0.015 + 1.5\,e^x$
10. $75\,e^x - 125 = 13 + 6\,e^x$
11. $e^{-4/t} = 0.8$
12. $e^{-15/t} = 0.5$
13. $e^{-1/(6t)} = 0.95$
14. $1 - e^{-7.4/t} = 0.26$
15. $1 + e^{8.6/t} = 3.54$
16. $2^x = 32$
17. $3^y = 243$
18. $5^t = 17.3$
19. $21^{1/a} = 0.21$
20. $15^{-1/b} = 0.95$

20–11 SEMILOG AND LOG–LOG PLOTS

Graphs are tools used to present mathematical data. As such, there is no one *right* way to draw a graph. The scales are dictated by use of the graph. Two scales useful in elec-

tronics are **semilog** and **log–log.** A semilog plot has one regular (linear) scale and one with logarithmic spacing. Figure 20–6 is an example of a semilog plot. In a log–log plot both scales have logarithmic spacing. Figure 20–7 is an example of a log–log plot.

In Figure 20–6 there are three curves. On rectangular (linear) graph paper, $y = 2x$ would be a straight line, but on semilog paper it is a curve. The graph of $y = x^2$ is a curve on both types of paper, but the curves do not look the same. The graph of $y = 2^x$ is a straight line on semilog paper. These last two curves show the usefulness of semilog plots. First, they compress large values (and expand values close to zero) so that more data may be presented on a single graph. Second, semilog graphs convert exponential curves into straight lines.

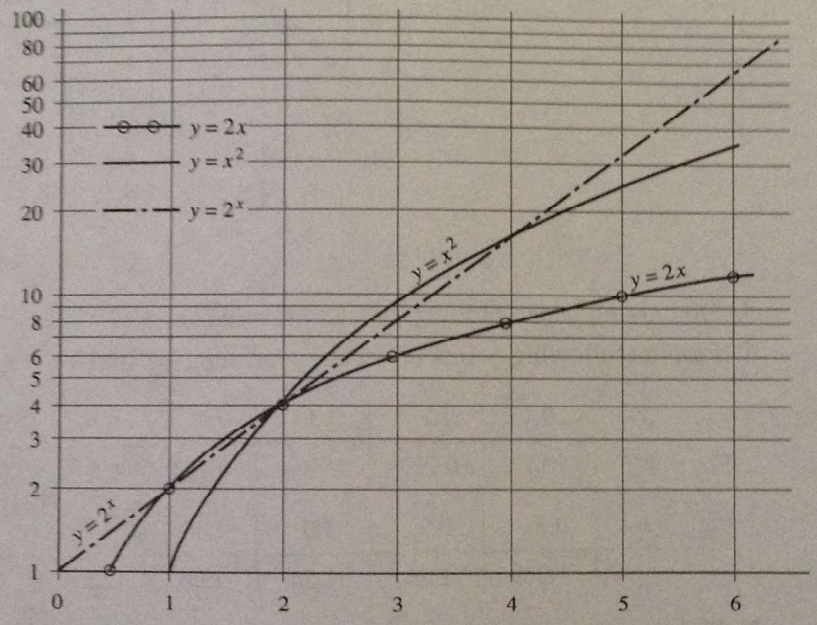


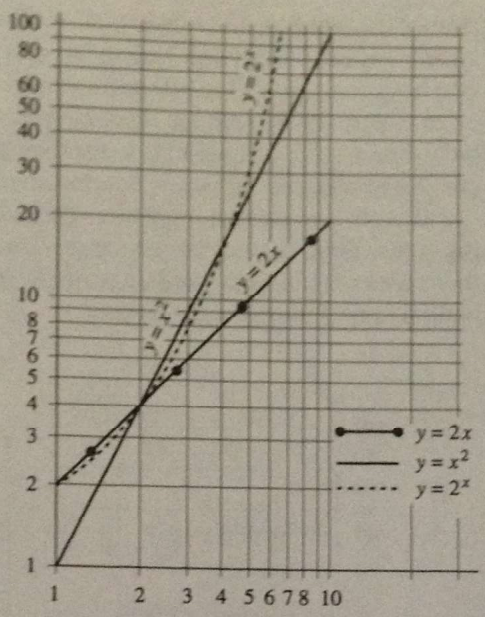
FIGURE 20–6 Examples of semilog plots.

In Figure 20–7, the same three curves are plotted on log–log paper. Here, $y = 2x$ and $y = x^2$ are straight lines, but $y = 2^x$ is a curve. Log–log plots are used to present wide ranges of data, especially data that have the form $y = ax^n$.

EXERCISE 20–11

1. Create a semilog plot of $y = e^x$.
2. Create a semilog plot of $y = 10^x$.
3. Create a log–log plot of $y = x^3$.
4. Create a log–log plot of $y = x^{1.5}$.

FIGURE 20–7 Examples of log–log plots.



5. On log–log paper, plot $y = \sqrt{x}$ and $y = \ln(x)$.
6. Plot the following points on rectangular, semilog, and log–log paper:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7
y	0.1	0.203	0.309	0.423	0.546	0.684	0.842

x	0.8	0.9	1.0	1.1	1.2	1.3	1.4
y	1.029	1.26	1.56	1.96	2.57	3.60	5.80

7. Plot the following data on semilog and log–log paper. Do they plot as a straight line?

x	0.5	1	1.5	2	2.5	3
y	0.5	2	4.5	8	12.5	18

8. Plot the following data on semilog and log–log paper. Do they plot as a straight line?

x	1	1.5	2	2.5	3	3.5	4	4.5	5
y	1.07	1.10	1.14	1.18	1.21	1.27	1.31	1.36	1.40

x	5.5	6	6.5	7	7.5	8	8.5	9	9.5	10
y	1.45	1.50	1.55	1.61	1.66	1.72	1.78	1.84	1.90	1.97

20–12 NOMOGRAPHS

In electronics and especially communications, it is often convenient to solve problems graphically. There are many graphical aides to solutions presented in handbooks and periodicals. These graphical aides are called **nomographs.** Figure 20–8 presents a simple nomograph consisting of a single straight line with different gradations on either side. To use this nomograph, find the power ratio on the top; directly under it is the corresponding number of decibels (dB). The reverse process will convert decibels to power ratio. For example, a power ratio of 3 corresponds to 4.8 decibels.

A second type of nomograph relates three variables. The formula for inductive reactance is $X_L = 2\pi fL$. A nomograph relating X_L, f, and L is presented in Figure 20–9. When two of the parameters are known, the third is found by placing a straight edge to join the two known values. The value of the third parameter is read from the appropriate scale under the straight edge.

EXAMPLE 20–32 Find the frequency if the inductance is 400 μH and the reactance is 2 kΩ.

Solution Use the nomograph in Figure 20–9. Place a mark at 400 μH and a second at 2 kΩ. Draw a straight line through these points. Read the frequency as 800 kHz.

Check Check the graphical solution:

$$X_L = 2\pi fL$$

Solve for f:

$$f = X_L/(2\pi L)$$

Substitute:

$$f = 2(10^3)/(2\pi 400 \times 10^{-6})$$

$$f = 796\ 000 \text{ Hz}$$

$$f = 796 \text{ kHz} \approx 800 \text{ kHz}$$

$\therefore$ The graphical solution checks.

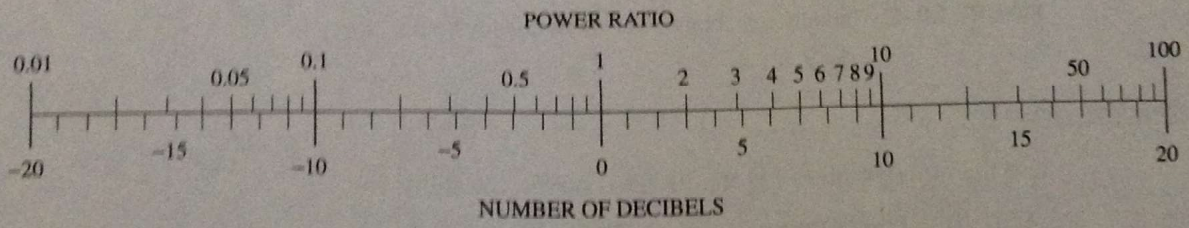


FIGURE 20–8 Nomograph for converting from power ratio to number of decibels.

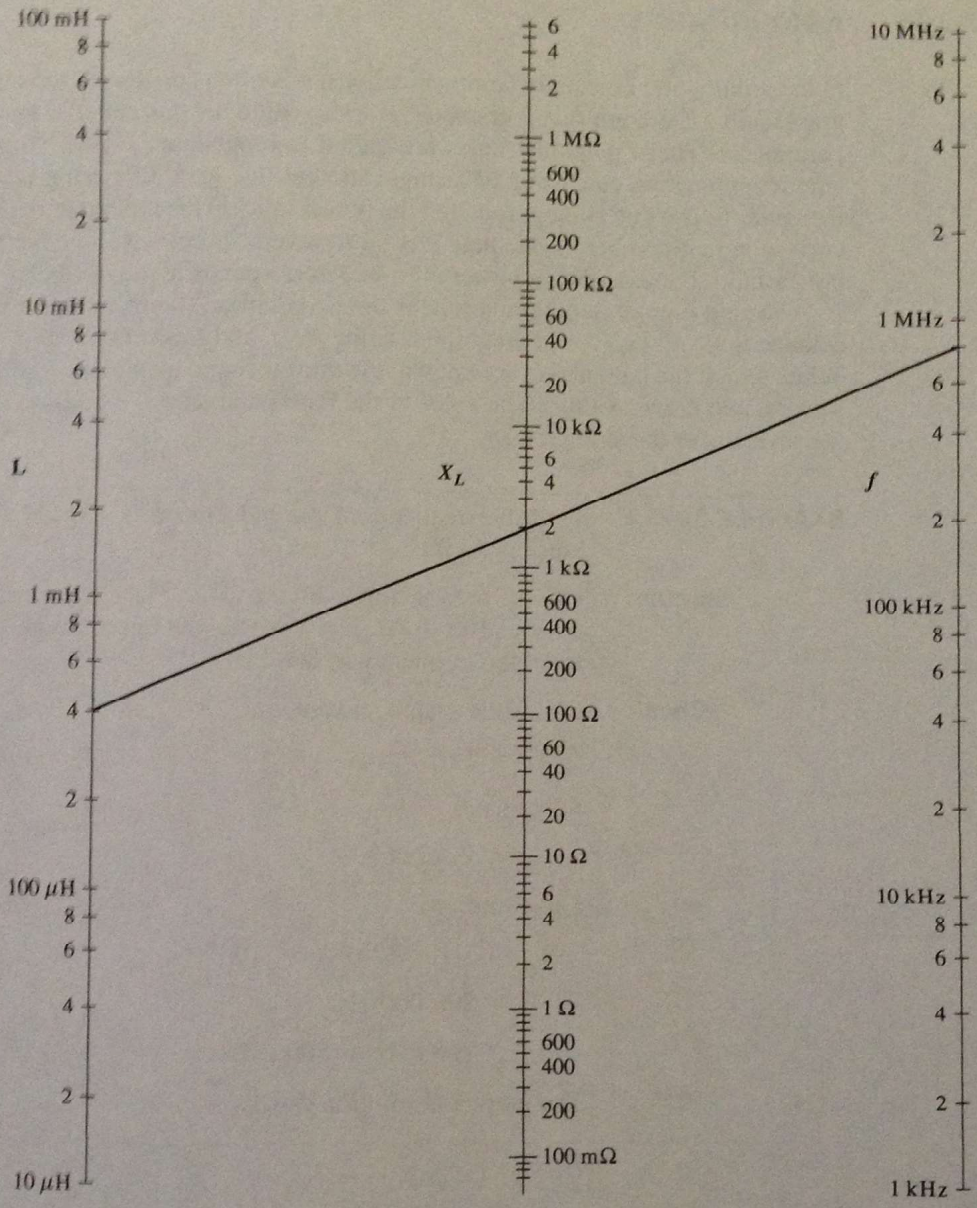


FIGURE 20–9 Inductive reactance versus frequency nomograph.

EXERCISE 20–12

Use the nomograph in Figure 20–8 to convert the following.

1. 6 dB to power ratio
2. 15 dB to power ratio
3. 0.02 to number of decibels
4. 20 to number of decibels

Use the nomograph in Figure 20–9 to find the third parameter.

5.	$L = 200\ \mu H$	$f = 1$ MHz	Find X_L.
6.	$L = 6$ mH	$f = 50$ kHz	Find X_L.
7.	$X_L = 60\ \Omega$	$f = 10$ kHz	Find L.
8.	$X_L = 10$ kΩ	$f = 10$ MHz	Find L.
9.	$X_L = 1$ MΩ	$L = 100$ mH	Find f.
10.	$L = 30$ mH	$X_L = 300\ \Omega$	Find f.

SELECTED TERMS

characteristic The integer part of a logarithm.

common antilogarithm The inverse of the logarithm function: when $a = \log(b)$ then $b = \text{antilog}\ (a)$.

common logarithm Base ten logarithm

logarithm The exponent to which the base must be raised to equal the original number.

mantissa The fractional part of a logarithm.

natural logarithm Base e (2.71828 . . .) logarithm.

19.1 LINEAR INEQUALITIES IN ONE VARIABLE

In Sec. 1.2 the inequality symbols $<$, $>$, $\leq$, and $\geq$ were introduced. In this section we shall deal with inequalities involving variables.

We begin with the notion of **intervals**, which are types of sets of real numbers that are defined in terms of inequality symbols. For example, the set of all real numbers x such that $3 \leq x \leq 5$ is called a **closed interval**. The term *closed* is used because the interval *includes* the numbers 3 and 5, which are called **endpoints** of the interval. We can denote this interval by writing $[3, 5]$, which is called **interval notation**. A bracket indicates that the endpoint *is included*. The set of all numbers x such that $a < x < b$ is called an **open interval** because the endpoints a and b are *not included*. In interval notation this set is written (a, b), where a parenthesis indicates that the endpoint is not included. Similarly, the interval $(1, 4]$, called a **half-open interval**, consists of all x such that $1 < x \leq 4$.

An interval is represented geometrically by darkening the portion of the number line that corresponds to the numbers (or points) in the interval (and by using parentheses and brackets as needed). In Fig. 19.1 the various types of intervals are shown. The infinity symbols ∞ and $-\infty$ that appear are not numbers but are merely convenient symbols for indicating that an interval extends indefinitely in some direction.

Inequalities that have their inequality symbols pointing in the same direction are said to have the *same sense*. If the symbols point in opposite directions, the

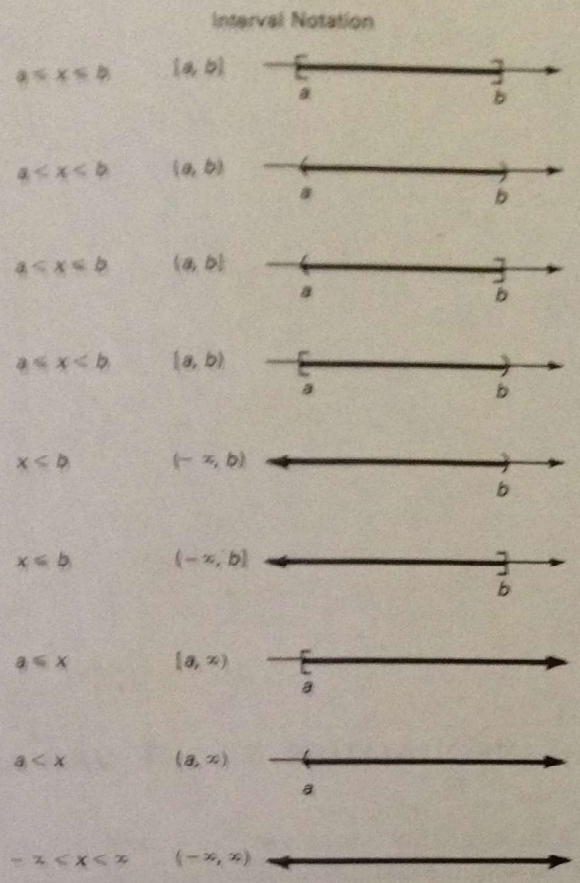


FIGURE 19.1

inequalities are said to be *opposite in sense*, or one is said to have the *reverse sense* of the other. Thus the inequalities $a < b$ and $c < d$ have the same sense, but $a < b$ and $c > d$ are opposite in sense.

As with equations, there are various types of inequalities. An **absolute inequality** is one that is true for *all* allowable values of the variables involved. A **conditional inequality** is one that is true for *some*, but not all, of the allowable values.

Example 1

a. $x^2 \ge 0$ is an absolute inequality because the square of *any* real number x is positive or zero.

b. $x^2 > 0$ is a conditional inequality because it is true if and only if x is not zero.

Solving an inequality, such as $2(x - 3) < 4$, means to find all values of the variable for which the inequality is true. This involves the application of certain rules which we now state.

Day: _____ Date: _____

1. **If the same number is added to or subtracted from both sides of an inequality, then the resulting inequality has the same sense as the original inequality.** Symbolically, [for the less than ($<$) relation],

$$\text{If } a < b, \text{ then } a + c < b + c \text{ and } a - c < b - c.$$

For example, since $7 < 10$, then $7 + 3 < 10 + 3$ or, simply, $10 < 13$.

2. **If both sides of an inequality are multiplied or divided by the same *positive* number, then the resulting inequality has the same sense as the original inequality.** Symbolically,

$$\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$

For example, since $3 < 7$ and $2 > 0$, then $3(2) < 7(2)$ and $\frac{3}{2} < \frac{7}{2}$.

3. **If both sides of an inequality are multiplied or divided by the same *negative* number, then the resulting inequality has the *reverse* sense of the original inequality.** Symbolically,

$$\text{If } a < b \text{ and } c > 0, \text{ then } a(-c) > b(-c) \text{ and } \frac{a}{-c} > \frac{b}{-c}.$$

For example, $4 < 7$ but $4(-2) > 7(-2)$ or, more simply, $-8 > -14$. Also, $\frac{4}{-2} > \frac{7}{-2}$.

4. **If the sides of an inequality are either both positive or both negative, then their respective reciprocals are unequal in the *reverse* sense.** For example,

$$2 < 4 \quad \text{but} \quad \tfrac{1}{2} \not< \tfrac{1}{4}.$$

5. **If both sides of an inequality are positive and we raise each side to the same positive power, then the resulting inequality has the same sense as the original inequality.** Symbolically,

$$\text{If } a > b > 0 \text{ and } n > 0, \text{ then } a^n > b^n \text{ and } \sqrt[n]{a} > \sqrt[n]{b}.$$

For example, $9 > 4$, so $9^2 > 4^2$ and $\sqrt{9} > \sqrt{4}$.

Whenever one of Rules 1–3 is applied to an inequality, the resulting inequality is equivalent to the given one. (**Equivalent inequalities**, like equivalent equations, have the same solutions.) Thus to solve an inequality, we apply Rules 1–3 until the values of the variable are obvious. We shall apply these rules to solve a *linear inequality*.

A **linear inequality** in the variable x is an inequality that can be written in the form

$$ax + b < 0 \qquad (\text{or } >, \leq, \geq),$$

where a and b are constants and $a \neq 0$.

In the following examples of solving linear inequalities, the given inequality will be replaced by equivalent inequalities until we isolate the variable on one side of the inequality. Whereas a linear *equation* has exactly one solution, you will see that a linear inequality has infinitely many solutions.

Example 2

Solve $2(x - 3) < 4$.

Solution

$$\begin{aligned}
2(x-3) &< 4 && \\
2x - 6 &< 4 && \text{[distributive law]} \\
2x - 6 + 6 &< 4 + 6 && \text{[adding 6 to both sides (Rule 1)]} \\
2x &< 10 && \text{[simplifying]} \\
\frac{2x}{2} &< \frac{10}{2} && \text{[dividing both sides by 2 (Rule 2)]} \\
x &< 5 && \text{[simplifying].}
\end{aligned}$$

Thus all real numbers less than 5 are solutions. We shall write our solution as $x < 5$ [in interval notation the solution is $(-\infty, 5)$]. For example, 1 is less than 5, so 1 is a solution. To check this we have

$$\begin{aligned}
2(1-3) &< 4 \\
2(-2) &< 4 \\
-4 &< 4.
\end{aligned}$$

Figure 19.2 shows the solution.

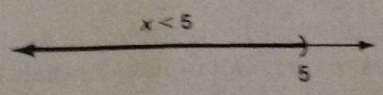


FIGURE 19.2

Example 3

Solve $3 - 2x \leq 6$.

Solution

$$3 - 2x \leq 6$$

$$-2x \leq 3 \qquad \text{[subtracting 3 from both sides (Rule 1)]}$$

$$x \geq -\frac{3}{2} \qquad \text{[dividing both sides by } -2 \text{ and changing direction of inequality (Rule 3)].}$$

The solution is $x \geq -\frac{3}{2}$. See Fig. 19.3.

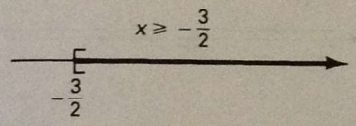


FIGURE 19.3

Example 4

Solve $\frac{3}{2}(s - 2) + 1 > -2(s - 4)$.

Solution To clear fractions, we multiply both sides by 2.

$$2[\tfrac{3}{2}(s - 2) + 1] > 2[-2(s - 4)] \qquad \text{[Rule 2]}$$

$$3(s - 2) + 2 > -4(s - 4)$$

$$3s - 4 > -4s + 16$$

$$7s - 4 > 16 \qquad \text{[adding } 4s \text{ to both sides (Rule 1)]}$$

$$7s > 20 \qquad \text{[adding 4 to both sides (Rule 1)]}$$

$$s > \frac{20}{7} \qquad \text{[dividing both sides by 7 (Rule 2)].}$$

See Fig. 19.4

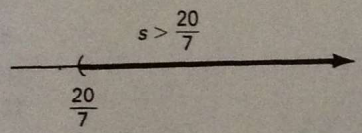


FIGURE 19.4

Example 5

Solve $2(x-4)-3>2x+1$.

Solution

$$\begin{aligned} 2(x-4)-3 &> 2x+1 \\ 2x-8-3 &> 2x+1 \\ -11 &> 1 \qquad \text{[subtracting } 2x \text{ from both sides]}. \end{aligned}$$

The inequality $-11>1$ is never true, so there is *no solution.*

Example 6

Solve $3z+7>8z-(5z-2)$.

Solution

$$\begin{aligned} 3z+7 &> 8z-(5z-2) \\ 3z+7 &> 3z+2 \\ 7 &> 2 \qquad \text{[subtracting } 3z \text{ from both sides]}. \end{aligned}$$

Since the inequality $7>2$ is always true (true for any value of z), the solution consists of *all real numbers.* We write this solution as $-\infty<z<\infty$. See Fig. 19.5.

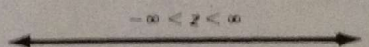


FIGURE 19.5

Problem Set 19.1

In Problems **1–34**, *solve the inequalities and indicate your answers geometrically on the real number line.*

1. $3x>12$.
2. $4x<-2$.
3. $4x-13\le 7$.
4. $3x\ge 0$.
5. $-4x\ge 2$.
6. $2y+1>0$.
7. $3-5s>5$.
8. $4s-1<-5$.
9. $3<2y+3$.
10. $6\le 5-3y$.
11. $2x-3\le 4+7x$.
12. $-3\ge 8(2-x)$.
13. $3(2-3x)>4(1-4x)$.
14. $8(x+1)+1<3(2x)+1$.
15. $2(3x-2)>3(2x-1)$.
16. $3-2(x-1)\le 2(4+x)$.
17. $x+2<\sqrt{3}-x$.
18. $\sqrt{2}(x+2)>\sqrt{8}(3-x)$.

19. $\frac{5}{3}x < 10.$

20. $-\frac{1}{2}x > 6.$

21. $\frac{9y+1}{4} \le 2y - 1.$

22. $\frac{4y-3}{2} \ge \frac{1}{3}.$

23. $4x - 1 \ge 4(x - 2) + 7.$

24. $0x \le 0.$

25. $\frac{1-t}{2} < \frac{3t-7}{3}.$

26. $\frac{3(2t-2)}{2} > \frac{6t-3}{5} + \frac{t}{10}.$

27. $2x + 3 \ge \frac{1}{2}x - 4.$

28. $4x - \frac{1}{2} \le \frac{3}{2}x.$

29. $\frac{2}{3}r < \frac{5}{6}r.$

30. $\frac{7}{4}t > -\frac{2}{3}t.$

31. $\frac{y}{2} + \frac{y}{3} > y + \frac{y}{5}.$

32. $9 - 0.1x \le \frac{2 - 0.01x}{0.2}.$

33. $0.1(0.03x + 4) \ge 0.02x + 0.434.$

34. $\frac{5y-1}{-3} < \frac{7(y+1)}{-2}.$

35. The amount of current needed for a certain electrical appliance is more than 7 A but less than 9.5 A. If 12 such appliances are to be used and I represents the total number of amperes needed, describe I by using inequalities.

36. Using inequalities, symbolize the following statement: The number of working hours x to produce a blueprint is at least 15 but not more than 18.

37. If a block slides down a rough inclined plane, it can be deduced from the principle of conservation of energy that the gravitational potential energy P of the block at the top of the inclined plane is greater than the kinetic energy K of the block at the bottom of the plane. If $K = 36$ J, geometrically indicate the possible values of P on the real number line.

19.2 NONLINEAR INEQUALITIES

Let us consider how we would go about solving a nonlinear inequality such as

$$x^2 - x - 2 > 0.$$

One method is to graph the equation $y = f(x) = x^2 - x - 2$ and observe when y is greater than 0. In Fig. 19.6 we see that $y > 0$ if $x < -1$ or $x > 2$. That is, the

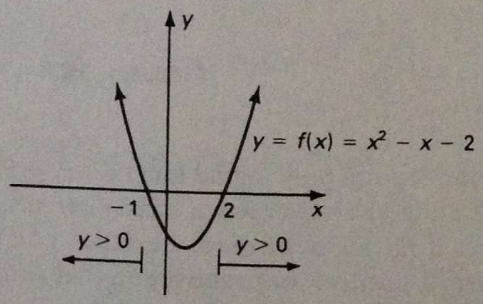


FIGURE 19.6

solution of $x^2 - x - 2 > 0$ is $x < -1$ or $x > 2$. Although this approach is easy to understand, the accuracy of the graphical technique is too limited. A more accurate method is needed.

Consider again the inequality $x^2 - x - 2 > 0$. To solve it another way, we first find the roots of the corresponding *equation* $x^2 - x - 2 = 0$. Factoring gives

$$(x + 1)(x - 2) = 0,$$

so the roots are -1 and 2, which are marked on a number line in Fig. 19.7.

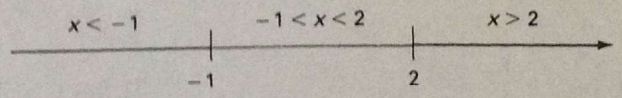


FIGURE 19.7

These two roots determine three open intervals (also shown in Fig. 19.7):

$$(-\infty, -1), \quad (-1, 2), \quad \text{and} \quad (2, \infty).$$

Referring back to Fig. 19.6, we see that if x is *any* number in the interval $(-\infty, -1)$, then $f(x) > 0$. If x is *any* number in the interval $(-1, 2)$, then $f(x) < 0$, and so on. That is, given any of these three intervals, $f(x)$ does not change sign *throughout* that interval. This means that to determine the sign of $f(x)$ on one of these intervals, it is sufficient to determine its sign at *any* point in that interval. For instance, -2 is in $(-\infty, -1)$ and $f(-2) = 4 > 0$. Thus $f(x) > 0$ on the *entire* interval $(-\infty, -1)$. Because zero is in $(-1, 2)$ and $f(0) = -2 < 0$, then $f(x) < 0$ on the *entire* interval $(-1, 2)$. Similarly, 3 is in $(2, \infty)$ and $f(3) = 4 > 0$, so $f(x) > 0$ on the *entire* interval $(2, \infty)$. A summary of our results is shown in Fig. 19.8. Therefore, from these results we conclude that $x^2 - x - 2 > 0$ for $x < -1$ or for $x > 2$, so we have solved the inequality. [In interval notation we can express the solution as $(-\infty, -1)$ and $(2, \infty)$].

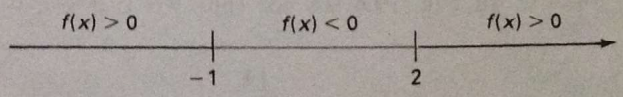


FIGURE 19.8

In general, for any polynomial function $P = P(x)$, the (real) zeros of P give rise to certain open intervals. On each of these intervals, $P(x)$ is always positive or always negative. Thus if $P(x)$ is positive (or negative) at a single point in one such interval, then $P(x)$ must be positive (or negative) on that *entire* interval. This fact allows us to solve inequalities without the need for graphing.

Example 1

Solve $x^2 - 4x - 12 < 0$.

Solution Let $f(x) = x^2 - 4x - 12$. First we factor $f(x)$.

$$f(x) = (x + 2)(x - 6).$$

Next we find the roots of the equation $f(x) = 0$.

$$\text{Roots of } (x + 2)(x - 6) = 0 \text{ are } -2 \text{ and } 6.$$

The roots are marked in Fig. 19.9.

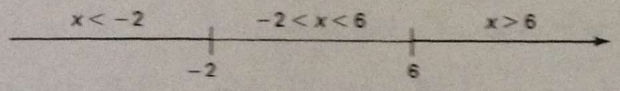


FIGURE 19.9

These two roots determine three open intervals:

$$(-\infty, -2), \quad (-2, 6) \quad \text{and} \quad (6, \infty).$$

On each interval, $f(x)$ [which is the product $(x + 2)(x - 6)$] must be always positive or always negative. We now test a value of x in each of them. For $-\infty < x < -2$, let us choose $x = -8$. Then the sign of $f(x)$ on $(-\infty, -2)$ is the same as the sign of $f(-8)$.

$$f(x) = (x + 2)(x - 6)$$
$$f(-8) = (-8 + 2)(-8 - 6)$$
$$= (-)(-) = +.$$

Note that it is not necessary actually to evaluate $f(-8)$ to find its sign. Knowing the signs of the factors allows us to find the sign of the product. Here $f(-8)$ is positive because there is an even number of negative factors. Thus $f(x) > 0$ for $x < -2$. For the other intervals we find that

$$f(0) = (+)(-) = -, \quad \text{so } f(x) < 0 \quad \text{for } -2 < x < 6,$$
$$f(10) = (+)(+) = +, \quad \text{so } f(x) > 0 \quad \text{for } x > 6.$$

A summary of our results is given by the *sign chart* in Fig. 19.10. Thus the solution of $x^2 - 4x - 12 < 0$ is $-2 < x < 6$ [or $(-2, 6)$] because that is the only one of the three intervals on which $f(x) < 0$.

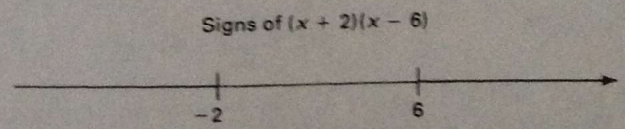


FIGURE 19.10

Example 2

Solve $x(x-1)(x+4) \le 0$.

Solution If $f(x) = x(x-1)(x+4)$, then the roots of the equation $f(x) = 0$ are 0, 1, and -4, which are shown in Fig. 19.11.

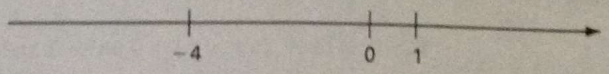


FIGURE 19.11

These three roots determine four intervals:

$$(-\infty, -4), \quad (-4, 0), \quad (0, 1) \quad \text{and} \quad (1, \infty).$$

Because -5 is in $(-\infty, -4)$, the sign of $f(x)$ on $(-\infty, -4)$ is the same as that of $f(-5)$. Given

$$f(x) = x(x-1)(x+4),$$

then

$$f(-5) = -5(-5-1)(-5+4) = (-)(-)(-) = -,$$

so $f(x) < 0$ for $x < -4$. For the other intervals we find that

$$f(-2) = (-)(-)(+) = +, \quad \text{so } f(x) > 0 \quad \text{for } -4 < x < 0,$$
$$f(\tfrac{1}{2}) = (+)(-)(+) = -, \quad \text{so } f(x) < 0 \quad \text{for } 0 < x < 1,$$
$$f(2) = (+)(+)(+) = +, \quad \text{so } f(x) > 0 \quad \text{for } x > 1.$$

Our results are summarized by the sign chart in Fig. 19.12. Thus $x(x-1)(x+4) < 0$ for

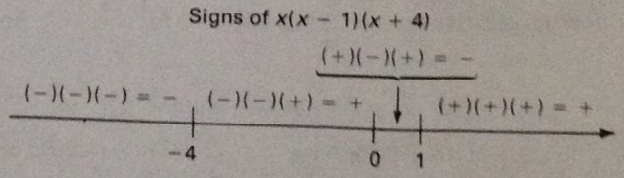


FIGURE 19.12

$x < -4$ or for $0 < x < 1$. To take care of the equality $(=)$ part of the inequality $(\le)$, we solve the equation

$$x(x-1)(x+4) = 0.$$

Since the roots are 0, 1, and -4, our solution to the inequality must include these numbers. Thus the solution consists of the intervals $(-\infty, -4]$ and $[0, 1]$; that is,

$$x \le -4 \quad \text{or} \quad 0 \le x \le 1.$$

Example 3

Solve $\dfrac{x^2 - 6x + 5}{x} \geq 0.$

Solution Let

$$f(x) = \frac{x^2 - 6x + 5}{x} = \frac{(x-1)(x-5)}{x}.$$

For a quotient we solve the inequality by considering the intervals determined by the roots of $f(x) = 0$, namely, 1 and 5, and those values of x for which f is undefined. Here f is undefined when $x = 0$. In Fig. 19.13 we have marked the points 0, 1, and 5. We thus consider the intervals

$$(-\infty, 0), \quad (0, 1), \quad (1, 5), \quad \text{and} \quad (5, \infty).$$

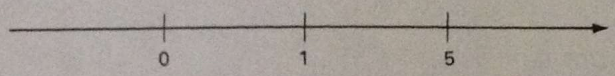


FIGURE 19.13

Determining the sign of $f(x)$ at a point in each interval, we find that

$$f(-1) = \frac{(-)(-)}{(-)} = -, \quad \text{so } f(x) < 0 \quad \text{for } x < 0,$$

$$f\left(\frac{1}{2}\right) = \frac{(-)(-)}{(+)} = +, \quad \text{so } f(x) > 0 \quad \text{for } 0 < x < 1,$$

$$f(2) = \frac{(+)(-)}{(+)} = -, \quad \text{so } f(x) < 0 \quad \text{for } 1 < x < 5,$$

$$f(6) = \frac{(+)(+)}{(+)} = + \quad \text{so } f(x) > 0 \quad \text{for } x > 5.$$

The sign chart is given in Fig. 19.14. Therefore, $f(x) \geq 0$ for $0 < x \leq 1$ or $x \geq 5$ (see Fig. 19.15). Why are 1 and 5 included, but zero excluded?

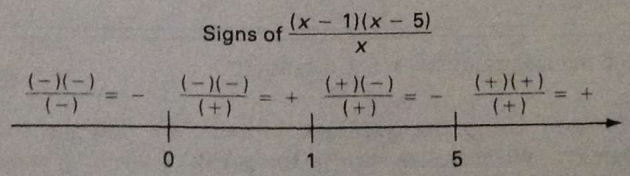


FIGURE 19.14

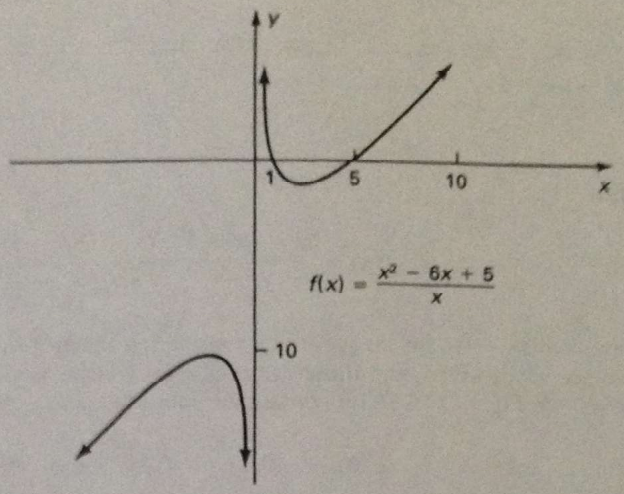


FIGURE 19.15

Example 4

Solve $t^3 + 16t > 8t^2$.

Solution We first rewrite the inequality so that one side is zero and then we factor.

$$t^3 - 8t^2 + 16t > 0,$$
$$t(t^2 - 8t + 16) > 0,$$
$$t(t-4)^2 > 0.$$

The roots of $t(t-4)^2 = 0$ are 0 and 4. From the sign chart in Fig. 19.16, we conclude that the solution is $0 < t < 4$ or $t > 4$.

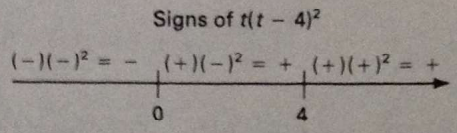


FIGURE 19.16

Example 5

a. Solve the inequality $x^2 + x + 1 > 0$.

Solution Using the quadratic formula, we find the roots of $x^2 + x + 1 = 0$ to be imaginary. We therefore turn to the graph of $y = x^2 + x + 1$ for any information it may yield (Fig. 19.17). Because for all x we have $y > 0$, the solution of $x^2 + x + 1 > 0$ is $-\infty < x < \infty$.

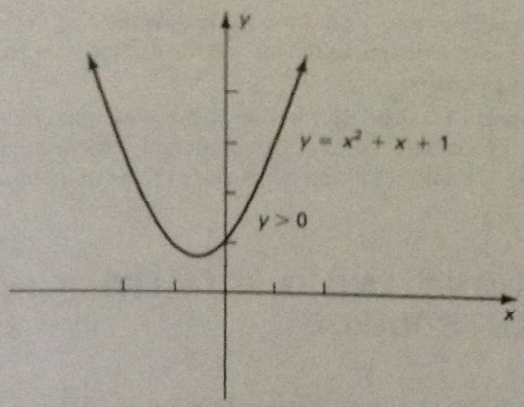


FIGURE 19.17

b. Solve $x^2 + x + 1 < 0$.

Solution From Part a, the expression $x^2 + x + 1$ is always positive. Thus the inequality $x^2 + x + 1 < 0$ has no solution.

Problem Set 19.2

In Problems **1–38**, *solve the inequalities.*

1. $(x - 1)(x - 5) < 0$.
2. $(x + 3)(x - 8) > 0$.
3. $(x + 1)(x - 3) > 0$.
4. $(x + 4)(x + 5) < 0$.
5. $x^2 - 1 < 0$.
6. $x^2 > 9$.
7. $x^2 - x - 6 > 0$.
8. $x^2 - 2x - 3 \le 0$.
9. $5s - s^2 \le 0$.
10. $t^2 + 9t + 18 \ge 0$.
11. $x^2 + 5x < -6$.
12. $x^2 - 12 > x$.
13. $x^2 + 4x - 5 \ge 3x + 15$.
14. $x^2 + 9x + 9 \le 2 - x^2$.
15. $2z^2 - 5z - 12 < 0$.
16. $5t^2 - 1 > 4t$.
17. $x^2 + 2x + 1 > 0$.
18. $x^2 + 9 \le 6x$.
19. $4(t^2 + t) - 1 \ge 2$.
20. $3s^2 \le 11s - 10$.
21. $(x + 2)(x - 1)(x - 4) > 0$.
22. $x(x + 3)(5 - x) < 0$.
23. $(x + 5)(x - 3)^2 < 0$.
24. $(x + 1)^2(x - 4)^2 > 0$.
25. $y^3 - y \le 0$.
26. $x^3 + 8x^2 + 15x \ge 0$.
27. $x^3 - 2x^2 \ge 0$.
28. $p^4 - 2p^3 - 3p^2 \le 0$.
29. $\dfrac{x - 4}{x + 8} > 0$.
30. $\dfrac{x + 3}{x + 5} < 0$.
31. $\dfrac{t}{3 - t} \le 0$.
32. $\dfrac{s - 1}{2s - 1} > 0$.
33. $\dfrac{5}{x + 2} > 0$.
34. $\dfrac{3}{9(x - 8)} < 0$.
35. $\dfrac{x + 3}{x^2 - 1} < 0$.
36. $\dfrac{x^2 - x - 6}{x + 4} > 0$.
37. $\dfrac{x^2 - 5x + 4}{x^2 + 5x + 4} \ge 0$.
38. $\dfrac{x^2 - 4}{x^2 + 2x + 1} \le 0$.

39. A company owns a 1 km by 2 km rectangular lot and needs to cut a uniform strip of trees along the outer edges of the lot. At most, how wide can the strip be if at least $\frac{3}{4}$ km^2 of the lot must remain?

40. A container manufacturer wishes to make an open box by cutting a 4-cm square from each corner of a square sheet of aluminum and then turning up the sides. The box is to contain at least 324 cm^3. Find the dimensions of the smallest sheet of aluminum that can be used.

19.3 EQUATIONS AND INEQUALITIES INVOLVING ABSOLUTE VALUE

Recall from Sec. 1.2 that the absolute value of a real number is its distance from zero on the real number line. We also gave an algebraic definition in Sec. 1.4:

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

In this definition, note that $-x$ is a positive number when x is negative. The next example shows how to solve some equations involving absolute value. In general, $|x| = d$ means $x = \pm d$, where $d > 0$.

Example 1

a. Solve $|x - 4| = 3$.

Solution This equation states that $x - 4$ is a number three units from zero. Thus either

$$x - 4 = 3 \quad \text{or} \quad x - 4 = -3.$$

If $x - 4 = 3$, then $x = 7$; if $x - 4 = -3$, then $x = 1$. Thus $x = 1, 7$.

b. Solve $|8 - 5x| = 6$.

Solution The equation is true if $8 - 5x = 6$ or if $8 - 5x = -6$. Solving these gives $x = \frac{2}{5}$ and $x = \frac{14}{5}$.

c. Solve $|x - 5| = -2$.

Solution Because the absolute value of a number is never negative, there is no solution.

The numbers 5 and 9 are 4 units apart. Also,

$$|9 - 5| = |4| = 4,$$
$$|5 - 9| = |-4| = 4.$$

In general, we may intepret $|a - b|$ or $|b - a|$ as the *distance* between a and b. For example, to solve the equation $|x - 4| = 3$ in Part a of Example 1 means to find all numbers x that are exactly 3 units from 4. Thus x can be 1 or 7. We also remark that $|a - b| = |b - a|$.

Let us turn now to inequalities. For example, if $|x| < 3$, then x is less than 3 units from 0. Thus x must lie between -3 and 3. That is, $-3 < x < 3$ [Fig. 19.18(a)]. However, if $|x| > 3$, then x must be more than 3 units from 0. Thus, one of two things must be true: Either $x > 3$ *or* $x < -3$ [Fig. 19.18(b)]. We can extend these ideas. If $|x| \leq 3$, then $-3 \leq x \leq 3$. If $|x| \geq 3$, then $x \geq 3$ or $x \leq -3$.

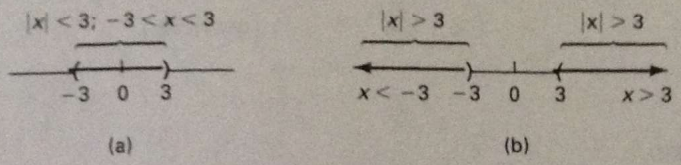


FIGURE 19.18

In general, the solutions of $|x| < d$ or $|x| \leq d$, where d is a positive number, consist of one interval. However, when $|x| > d$ or $|x| \geq d$, there are two intervals in the solution. More precisely we have the following rules.

ABSOLUTE-VALUE INEQUALITIES

If $d > 0$, then

1.	$\|x\| < d$	means	$-d < x < d$;
	$\|x\| \leq d$	means	$-d \leq x \leq d$.
2.	$\|x\| > d$	means	$x < -d$ or $x > d$;
	$\|x\| \geq d$	means	$x \leq -d$ or $x \geq d$.

Example 2

Solve the following absolute value inequalities.

a. $|x - 2| < 4$.

Solution The number $x - 2$ must be less than 4 units from 0. By Rule 1, this means that $x - 2$ lies between -4 and 4. Thus $-4 < x - 2 < 4$. We may set up the procedure for solving this inequality as follows:

$$-4 < x - 2 < 4$$

$$-4 + 2 < x < 4 + 2 \qquad \text{[adding 2 to each member]}$$

$$-2 < x < 6.$$

Day: Date:

b. $|3 - 2x| \leq 5$.

Solution

$$-5 \leq 3 - 2x \leq 5 \qquad \text{[Rule 1]}$$

$$-5 - 3 \leq -2x \leq 5 - 3 \qquad \text{[subtracting 3 from each member]}$$

$$-8 \leq -2x \leq 2,$$

$$4 \geq x \geq -1 \qquad \text{[dividing each member by } -2 \text{ and changing the direction of inequalities]}$$

$$-1 \leq x \leq 4 \qquad \text{[rewriting].}$$

Example 3

Solve the following inequalities.

a. $|x + 5| \geq 7$.

Solution The number $x + 5$ must be *at least* 7 units from 0. By Rule 2, either $x + 5 \geq 7$ *or* $x + 5 \leq -7$. Solving these inequalities, we have

$$x \geq 2 \quad \text{or} \quad x \leq -12.$$

b. $|3x - 4| > 1$.

Solution By Rule 2, either $3x - 4 > 1$ or $3x - 4 < -1$. Thus either $3x > 5$ or $3x < 3$. The solution is

$$x > \frac{5}{3} \quad \text{or} \quad x < 1.$$

Example 4

Use absolute value notation to express each statement.

a. x is less than four units from 7.

Solution Since the distance between x and 7 is less than 4, we must have

$$|x - 7| < 4.$$

b. x differs from 2 by at least 3.

Solution

$$|x - 2| \geq 3.$$

c. $x < 9$ and $x > -9$ simultaneously.

Solution

$$|x| < 9.$$

d. x is more than two units from -3.

Solution

$$|x - (-3)| > 2,$$
$$|x + 3| > 2.$$

e. x is less than σ (sigma) units from μ (mu).

Solution

$$|x - \mu| < \sigma.$$

The following are four basic properties of absolute value:

BASIC PROPERTIES OF ABSOLUTE VALUE

1. $|ab| = |a| \cdot |b|$.
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.
3. $|a - b| = |b - a|$.
4. $|a + b| \leq |a| + |b|$.

Example 5

a. $|(-7) \cdot 3| = |-7| \cdot |3| = 21$; $|(-7)(-3)| = |-7| \cdot |-3| = 21$.

b. $|4 - 2| = |2 - 4| = 2$.

c. $|7 - x| = |x - 7|$.

d. $\left|\frac{-7}{3}\right| = \frac{|-7|}{|3|} = \frac{7}{3}$; $\left|\frac{-7}{-3}\right| = \frac{|-7|}{|-3|} = \frac{7}{3}$.

e. $\left|\frac{x-3}{-5}\right| = \frac{|x-3|}{|-5|} = \frac{|x-3|}{5}$.

f. $|2 + 3| = 5 = |2| + |3|$, and $|2 + (-3)| = 1 < |2| + |-3|$.

Problem Set 19.3

In Problems **1–30**, *solve the given equation or inequality.*

1. $|x| = 6$.
2. $|-x| = 3$.
3. $\left|\frac{x}{3}\right| = 2$.
4. $\left|\frac{4}{x}\right| = 8$.
5. $|x - 5| = 8$.
6. $|4 + 3x| = 2$.
7. $|5x - 2| = 0$.
8. $|7x + 3| = x$.
9. $|7 - 4x| = 5$.
10. $|1 - 2x| = 1$.
11. $|x| < 3$.
12. $|x| < 10$.
13. $|x| > 6$.
14. $|x| > 3$.
15. $|2x| \leq 2$.
16. $|4x| \leq 3$.
17. $|3x| \geq 9$.
18. $\left|\frac{x}{2}\right| \geq 3$.
19. $|x - 4| < 16$.
20. $|y + 5| \leq 6$.
21. $|y + 1| \geq 6$.
22. $|x - 2| \geq 4$.
23. $|3x - 5| \leq 1$.
24. $\left|\frac{x}{3} - 5\right| < 4$.
25. $|1 - 3x| < 2$.
26. $|4x - 1| > 7$.
27. $|\frac{1}{2} - t| > \frac{1}{2}$.
28. $|5 - 2x| > 1$.
29. $\left|\frac{3x - 8}{2}\right| \geq 4$.
30. $\left|\frac{x - 8}{4}\right| \leq 2$.
31. Why does $|x - 5| < -12$ have no solution?
32. Why does every value of x satisfy $|2x + 1| > -1$?
33. Using the absolute value symbol, express each fact.
    (a) x is less than three units from 7.
    (b) x differs from 2 by less than 3.
    (c) x is no more than five units from 7.
    (d) The distance between 7 and x is 4.
    (e) $x + 4$ is less than two units from the origin.
    (f) x is between -3 and 3 but is not equal to 3 or -3.
    (g) $x < -6$ or $x > 6$.
    (h) $x - 6 > 4$ or $x - 6 < -4$.
    (i) The number of hours, x, that a machine will operate efficiently differs from 105 by less than 3.
34. Show that if $|x - \mu| \leq 2\sigma$, then $-2\sigma + \mu \leq x \leq 2\sigma + \mu$.
35. In the manufacture of a certain machine, the average dimension of a part is 0.01 cm. Using the absolute value symbol, express the fact that an individual measurement x of a part does not differ from the average by more than 0.005 cm.

19.4 LINEAR INEQUALITIES IN TWO VARIABLES

Suppose that a chemical manufacturer has 60 kg of a certain chemical element and uses it all to manufacture x units of compound A and y units of compound B. If each unit of A requires 2 kg of the element and each unit of B requires 3 kg, then the possible combinations of A and B that can be produced must satisfy the equation

$$2x + 3y = 60, \qquad \text{where } x, y \geq 0.$$

The solution is represented by the line segment in Fig. 19.19. For example, if 15 units of A are made, these require 30 kg of the element. Since 60 kg of the element must be used, 10 kg of B must be made (these require 30 kg of the element). Thus the point (15, 10) is on the line.

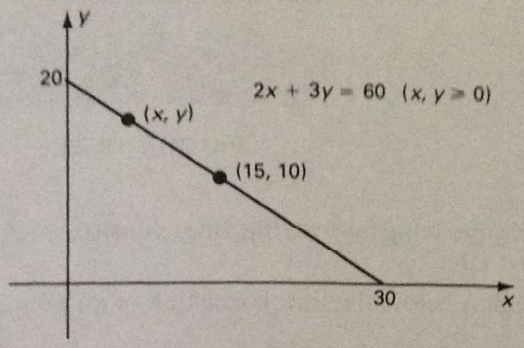


FIGURE 19.19

On the other hand, suppose the manufacturer does not necessarily wish to use the *total* supply of the element. In this case, the possible combinations are described by the inequality

$$2x + 3y \leq 60, \qquad \text{where } x, y \geq 0. \tag{1}$$

When linear inequalities in *one* variable were discussed in Sec. 19.1, their solutions were represented geometrically by *intervals* on the real number line. However, for an inequality in *two* variables, as in Inequality 1, the solution is usually represented by a *region* in the coordinate plane. We shall find the region corresponding to (1) after considering such inequalities in general.

> **A linear inequality** in the variables x and y is an inequality that can be written in the form
>
> $$ax + by + c < 0 \qquad (\text{or } \leq 0, \geq 0, > 0),$$
>
> where a, b, and c are constants and a and b are not both zero.

Geometrically, the solution of a linear inequality in x and y consists of all points in the plane whose coordinates satisfy the inequality. In particular, the graph of a nonvertical line $y = mx + b$ separates the plane into three distinct parts (see Fig. 19.20).

1. The line itself, consisting of all points (x, y) with coordinates that satisfy $y = mx + b$.

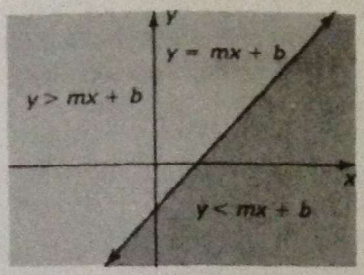


FIGURE 19.20

2. The region lying above the line, consisting of all points (x, y) that satisfy $y > mx + b$.
3. The region below the line, consisting of all points (x, y) satisfying $y < mx + b$.

For a vertical line $x = a$, we speak of regions to the right ($x > a$) or the left ($x < a$) of the line (see Fig. 19.21).

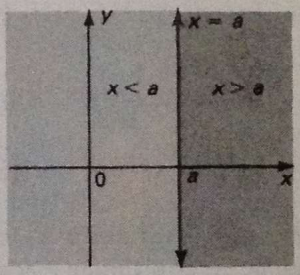


FIGURE 19.21

To apply these facts, we shall solve $x + y < 4$. We first sketch the corresponding *line* $x + y = 4$ by choosing two points on it, for instance the intercepts $(4, 0)$ and $(0, 4)$ (see Fig. 19.22). By writing the inequality in the equivalent form $y < -x + 4$,

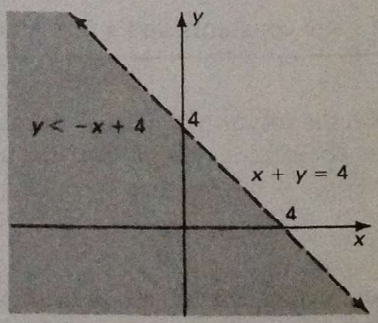


FIGURE 19.22

we conclude from number 3 in the previous list that the solution consists of all points below the line. Part of this region is shaded in Fig. 19.22. Thus if (x, y) is *any* point in this region, then y is less than the number $-x + 4$. For example, the point $(-2, 1)$ is in the region and $1 < -(-2) + 4$. You must realize that this check with a single point does *not* guarantee that our solution is correct. However, if the given inequality is *not* satisfied by the chosen point, then our solution is not correct. Moreover, if we had required that $y \leq -x + 4$ instead of $y < -x + 4$, then the line $y = -x + 4$ would also have been included in the solution as indicated by the solid line in Fig. 19.23. We shall adopt the conventions that **a solid line *is included* and that a broken line *is not* included in the solution.**

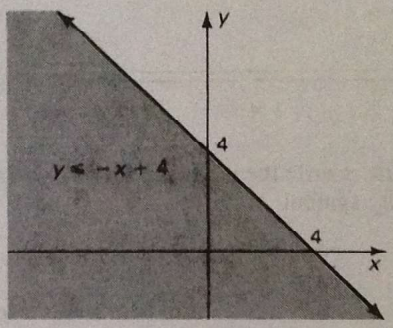


FIGURE 19.23

Example 1

Find the region described by $y \leq 3$.

Solution Since x does not appear, the inequality is assumed to be true for all values of x. The solution consists of the line $y = 3$ *and* the region below it (see Fig. 19.24), because the y-coordinate of each point in that region is less than 3.

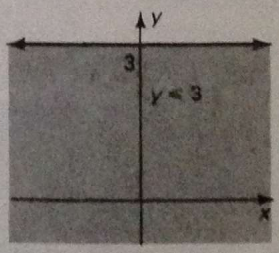


FIGURE 19.24

Example 2

Find the region described by $x \leq 4$.

Solution Because y does not appear, the inequality is assumed to be true for all values of y. The solution consists of the line $x = 4$ and the region to the left of the line (see Fig. 19.25).

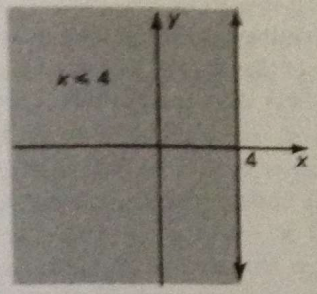


FIGURE 19.25

Example 3

Solve $2(2x - y) < 2(x + y) + 4$.

Solution We first rewrite the inequality in an equivalent form so that y alone appears to the left of the inequality symbol:

$$2(2x - y) < 2(x + y) + 4$$

$$4x - 2y < 2x + 2y + 4$$

$$-4y < -2x + 4$$

$$-y < -\frac{x}{2} + 1$$

$$y > \frac{x}{2} - 1.$$

In the last step, the sense of the original inequality was changed because we multiplied both sides by -1, a negative number. We now sketch the line $y = (x/2) - 1$ by using the points $(2, 0)$ and $(0, -1)$. Then we shade the region above the line (see Fig. 19.26). Every point in this region is a solution.

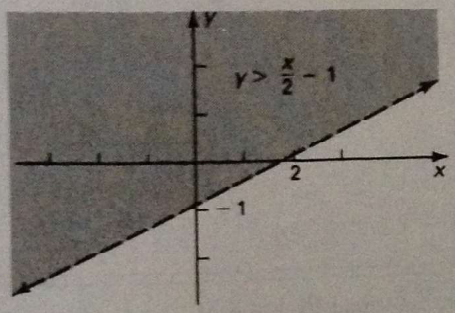


FIGURE 19.26

The solution of a *system* of inequalities consists of all points whose coordinates simultaneously satisfy all the given inequalities. Geometrically, it is the region that is common to all the regions determined by the given inequalities. For example, let us solve the system

$$\begin{cases} y - 3x < 6 \\ x - y \leq -3. \end{cases}$$

This system is equivalent to

$$\begin{cases} y < 3x + 6 \\ y \geq x + 3. \end{cases}$$

Note that each inequality has been written so that y is isolated on one side. Thus the appropriate regions with respect to the corresponding lines will be apparent. We first sketch the lines $y = 3x + 6$ and $y = x + 3$ and then shade the region that is simultaneously *below* the first line and *on or above* the second line (see Fig. 19.27). This region is the solution. When sketching the lines, **you should draw dashed lines everywhere until it is clear which portions of the lines are to be included in the solution.**

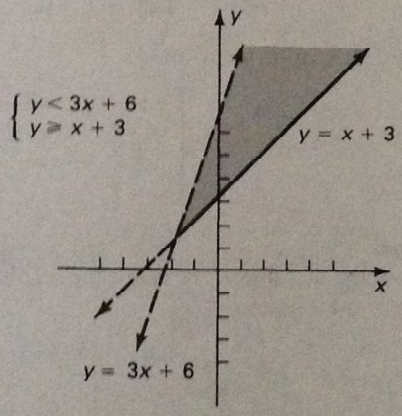


FIGURE 19.27

Example 4

Find the region described by

$$\begin{cases} 2x + 3y \leq 60 \\ x \geq 0 \\ y \geq 0. \end{cases}$$

Solution This system relates to Inequality 1 in the discussion at the beginning of this section. The inequalities $x \geq 0$ and $y \geq 0$ restrict the solution to points that are both on or to the right of the y-axis *and* also on or above the x-axis. The desired region is indicated in Fig. 19.28.

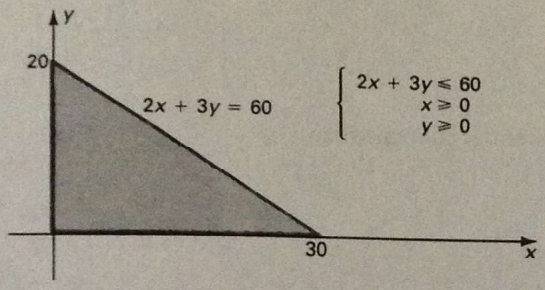


FIGURE 19.28

Example 5

Solve the system

$$\begin{cases} 2x + y > 3 \\ x \geq y \\ 2y - 1 > 0. \end{cases}$$

Solution The system is equivalent to

$$\begin{cases} y > -2x + 3 \\ y \leq x \\ y > \frac{1}{2}. \end{cases}$$

We sketch the lines $y = -2x + 3$, $y = x$, and $y = \frac{1}{2}$ and then shade the region that is simultaneously *above* the first line, *on or below* the second line, and *above* the third line (see Fig. 19.29). This region is the solution.

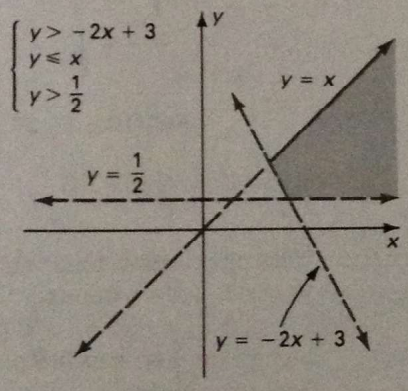


FIGURE 19.29

Problem Set 19.4

Sketch the region described by the following inequalities.

1. $y > 2x$.
2. $y < 3x - 4$.
3. $2x + 3y \leq 6$.
4. $x + 3y > 12$.
5. $x + 5y < -5$.
6. $3x + y \leq 0$.
7. $-x \geq 2y - 4$.
8. $2x + y \geq 10$.
9. $2(x + y) > 2(2x - y)$.
10. $-x < 2$.
11. $\frac{3}{2}x + \frac{4}{3}y > \frac{3}{2}(x - y)$.
12. $2(x^2 + 4x + y) > 4(x^2 - y + 1) - 2x^2$.
13. $\begin{cases} y \leq 2x \\ x > 2y. \end{cases}$
14. $\begin{cases} x - y < 1 \\ y - x \leq 1. \end{cases}$
15. $\begin{cases} 3x - 2y < 6 \\ x - 3y > 9. \end{cases}$
16. $\begin{cases} 2x + 3y > -6 \\ 3x - y < 6. \end{cases}$
17. $\begin{cases} 2x + 3y \leq 6 \\ x \geq 0. \end{cases}$
18. $\begin{cases} 2y - 3x < 6 \\ x < 0. \end{cases}$
19. $\begin{cases} 2x - 2 \geq y \\ 2x \leq 3 - 2y. \end{cases}$
20. $\begin{cases} \frac{3}{2}x - \frac{3}{4}y \geq 1 \\ x(x + 1) - 5 \leq xy - x(y - x). \end{cases}$
21. $\begin{cases} x - y > 4. \\ x < 2 \\ y > 1. \end{cases}$
22. $\begin{cases} 2x + y < -1 \\ y > -x \\ 2x + 4 < 0. \end{cases}$
23. $\begin{cases} y < 2x + 1 \\ y > 1 \\ x > \frac{1}{2}. \end{cases}$
24. $\begin{cases} 4x + 3y \geq 12 \\ y \geq x \\ 2y \leq 3x + 6. \end{cases}$
25. $\begin{cases} 5y - 2x \leq 10 \\ 4x - 6y \leq 12 \\ y \geq 2. \end{cases}$
26. $\begin{cases} 3x + y > -6 \\ x - y > -5 \\ x \geq 0. \end{cases}$

19.5 REVIEW

Important Terms and Symbols

Section 19.1 Interval, closed interval, endpoint, interval notation, open interval, half-open interval, $[a, b]$, (a, b), $(a, b]$, $[a, b)$, $(-\infty, b)$, $(-\infty, b]$, $[a, \infty)$, (a, ∞), $(-\infty, \infty)$, sense of inequality, absolute inequality, conditional inequality, equivalent inequalities, linear inequality in x.

Section 19.4 Linear inequality in x and y, system of inequalities.

Formula Summary

LINEAR INEQUALITIES

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
3. If $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

Multiplication or division by a negative number reverses the sense of the inequality.

ABSOLUTE-VALUE EQUATIONS AND INEQUALITIES

If $d > 0$, then

1. $|x| = d$ means $x = \pm d$.
2. $|x| < d$ means $-d < x < d$.
3. $|x| > d$ means $x < -d$ or $x > d$.

PROPERTIES OF INEQUALITIES

1. $|ab| = |a||b|$.
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.
3. $|a - b| = |b - a|$.
4. $|a + b| \leq |a| + |b|$.

Review Questions

1. True or false: If $x > -3$, then $-x > 3$. (a) _____

   If $x > 4$, then $2x > 8$. (b) _____

   If $2 < x < 3$, then both $x > 2$ and $x < 3$. (c) _____

2. The solution of $4x \leq 12$ is _____.
3. The solution of $x + 1 < x + 2$ is _____.
4. The solution of $x - 2 \geq x + 1$ is _____.
5. The solution of $|x - 5| = 0$ is _____.
6. The solution of $-3 \leq x - 4 \leq 3$ is _____.

7. If $-x > 0$, then $|x| =$ _____.

8. In absolute value notation, the fact that $2x$ is less than seven units of 4 would be written _____.

9. True or false: The inequality $x > 2$ is equivalent to $x^2 > 4$. _____

10. The inequality $x \geq 4$ is satisfied by all points to the $\underset{\text{(a)}}{\text{(right)(left)}}$ of the line $x = 4$ and $\underset{\text{(b)}}{\text{(includes)(excludes)}}$ the line itself.

11. If $y_1 \geq m_1x + b_1$ and $y_2 \leq m_2x + b_2$ is a system of inequalities, which of the regions 1, 2, 3, or 4 in Fig. 19.30 would correspond to the solution? _____

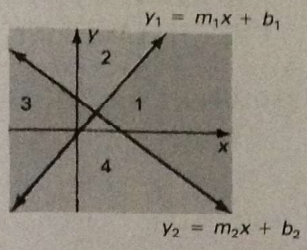


FIGURE 19.30

Answers to Review Questions

1. (a) False, (b) True, (c) True. **2.** $x \leq 3$. **3.** $-\infty < x < \infty$. **4.** $\varnothing$. **5.** 5. **6.** $1 \leq x \leq 7$. **7.** $-x$ **8.** $|2x - 4| < 7$. **9.** False. **10.** (a) Right, (b) Includes. **11.** 3.

Review Problems

In Problems **1–34**, *solve the inequalities and equations.*

1. $3x + 5 < 6$.

2. $3(x + 1) > 9$.

3. $4 - 2x \geq 8$.

4. $-2(x + 6) \leq x + 4$.

5. $3(t + 4) < 9 + 6t$.

6. $5(s + 3) > 2(s - 1)$.

7. $\frac{1}{3}(x + 2) \geq \frac{1}{4}x + 4$.

8. $\dfrac{x + 1}{5} \leq \dfrac{x}{10} + 2$.

9. $x^2 + 4x - 12 < 0$.

10. $x^2 + 11x + 28 > 0$.

11. $y^2 > 6y$.

12. $z^2 + 6z < -5$.

13. $2x^2 + 5x \geq x^2 - 4x - 20$.

14. $3x(x - 4) \leq 2x^2 - 27$.

15. $x(3x + 2) < 1$.

16. $6x(x - 1) > 1 - 5x$.

17. $(x + 4)(x - 5)(x - 9) > 0$.

18. $(x + 2)(x + 4)^2 < 0$.

19. $p^3 - 8p^2 \leq 0$.

20. $r^3 - 9r \geq 0$.

21. $\dfrac{x + 9}{x + 2} \geq 0$.

22. $\dfrac{x + 3}{x^2 - 3x + 2} \leq 0$.

23. $\dfrac{x^2 - 6x + 9}{x^2 + 7x + 10} > 0$.

Day: Date:

24. $\dfrac{x^2 + 2x - 8}{x^2 - 25} < 0.$

25. $|3 - 2x| = 7.$

26. $\left|\dfrac{5x - 8}{13}\right| = 0.$

27. $|3x| > 6.$

28. $|x + 4| \leq 6.$

29. $|4x - 1| < 1.$

30. $\left|\dfrac{5x - 8}{12}\right| \geq 1.$

31. $|4 - 2x| \geq 4.$

32. $|-1 - x| < 1.$

33. $|x + \frac{1}{2}| \leq \frac{3}{2}.$

34. $|\frac{2}{3}x - 5| > 4.$

In Problems **35–38**, *sketch the region described by the given system.*

35. $\begin{cases} 2x + y < 4 \\ -y + 2x > 5. \end{cases}$

36. $\begin{cases} 3x + 2y > 5 \\ -3y + 5x < 7. \end{cases}$

37. $\begin{cases} x - y < 4 \\ y - x < 4. \end{cases}$

38. $\begin{cases} y \geq 5 \\ 2x - y < -2. \end{cases}$

39. $\begin{cases} x + y > 1 \\ 3x - 5 \geq y \\ y > 2x. \end{cases}$

40. $\begin{cases} 3x + y > -4 \\ x - y < -5 \\ x > 0. \end{cases}$

41. Show that if $|y - L| < \epsilon$, then $L - \epsilon < y < L + \epsilon$.